

A Bialgebra Framework for Positivity in Schubert Calculus

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Schubert polynomials

- ▶ Multivariate polynomials representing cohomology classes of Schubert varieties in the complete flag variety.
- ▶ Indexed by permutations, with the Schubert polynomial $\mathfrak{S}_w$ corresponding to a permutation w .
- ▶ Generalize the Schur polynomials, so their structure constants generalize the Littlewood-Richardson coefficients.
- ▶ Structure constants are nonnegative integers, but no positive combinatorial formula is known for them in general.

$$\mathfrak{S}_{41523}(x_1, x_2, x_3) = x_1^3 x_2^2 + x_1^3 x_2 x_3 + x_1^3 x_3^2$$

A run of the mill bialgebra

Define

$$\mathbf{S} = \bigoplus_{n \geq 0} \mathbf{S}_n, \quad \mathbf{S}_n = \mathbb{Q}[x_1, \dots, x_n].$$

Multiplication in $\mathbf{S}$ is componentwise:

- ▶ if $f, g \in \mathbf{S}_n$, then $f \cdot g$ is the usual product in $\mathbb{Q}[x_1, \dots, x_n]$;
- ▶ if $f \in \mathbf{S}_m$, $g \in \mathbf{S}_n$ with $m \neq n$ and $m, n > 0$, then $f \cdot g = 0$.

So each graded piece is a familiar polynomial ring, and cross-component products are trivial. The coproduct is

$$\Delta(x_c) = \sum_{ab=c} x_a \otimes x_b$$

where ab is the concatenation of weak compositions. This makes $\mathbf{S}$ into a graded bialgebra.

$$\Delta(x_1^2 x_3 x_5) = 1 \otimes x_1^2 x_3 x_5 + x_1^2 \otimes x_3 x_5 + x_1^2 x_3 \otimes x_5 + x_1^2 x_3 x_5 \otimes 1.$$

Double grading

In addition to the degree as a polynomial, $\mathbf{S}$ is also graded by inclusion in the direct summand $\mathbf{S}_n$, which we interpret as length.

We define the dual algebra $\mathcal{DS}$ to be the graded dual of $\mathbf{S}$ with respect to this $\mathbb{N} \times \mathbb{N}$ grading. For the monomial basis, which is indexed by weak compositions, we define the dual basis $[\alpha]$ for weak compositions α by

$$\langle x_\alpha, [\beta] \rangle = \delta_{\alpha, \beta}$$

Hence $\mathcal{DS}$ is also graded by length and degree, with the latter being the sum of the composition, and the product and coproduct are dual to those of $\mathbf{S}$ with respect to the degree grading.

Dual indexing and product

We have that the dual product is concatenation:

$$[\alpha] \cdot [\beta] = [\alpha \beta].$$

Hence $\mathcal{DS}$ is a free associative algebra (tensor algebra on the generators $[k]$, $k \geq 0$).
The coproduct is

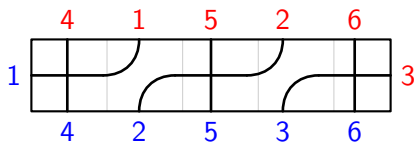
$$\Delta([\gamma]) = \sum_{\alpha+\beta=\gamma} [\alpha] \otimes [\beta].$$

We now define and investigate a curious combinatorially defined basis of $\mathcal{DS}$ indexed by code-length bounded permutations.

Crossing diagrams

Think of a one-row strip of $n - 1$ boxes.

- ▶ Label the left edge by 1.
- ▶ Label the bottom by a permutation of the numbers $2, 3, \dots, n$ (left to right).
- ▶ In each box, place either a *cross* $\boxplus$ or a *bump* $\boxminus$.



Follow the strands through the row. The admissibility condition is that it cannot undo inversions, only create new ones.

A nice combinatorial basis

Let Ξ_w^k be the basis indexed by permutations w such that $w(p) < w(p+1)$ for all $p > k$ satisfying the following:

- ▶ $\Xi_w^1 = [w(1) - 1]$
- ▶ For a nonnegative integer $[p]$, we have that

$$[p] \cdot \Xi_w^k = \sum_{w'} \Xi_{w'}^{k+1}$$

where w' is obtained from w through a valid crossing diagram with p crosses after adding 1 to each element of w and adding a new first element 1.

I can provide an explicit (but mildly cancellative) formula for Ξ_w^k in the word basis, as well as a positive formula for the product of Ξ_u^p and Ξ_v^q in terms of crossing diagrams. The coproduct structure constants are also nonnegative integers, but I do not have a positive combinatorial formula for them yet. . .

Taking off the mask: The dual Schubert basis

The basis satisfies

$$\langle \mathfrak{S}_u(x_1, \dots, x_n), \Xi_v^n \rangle = \delta_{u,v}$$

where $\mathfrak{S}_u(x_1, \dots, x_n)$ is the Schubert polynomial corresponding to the permutation u .
The coefficients $c_{u,v}^w$ of the coproduct formula

$$\Delta \Xi_w^n = \sum_{u,v} c_{u,v}^w \Xi_u^n \otimes \Xi_v^n$$

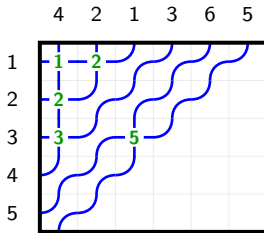
are the Littlewood-Richardson coefficients for Schubert polynomials!

Solution of the classical LR problem

- ▶ One solution of the classical LR problem is the Knuth relations on words of tableaux
- ▶ This combinatorial construction creates an algebra with clearly nonnegative structure constants
- ▶ The subalgebra generated by sums of tableaux of a given shape behave precisely like the Schur polynomials, giving the solution
- ▶ I will do this here for Schubert polynomials! With a catch

Bounded pipe dreams

Pipe dreams are well-studied objects introduced in the 1990s by Bergeron and Billey to give a nicer combinatorial formula for Schubert polynomials equivalent to an earlier one by Billey, Jockusch, and Stanley involving reduced words/compatible sequences. Bounded pipe dreams have an additional length parameter tacked on.



Theorem (Bergeron, Billey, 1993)

The Schubert polynomial $\mathfrak{S}_w$ is equal to

$$\sum_{R \in \text{Pipe}(w)} x_{\text{wt}(R)}$$

Transition algorithm on bounded pipe dreams

There is an algorithm that takes a bounded pipe dream of length n with row n empty to a unique pipe dream of length $n - 1$ in a way that matches the Schubert transition formula in a strong, weight-preserving, bijective form.

Input: A bounded pipe dream R with $\text{length}(R) = n$ and row n empty.

Output: A bounded pipe dream R' with $\text{length}(R') = n - 1$, $|R'| = |R|$ (same number of crossing in each row), and $w_R \xrightarrow{n} w_{R'}$.

if R with last row removed is a valid bounded pipe dream **then**

return R with last row removed

else

 Locate crossings corresponding to maximal right descents of $w(R)$ until the maximal right descent is $n - 1$, say at rows $i_1 < i_2 < \dots < i_p$ with corresponding columns $j_1, j_2, \dots, j_p$.

$R^- \leftarrow R \setminus \{(i_1, j_1), \dots, (i_p, j_p)\}$.

 Let $I \leftarrow (i_1, i_2, \dots, i_p)$.

$D \leftarrow R^- \cup \{(n, 1), (n, 2), \dots, (n, p)\}$.

$D' \leftarrow I \overset{n-1}{\rightsquigarrow} D$ (an insertion algorithm similar to existing ones for pipe dreams)

$R' \leftarrow \{(i, j) \in D' \mid i < n\}$ as a bounded pipe dream with $n - 1$ rows.

return R' .

end if

Bounded pipe dream ring product

Given bounded pipe dreams R_1, R_2 , define $R_1 \star R_2$ by taking the sum of all R obtained by

- ▶ *untransitioning* R_1 in all possible valid ways to create space while remaining a pipe dream,
- ▶ shifting R_2 upward,
- ▶ placing the shifted R_2 into the new empty region, unchanged, if doing so yields a valid pipe dream.

This allows us to lift the product of the dual algebra $\mathcal{DS}$ to a product on formal linear combinations of bounded pipe dreams.

Theorem (S, 2026+)

The product $\star$ is associative and gives a well-defined ring structure on the vector space Pipe spanned by bounded pipe dreams.

Pipe dream ring as a lift of $\mathcal{DS}$

There are two surjective linear maps from the pipe dream ring $\mathcal{R}$ to the dual algebra $\mathcal{DS}$:

$$\begin{aligned}\pi_{\text{Sch}} : \mathcal{R} &\twoheadrightarrow \mathcal{DS}, & \pi_{\text{Sch}}(G) &= \Xi_{\text{perm}(G)}^{(n)}, \\ \pi_{\text{wt}} : \mathcal{R} &\twoheadrightarrow \mathcal{DS}, & \pi_{\text{wt}}(G) &= [\text{wt}(G)].\end{aligned}$$

By virtue of the fact that $\mathcal{DS}$ is free, there is a section $\iota : \mathcal{DS} \hookrightarrow \mathcal{R}$ of π_{wt} that sends $[i]$ for an integer i to the unique pipe dream r_i with $\text{wt}(r_i) = (i)$. It turns out that these are all ring homomorphisms and ι is also a section of π_{Sch} .

Theorem (S, 2026+)

We have that π_{Sch} and π_{wt} are surjective ring homomorphisms to $\mathcal{DS}$ and ι is an injective ring homomorphism from $\mathcal{DS}$ to $\mathcal{R}$ such that

$$\pi_{\text{Sch}} \circ \iota = \pi_{\text{wt}} \circ \iota = \text{id}_{\mathcal{DS}}.$$

Littlewood-Richardson rule for (dual) Schubert polynomials

We immediately obtain a formula for the product of dual Schubert elements.

Theorem (S, 2026+)

For dual Schubert elements $\Xi_u^{n_1}, \Xi_v^{n_2}$, we have

$$\Xi_u^{n_1} \cdot \Xi_v^{n_2} = \sum_w d_{u,v}^w(n_1, n_2) \Xi_w^{n_1+n_2},$$

where, fixing pipe dreams R_u for u with $\text{length}(R_u) = n_1$ and R_v for v with $\text{length}(R_v) = n_2$, $d_{u,v}^w(n_1, n_2)$ is the number of bounded pipe dreams $R_w \in \text{Pipe}(w)$ such that R_w has a coefficient of 1 in the product $R_u \star R_v$.

Equivalently,

$$\mathfrak{S}_w(x_1, \dots, x_{n_1+n_2}) = \sum_{u,v} d_{u,v}^w(n_1, n_2) \mathfrak{S}_u(x_1, \dots, x_{n_1}) \mathfrak{S}_v(x_{n_1+1}, \dots, x_{n_1+n_2}).$$

What about a coproduct?

- ▶ What we want now is to construct a coproduct on this algebra of pipe dreams that lifts the coproduct on $\mathcal{DS}$.
- ▶ I don't know how to do this. However, there is a specialized product that can be derived from the $\star$ product that does lift the coproduct on $\mathcal{DS}$.

The squash product

- ▶ Given $R_1, R_2 \in \text{Pipe}_n$, we can define $R_1 * R_2$ to be the union of R_1 with the pipe dream obtained from R_2 by shifting all columns to the right by the length of $\text{perm}(R_1)$
- ▶ Applying the combinatorial transition to cut back to the same number of rows, we obtain a bijective product on Pipe_n that lifts the coproduct on $\mathcal{DS}$, *provided that* R_2 is n -Grassmannian
- ▶ This is a right module action of the (opposite) plactic algebra on Pipe_n

Ghost pipe dreams

- ▶ There is a larger algebra based on the squash product that is essentially a tensor product of opposite plactic algebras with increasing numbers of rows that maps onto Pipe
- ▶ It has an associative product by declaring that n -plactic and m -plactic elements commute for $n \neq m$ and that the product of an n -plactic element with an n -plactic element is given by the squash product. This is compatible with the crystal raising/lowering operators
- ▶ This algebra has “Schubert elements” $\mathfrak{S}_w$ that map to the sum of all pipe dreams with permutation w and have the same product structure constants as the Schubert polynomials (!!!)
- ▶ However, the kernel of the map to Pipe_n is not an ideal
- ▶ Thus there are signed “ghost” pipe dreams that are necessary to maintain the product structure but cancel in Pipe

The representation theorem

Theorem

Let P be an n -row pipe dream. Then there is a unique way to write P as a product

$$P = P_{(n)} \cdot P^{(n)}$$

where $\text{perm}(P_{(n)}) \in S_n$ and $P^{(n)}$ is n -Grassmannian such that for every $P_1, P_2 \in \text{Pipe}_n$ we have

$$(P_1 \cdot P_2)_{(n)} (P_1 \cdot P_2)^{(n)} = ((P_1)_{(n)} \cdot (P_2)_{(n)}) (P_1^{(n)} \cdot P_2^{(n)})$$

Corollary

Every Schubert element $\mathfrak{S}_w$ with n rows can be uniquely expressed as

$$\mathfrak{S}_w = \sum_u c_{w,u}^\lambda \mathfrak{S}_u \cdot G_\lambda$$

where each $u \in S_n$ and each G_λ is an n -Grassmannian Schubert element, with the coefficients $c_{w,u}^\lambda$ being (possibly negative) integers.

Positive formula

Theorem (S, 2026+)

SCHUB $_n \in \#P$ for all n .

Proof.

Using properties of Demazure crystals and the representation theorem, we can deduce with finite precomputation that the highest weight pipe dreams of the product of two n -row Schubert elements have nonnegative coefficients. This reduces the problem to the squash product, which is equivalent to the Littlewood-Richardson rule. $\square$

Since the precomputation is known to come to a positive result, we have essentially skirted the restriction on a positive formula by reducing the cancellation to a finite computation that applies *uniformly*.