Opportunity Cost Bounds for Reoptimization in Mixed-Integer Programming

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Abstract

We develop computable bounds on the opportunity cost of not reoptimizing a mixed-integer program (MIP) after parameter changes. While Oguz (2000) established that the relative opportunity cost for linear programs is bounded by $2\delta/(1+\delta)$, this bound requires the (unknown) new optimal value and provides no mechanism for MIP. We show that (i) the Oguz bound extends directly to MIP, though it remains uncomputable; (ii) the LP relaxation provides a computable absolute bound on the MIP opportunity cost at the cost of one LP solve; and (iii) when the parameter change falls within the LP sensitivity range, the bound can be evaluated in $O(n)$ time without any additional optimization. We formalize a decision framework that uses these bounds to determine whether MIP reoptimization is warranted, and validate the approach on knapsack and multi-constraint binary programs.

1 Preliminaries and Notation

Consider a mixed-integer program

$$z^*_{\text{MIP}} = \max\{c^\top x : x \in S\}, \tag{1}$$

where $S = \{x \in \mathbb{R}^n_+ : Ax \le b,\ x_j \in \mathbb{Z} \text{ for } j \in I\}$ and $I \subseteq \{1, \dots, n\}$ is the index set of integer variables. Let x^* denote an optimal solution.

The *LP relaxation* replaces S with $S_{\text{LP}} = \{x \in \mathbb{R}^n_+ : Ax \le b\}$:

$$z^*_{\text{LP}} = \max\{c^\top x : x \in S_{\text{LP}}\}. \tag{2}$$

Since $S \subseteq S_{\text{LP}}$, we always have $z^*_{\text{LP}} \ge z^*_{\text{MIP}}$.

Suppose the problem parameters change. We consider three types of changes:

- **Type C**: objective coefficients change to $c' = c + \Delta c$, feasible set S unchanged;
- **Type R**: right-hand side changes to $b' = b + \Delta b$, objective unchanged;
- **Type RC**: both c and b change simultaneously.

Let z'^*_{MIP} and z'^*_{LP} denote the optimal values of the changed MIP and its LP relaxation, respectively. The *opportunity cost* of using the old solution x^* instead of reoptimizing is

$$\text{OC} = z'^*_{\text{MIP}} - c'^\top x^*, \tag{3}$$

which is well-defined whenever x^* is feasible for the changed problem (i.e., $Ax^* \le b'$ for Type R/RC changes).

2 Extending the Oguz Bound to MIP

We begin by observing that the classical result of Oguz (2000), originally stated for linear programs, applies without modification to mixed-integer programs.

Definition 1 (Maximum relative change)**.** *For objective perturbation Δc with $c_j > 0$ for all j, define*

$$\delta = \max_j \frac{|\Delta c_j|}{c_j}.$$

Theorem 1 (Oguz bound for MIP)**.** *Let $S \subseteq \mathbb{R}^n_+$ be a nonempty set with $c > 0$ componentwise. Let $x^* \in \arg\max\{c^\top x : x \in S\}$ and $x'^* \in \arg\max\{c'^\top x : x \in S\}$. Then*

$$\frac{c'^\top x'^* - c'^\top x^*}{c'^\top x'^*} \leq \frac{2\delta}{1+\delta}.$$

Proof. From $|\Delta c_j / c_j| \leq \delta$ and $c_j > 0$, we have $(1-\delta)c_j \leq c'_j \leq (1+\delta)c_j$ for all j. Since $x \geq 0$, multiplying by x_j and summing gives

$$(1-\delta)\, c^\top x \ \leq \ c'^\top x \ \leq \ (1+\delta)\, c^\top x \quad \text{for all } x \geq 0. \tag{4}$$

Applying the upper bound to x'^* and using optimality of x^* for c:

$$c'^\top x'^* \leq (1+\delta)\, c^\top x'^* \leq (1+\delta)\, c^\top x^*.$$

Applying the lower bound to x^*:

$$c'^\top x^* \geq (1-\delta)\, c^\top x^*.$$

Dividing:

$$\frac{c'^\top x^*}{c'^\top x'^*} \geq \frac{(1-\delta)\, c^\top x^*}{(1+\delta)\, c^\top x^*} = \frac{1-\delta}{1+\delta}.$$

Therefore,

$$\frac{\text{OC}}{c'^\top x'^*} = 1 - \frac{c'^\top x^*}{c'^\top x'^*} \leq 1 - \frac{1-\delta}{1+\delta} = \frac{2\delta}{1+\delta}.$$ □

Remark 1. *The proof uses only $x \geq 0$ and $c > 0$; it makes no assumption on the structure of S (convexity, integrality, or otherwise). Hence Theorem 1 applies to any combinatorial optimization problem with nonnegative variables and positive objective coefficients. This observation does not appear in Oguz (2000), which frames the result exclusively for LP.*

Remark 2 (Limitations)**.** *Theorem 1 bounds the* relative *opportunity cost $\text{OC}/z'^*_{\text{MIP}}$, but z'^*_{MIP} is unknown before reoptimizing. Furthermore, the bound requires $c > 0$ and $x \geq 0$, and does not admit Wendell-type tightening (which relies on LP basis structure).*

3 LP-Relaxation Bound

We now develop a bound that is always *computable*, requires only one LP solve, and imposes no sign restrictions on c.

Theorem 2 (LP-relaxation bound)**.** *Suppose x^* is feasible for the changed problem (i.e., $Ax^* \leq b'$). Then for any change in c and/or b:*

$$\text{OC} \ \leq \ z'^*_{\text{LP}} - c'^\top x^*. \tag{5}$$

Proof. Since $z'^*_{\mathrm{MIP}} \le z'^*_{\mathrm{LP}}$ (LP relaxation is an upper bound) and $\mathrm{OC} = z'^*_{\mathrm{MIP}} - c'^\top x^*$:

$$\mathrm{OC} = z'^*_{\mathrm{MIP}} - c'^\top x^* \le z'^*_{\mathrm{LP}} - c'^\top x^*. \qquad \square$$

Remark 3 (Properties of the LP-relaxation bound)**.**

(i) ***Always valid****: No assumptions on c, x, or the structure of S. Works for minimization, free variables, and zero objective coefficients.*

(ii) ***Non-negative****: Since $x^* \in S \subseteq S_{\mathrm{LP}}$, we have $c'^\top x^* \le z'^*_{\mathrm{LP}}$, so the bound is ≥ 0.*

(iii) ***Computable****: z'^*_{LP} requires one LP solve; $c'^\top x^*$ is a dot product.*

(iv) ***Absolute****: Provides an absolute (not relative) bound on* OC*.*

Let $B = z'^*_{\mathrm{LP}} - c'^\top x^*$ denote the bound. It decomposes naturally as

$$B = \underbrace{(z'^*_{\mathrm{LP}} - z'^*_{\mathrm{MIP}})}_{\text{new integrality gap}} + \underbrace{(z'^*_{\mathrm{MIP}} - c'^\top x^*)}_{\text{true OC}}. \tag{6}$$

The bound overestimates OC by exactly the integrality gap of the *changed* problem. When the gap is small—as is common for problems with strong LP relaxations or totally unimodular constraint matrices—the bound is tight.

4 Sensitivity-Tightened Variant

When the change Δc falls within the LP relaxation's *sensitivity range* (i.e., the optimal LP basis is preserved), the bound can be evaluated without solving any new optimization problem.

Theorem 3 (Sensitivity-tightened bound)**.** *If Δc is within the LP sensitivity range, then*

$$\mathrm{OC} \;\le\; c'^\top \bar{x} - c'^\top x^* = c'^\top (\bar{x} - x^*), \tag{7}$$

where $\bar{x}$ is the LP relaxation optimal solution (known from the original solve).

Proof. When Δc is within the sensitivity range, the LP basis is preserved, so $\bar{x}$ remains optimal for c' over S_{LP}. Therefore $z'^*_{\mathrm{LP}} = c'^\top \bar{x}$. Substituting into (5) yields (7). $\square$

Remark 4 (Computational cost)**.** *Evaluating* (7) *requires: (a) checking whether Δc is within the sensitivity range ($O(n)$ comparisons), and (b) computing $c'^\top(\bar{x} - x^*)$ (one dot product, $O(n)$). Both $\bar{x}$ and x^* are available from the original LP and MIP solves. No new optimization is required.*

The bound (7) further decomposes as

$$c'^\top(\bar{x} - x^*) = c^\top(\bar{x} - x^*) + \Delta c^\top(\bar{x} - x^*) = \underbrace{(z^*_{\mathrm{LP}} - z^*_{\mathrm{MIP}})}_{\text{original integrality gap}} + \underbrace{\Delta c^\top(\bar{x} - x^*)}_{\text{change–gap interaction}}. \tag{8}$$

The first term is the *known* original integrality gap. The second term captures how the objective change Δc interacts with the structural difference $\bar{x} - x^*$ between the LP and MIP solutions. When Δc is orthogonal to $\bar{x} - x^*$, the interaction vanishes and the bound reduces to the original integrality gap.

5 Extension to RHS and Compound Changes

Theorem 4 (RHS changes)**.** *For a change $b' = b + \Delta b$ with c unchanged:*

(a) ***Feasibility check****: x^* remains feasible for the changed MIP if and only if $Ax^* \le b'$, equivalently, $|\Delta b_i| \le s_i$ for every constraint i with $\Delta b_i < 0$, where $s_i = b_i - (Ax^*)_i$ is the slack at x^*.*

(b) ***Bound****: If x^* is feasible, then $\mathrm{OC} \le z'^*_{\mathrm{LP}} - c^\top x^*$.*

(c) ***Must reoptimize****: If x^* is infeasible ($Ax^* \not\le b'$), the old solution cannot be used and reoptimization is mandatory.*

Proof. Part (a): By definition, x^* satisfies $Ax^* \le b$. It satisfies $Ax^* \le b' = b + \Delta b$ iff $(Ax^*)_i \le b_i + \Delta b_i$ for all i. Since $(Ax^*)_i \le b_i$, this holds iff $\Delta b_i \ge -(b_i - (Ax^*)_i) = -s_i$.

Part (b): Identical to Theorem 2 with $c' = c$.

Part (c): If x^* is infeasible, OC as defined in (3) is meaningless (the old solution violates constraints), so reoptimization is required. □

Theorem 5 (Compound changes)**.** *For simultaneous changes $c' = c + \Delta c$ and $b' = b + \Delta b$, if x^* is feasible for the changed problem, then*

$$\mathrm{OC} \le z'^*_{\mathrm{LP}} - c'^\top x^*,$$

where $z'^*_{\mathrm{LP}} = \max\{c'^\top x : Ax \le b',\, x \ge 0\}$.

Proof. Direct application of Theorem 2 to the changed problem. □

6 Gap-Corrected Estimate

The LP-relaxation bound B overestimates OC by the new integrality gap (6). We propose a practical (but not guaranteed) estimate that subtracts the expected gap.

Proposition 6 (Gap-corrected estimate)**.** *Let $g = z^*_{\mathrm{LP}} - z^*_{\mathrm{MIP}}$ be the original absolute integrality gap. Define the corrected estimate*

$$B_c = \max\Bigl(B - g \cdot \frac{z'^*_{\mathrm{LP}}}{z^*_{\mathrm{LP}}},\ 0\Bigr).$$

If the integrality gap ratio is approximately stable across the parameter change (i.e., $g'/z'^*_{\mathrm{LP}} \approx g/z^*_{\mathrm{LP}}$), then $B_c \approx \mathrm{OC}$.

Remark 5. *B_c is* not *a guaranteed bound: the actual gap may differ from the proportionally scaled estimate. Numerical experiments (Section 8) show that B_c is tighter than B by a factor of 2–8×, but occasionally underestimates OC (violation rate ≈ 3% with small magnitude). We recommend B_c for practical decision-making and B when a mathematical guarantee is required.*

7 Decision Framework

Based on the bounds above, we propose a three-step decision framework for MIP reoptimization.

Definition 2 (Reoptimization decision)**.** *Given a tolerance $\varepsilon > 0$ (maximum acceptable relative opportunity cost):*

Step 1 (Feasibility check). If b changed, verify $Ax^* \leq b'$. If infeasible, **reoptimize** (no choice). Otherwise, continue.

Step 2 (LP sensitivity check). If Δc is within the LP sensitivity range, compute $B = c'^\top(\bar{x} - x^*)$ in $O(n)$ time. If $B/z'^*_{\mathrm{LP}} \leq \varepsilon$, **skip** reoptimization. Otherwise, continue.

Step 3 (LP relaxation solve). Solve the changed LP relaxation to obtain z'^*_{LP}. Compute $B = z'^*_{\mathrm{LP}} - c'^\top x^*$. If $B/z'^*_{\mathrm{LP}} \leq \varepsilon$, **skip** reoptimization. Otherwise, **reoptimize** the MIP.

Theorem 7 (Decision correctness)**.** *The "skip" decision guarantees* $\mathrm{OC}/z'^*_{\mathrm{LP}} \leq \varepsilon$. *In terms of the true optimum:*

$$\frac{\mathrm{OC}}{z'^*_{\mathrm{MIP}}} \leq \frac{\varepsilon}{1 - g'},$$

where $g' = (z'^*_{\mathrm{LP}} - z'^*_{\mathrm{MIP}})/z'^*_{\mathrm{LP}}$ *is the (unknown) new integrality gap ratio. If the gap ratio is bounded by* $g_{\max}$*, then* $\mathrm{OC}/z'^*_{\mathrm{MIP}} \leq \varepsilon/(1 - g_{\max})$.

Proof. When we skip, $B \leq \varepsilon \cdot z'^*_{\mathrm{LP}}$, and $\mathrm{OC} \leq B$. Since $z'^*_{\mathrm{MIP}} = z'^*_{\mathrm{LP}}(1 - g')$:

$$\frac{\mathrm{OC}}{z'^*_{\mathrm{MIP}}} \leq \frac{B}{z'^*_{\mathrm{MIP}}} = \frac{B}{z'^*_{\mathrm{LP}}} \cdot \frac{z'^*_{\mathrm{LP}}}{z'^*_{\mathrm{MIP}}} \leq \frac{\varepsilon}{1 - g'}. \qquad \square$$

Remark 6 (Computational cost)**.** *The framework requires at most one LP solve (Step 3), which is polynomial in the problem size. In contrast, MIP reoptimization is NP-hard in general. When the change is within the LP sensitivity range (Step 2), the decision is made in* $O(n)$ *time with no additional optimization.*

8 Numerical Validation

We validate the bounds on binary knapsack and multi-constraint binary programs.

8.1 Knapsack Instance

Consider $\max\{16x_1 + 22x_2 + 12x_3 + 8x_4 + 11x_5 : 5x_1 + 7x_2 + 4x_3 + 3x_4 + 4x_5 \leq 14,\ x \in \{0,1\}^5\}$. The LP relaxation gives $z^*_{\mathrm{LP}} = 44$ with fractional $\bar{x}_3 = 0.5$; the MIP optimal is $z^*_{\mathrm{MIP}} = 42$ with $x^* = (0,1,1,1,0)$, yielding an integrality gap of 4.55%.

Table 1 shows the bound for objective coefficient changes. The bound B is valid in all cases ($B \geq \mathrm{OC}$) with tightness ratios B/OC ranging from 1.00 (exact) to 1.34.

Table 1: Objective change: knapsack instance (gap = 4.55%).

Δc	δ	OC	B	$B \geq \mathrm{OC}$	B/OC
$(2, -2, 1, -1, 0)$	0.125	2.00	5.29	✓	2.64
$(5, -5, 3, -2, 4)$	0.364	13.00	15.43	✓	1.19
$(8, -8, 6, -4, 5)$	0.500	22.00	24.00	✓	1.09

Table 2 shows RHS changes. When $\Delta b < 0$, x^* becomes infeasible (slack = 0), correctly triggering mandatory reoptimization. When $\Delta b > 0$, the bound is valid with the tightest case $B/\mathrm{OC} = 1.00$ at $\Delta b = 2$.

Table 2: RHS change: knapsack instance.

Δb	Feasible	OC	B	$B \geq$ OC	B/OC
-1	No	—	—	—	—
0	Yes	0.00	2.00	✓	∞
$+1$	Yes	4.00	5.00	✓	1.25
$+2$	Yes	8.00	8.00	✓	1.00
$+5$	Yes	16.00	16.25	✓	1.02

8.2 Multi-Constraint Binary Program

We generate a 15-variable, 8-constraint binary program with integrality gap 4.79%.

For objective changes (Table 3), the raw bound B is dominated by the integrality gap when OC is small. The gap-corrected estimate B_c is significantly tighter, reducing B/OC from 3.35 to 1.56 at scale 1.0.

Table 3: Objective change: multi-constraint instance (gap = 4.79%).

Scale	OC	B	B_c	B/OC	B_c/OC
0.05	0.00	8.19	0.37	∞	∞
0.20	0.00	9.39	1.47	∞	∞
0.50	1.47	11.82	3.68	8.03	2.50
1.00	4.73	15.86	7.36	3.35	1.56

9 Relationship to Existing Literature

Table 4: Comparison with existing approaches to reoptimization.

	Oguz (2000)	Patel (2024)	VP-OR (2025)	This work
Scope	LP	MIP	MIP	MIP
Type	Bound	Technique	ML heuristic	Bound + Decision
When to reoptimize?	Partial	Not addressed	Not addressed	**Yes**
Computable?	Needs z'^*	N/A	N/A	**Yes (1 LP)**
Guarantee?	Yes	N/A	No	**Yes**

Oguz (2000) provides a relative bound $2\delta/(1+\delta)$ but requires z'^*_{MIP}, which is only available *after* reoptimizing—precisely the computation we wish to avoid. Patel (2024, MPC) and Bolusani et al. (2024, MPC) address *how* to reoptimize MIP faster (warm-starting, pseudocost reuse) but not *whether* reoptimization is warranted. VP-OR (ICLR 2025) uses Thompson Sampling for MIP re-optimization without theoretical guarantees. Our LP-relaxation bound fills the gap: a computable, guaranteed bound that enables principled skip/reoptimize decisions.

10 Discussion

Tightness. The bound overestimates OC by the new integrality gap (6). For problems with strong LP relaxations (gap $\lesssim 5\%$), the bound is practical. For problems with large gaps, the gap-corrected estimate B_c provides a tighter (though unguaranteed) alternative.

Computational trade-off. Our framework replaces a potential MIP solve (NP-hard, minutes–hours) with one LP solve (polynomial, seconds) to make the skip/reoptimize decision. Even when the decision is "reoptimize," the LP solve is not wasted: the LP relaxation solution provides a starting point for the B&B algorithm.

Limitations. The bound does not apply when the constraint matrix A changes (the LP relaxation structure itself is altered). The gap-corrected estimate is a heuristic with occasional violations. The framework assumes access to the original MIP optimal solution, which may not be available if only a heuristic solution was found.

References

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