

# A LITTLEWOOD–RICHARDSON RULE FOR FOREST POLYNOMIALS VIA THE SCHUBERT BIALGEBRA

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**ABSTRACT.** The forest polynomials  $\mathfrak{P}_a$  of Nadeau–Tewari form a  $\mathbb{Z}$ -basis of  $\mathbb{Z}[x_1, x_2, \dots]$  whose role for the cohomology of the quasisymmetric flag variety parallels that of Schubert polynomials for the classical flag variety. Nonnegativity of the structure constants  $\beta_{a,b}^c$  in  $\mathfrak{P}_a \mathfrak{P}_b = \sum_c \beta_{a,b}^c \mathfrak{P}_c$  is known, but no Littlewood–Richardson-style enumerative rule has been available. We give such a rule:  $\beta_{a,b}^c$  counts pairs of forest RC graphs of forest-codes  $a$  and  $b$  whose lift product lands on a forest RC graph of forest-code and weight both equal to  $c$ . The same rule descends to the cup product on  $H^\bullet(\mathrm{QFl}_n)$ . The proof introduces a *Schubert bialgebra*  $\mathcal{A}$  and lifts the multiplication on its graded dual  $\mathcal{D}$  to a product on a free abelian group  $\mathcal{BRC}$  of bounded RC graphs; the same machinery yields enumerative LR rules for the dual Schubert, dual key, dual forest, and dual slide bases of  $\mathcal{D}$ .

## 1. INTRODUCTION

### 2. THE SCHUBERT BIALGEBRA AND ITS DUAL

#### 2.1. Some preliminaries.

**Definition 2.1.1** (Two Pieri relations  $\xrightarrow{k}$  and  $\xRightarrow{k}$ ). The relations  $\xrightarrow{k}$  and  $\xRightarrow{k}$  are defined on  $S_\infty$  as follows, with different notation, in [8]. For  $v, w \in S_\infty$ , we have  $v \xrightarrow{k} w$  if there is a sequence of transpositions  $t_{a_1 b_1}, \dots, t_{a_m b_m}$  such that the following hold:

- (1)  $a_i \leq k < b_i$  for all  $i$ .
- (2)  $\ell(ut_{a_1 b_1} \cdots t_{a_i b_i}) = \ell(u) + i$  for all  $i$ .
- (3)  $w = ut_{a_1 b_1} \cdots t_{a_m b_m}$ .
- (4) The integers  $a_1, \dots, a_m$  are pairwise distinct.

For  $\xRightarrow{k}$ , only (4) changes:

- (4') The integers  $b_1, \dots, b_m$  are pairwise distinct.

Notice that necessarily  $m \leq k$  for  $\xrightarrow{k}$ , but there is no upper bound on  $m$  for  $\xRightarrow{k}$ . This reflects the fact that the following theorem holds:

**Theorem 2.1** ([8, Theorem 1]). *Let  $p \geq 0$ ,  $k > 0$  be integers and let  $u$  be a permutation.*

*I. We have the formula*

$$\mathfrak{S}_u(x) h_{p,k}(x) = \sum_{\substack{u \xRightarrow{k} w \\ \ell(u,w)=p}} \mathfrak{S}_w(x)$$

*where  $h_{p,k}(x)$  is the complete homogeneous symmetric polynomial of degree  $p$  in the first  $k$  variables.*

*II. We have the formula*

$$\mathfrak{S}_u(x) e_{p,k}(x) = \sum_{\substack{u \xrightarrow{k} w \\ \ell(u,w)=p}} \mathfrak{S}_w(x)$$

*where  $e_{p,k}(x)$  is the elementary symmetric polynomial of degree  $p$  in the first  $k$  variables.*

**Definition 2.1.2** (Lehmer code, dual code, dominant permutations). For a permutation  $w \in S_\infty$ , the (right) *Lehmer code* of  $w$  is the sequence  $\mathfrak{c}(w) = (c_1, c_2, \dots)$  defined by

$$c_i = \#\{j > i : w(j) < w(i)\}.$$

This is a weak composition with  $\sum_i c_i = \ell(w)$ , and the map  $w \mapsto \mathbf{c}(w)$  is a bijection from  $S_\infty$  onto the set of weak compositions of finite support. We will sometimes write  $\mathbf{c}_i(w) := c_i$ . The *dual code* of  $w$  is

$$\mathbf{c}^*(w) := \mathbf{c}(w^{-1}).$$

A permutation  $\mu \in S_\infty$  is called *dominant* if  $\mathbf{c}(\mu)$  is a partition (a weakly decreasing sequence of nonnegative integers, padded by 0s). For a partition  $\lambda$ , we write  $\mathfrak{d}_\lambda$  for the dominant permutation with

$$\mathbf{c}^*(\mathfrak{d}_\lambda) = \lambda.$$

**Definition 2.1.3** (Pulling out variables from double Schubert polynomials, the relation  $\searrow^i$ ). For a permutation  $v'$  such that  $v' \in S_n$ , define

$$\varphi_{i,n}(v')(j) = \begin{cases} v'(j) & \text{if } j < i \\ n+1 & \text{if } j = i \\ v'(j-1) & \text{if } i < j \leq n+1 \\ j & \text{if } j > n+1 \end{cases}$$

Now fix  $v \in S_n$  and  $i \geq 1$  an integer, we define a relation  $\searrow^i$  by declaring that  $v \searrow^i v'$  if

$$v \xRightarrow{i} \varphi_{i,n}(v')$$

Equivalently,  $v \searrow^i v'$  if whenever  $v \in S_n$  and  $n$  is minimal, we have that  $v$  satisfies the relation  $\xRightarrow{i}$  with respect to the permutation obtained from  $v'$  by inserting  $n+1$  at position  $i$ . We note that this concept was introduced by Bergeron and Sottile in [3].

If  $v \searrow^i v'$ , we define a set of integers  $Q_i(v', v)$  by

$$Q_i(v', v) = \{v(j) \mid j > i \text{ and } v'(j-1) = v(j)\}$$

Given the fact that any element of  $S_\infty$  fixes all but finitely many positive integers, it follows that  $Q_i(v', v)$  is a finite set.

The permutations  $v'$  such that  $v \searrow^i v'$  arise when pulling a variable out of a Schubert polynomial and expressing the coefficients of the powers of this variable as Schubert polynomials in the remaining variables, as is done in [2], from which this definition essentially comes.

**Definition 2.1.4** (Upward shifting of permutations, maximum right descent). For a permutation  $v \in S_\infty$ , we define  $\uparrow v$  to be the permutation defined by

$$\uparrow v(i) = \begin{cases} v(i-1) + 1 & \text{if } i > 1 \\ 1 & \text{if } i = 1 \end{cases}$$

Recursively, we define  $\uparrow^k(v)$  to be  $\uparrow(\uparrow^{k-1}(v))$ . In the literature, this is sometimes written as

$$\uparrow^k v = 1^k \times v$$

For a permutation  $w$ , we define  $\mathbf{m}(w)$  to be the maximum right descent of  $w$ . That is

$$\mathbf{m}(w) = \max\{i \mid w(i) > w(i+1)\}$$

The next proposition specializes, in the case of ordinary Schubert polynomial  $\mathfrak{S}_v(x)$ , to a formula that isolates a chosen index  $i$ : one can express  $\mathfrak{S}_v(x)$  as a sum of terms of the form  $x_i^p \mathfrak{S}_{v'}(x^{(i)})$ , thereby effectively extracting the variable  $x_i$  and leaving Schubert polynomials in the remaining variables [3, Theorem 5.1]. Proposition 2.1.5 is the double Schubert polynomial version of this, which, as far as we know, is new.

**Proposition 2.1.5.** *Let  $v \in S_\infty$  and let  $i > 0$  be an integer. Then we have*

$$\mathfrak{S}_v(x; y) = \sum_{v \searrow^i v'} \mathfrak{S}_{v'}(x^{(i)}; y) \prod_{b \in Q_i(v', v)} (x_i - y_b)$$

See the Appendix for a proof.

**2.2. Definition.** We define a commutative algebra  $\mathcal{A}$  over the integers as follows. For each  $n$ , define  $\mathcal{A}_n$  to be the polynomial ring over  $\mathbb{Z}$  in the variables  $x_1, \dots, x_n$ . Then define

$$\mathcal{A} = \bigoplus_{n=0}^{\infty} \mathcal{A}_n$$

The multiplication within  $\mathcal{A}_n$  is as usually defined for the polynomial ring. However, if  $a \in \mathcal{A}_m$  and  $b \in \mathcal{A}_n$  with  $m \neq n$  and  $m, n > 0$ , then

$$ab = 0$$

Otherwise, the component for  $n = 0$  is identified with the coefficient ring.

*Remark (Fat identities).* The constant polynomial  $1 \in \mathcal{A}_n$  for  $n > 0$  acts as a unit on  $\mathcal{A}_n$  itself but is *not* a unit of  $\mathcal{A}$  as a whole: by the rule above it annihilates every  $\mathcal{A}_m$  with  $m \neq n$  and  $m > 0$ . We will refer to these length- $n$  units informally as “fat identities”. This is the conceptual hinge that makes  $\mathcal{A}$  a bialgebra rather than a Hopf algebra. The triviality of cross-length products is precisely what allows the deconcatenation coproduct  $\Delta$  defined below to be a homomorphism of rings; conversely, it forces  $\mathcal{A}$  to fail to be connected, so  $\mathcal{A}$  admits no antipode. Every theorem in this article should be read in light of this length-grading: the length component never mixes under multiplication, only under the coproduct.

Each  $\mathcal{A}_n$  has a basis consisting of elements  $x_a$ , where  $a$  is a weak composition of length  $n$ , and the notation indicates that

$$x_a = x_1^{a_1} \cdots x_n^{a_n}$$

The direct sum therefore has a basis that can canonically be identified with the union of these.

We define a coproduct  $\Delta : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$  on the basis  $x_c^{(n)}$  by

$$\Delta(x_c) = \sum_{ab=c} x_a \otimes x_b$$

where the equation  $ab = c$  indicates that  $a$  concatenated with  $b$  is equal to  $c$ . We also define a counit  $\varepsilon : \mathcal{A} \rightarrow \mathbb{Z}$  by  $\varepsilon(x_a) = 0$  unless  $a$  is the empty composition.

**Lemma 2.2.1.** *With  $\Delta$  and  $\varepsilon$ ,  $\mathcal{A}$  is a coassociative, counital coalgebra.*

*Proof.* We have

$$\Delta(x_d^{(n)}) = \sum x_a^{(p)} \otimes x_c^{(q)}$$

Applying  $\Delta$  to either tensor factor results in

$$\sum x_a^{(p)} \otimes x_b^{(q)} \otimes x_c^{(r)}$$

The symmetry of this is exactly the coassociativity condition. Seeing that we may choose  $p = n$  or  $q = n$ , the definition of the counit gives us the result that  $\mathcal{A}$  is counital as well under  $\Delta$  and  $\varepsilon$ .  $\square$

**Lemma 2.2.2.**  *$\Delta : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$  is a homomorphism of rings.*

*Proof.* This is where the condition that  $x_a^{(p)} x_b^{(q)} = 0$  unless  $p = q$  when both  $p, q > 0$  comes in. It ensures that only monomials of the same length have nonzero products and preserves the structure of the coproduct as a homomorphism of rings.  $\square$

**Lemma 2.2.3.**  *$\nabla : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$  is a homomorphism of coalgebras.*

*Proof.* Consider the basis element  $x_a \otimes x_b$  of  $\mathcal{A} \otimes \mathcal{A}$ . The product is

$$x_{a+b}$$

Applying  $\Delta$  results in

$$\sum_{a'b'=a+b} x_{a'} \otimes x_{b'}$$

Conversely, applying  $\Delta \otimes \Delta$  yields

$$\sum_{a_1 a_2 = a, b_1 b_2 = b} x_{a_1} \otimes x_{a_2} \otimes x_{b_1} \otimes x_{b_2}$$

and applying  $\nabla \otimes \nabla$  yields

$$\sum x_{a_1+b_1} \otimes x_{a_2+b_2}$$

The definition of  $\nabla$  ensures that  $a_1$  and  $b_1$  are the same length for a nonzero term, and similarly for  $a_2$  and  $b_2$ . Setting  $a_1 + b_1 = a'$ ,  $a_2 + b_2 = b'$  yields the result.  $\square$

**Corollary 2.2.4.**  *$\mathcal{A}$  is a bialgebra over  $\mathbb{Z}$ .*

$\mathcal{A}$  is afforded a grading into finite dimensional components by observing that each  $\mathcal{A}_n$  itself is a graded ring with each homogeneous component being a finitely generated free module. Considering the pair  $(n, d)$ , where  $n$  is the number of variables and  $d$  is the degree, as a  $\mathbb{Z}^2$  grading, we may take the graded dual module  $\mathcal{D}$ , which, by virtue of the grading, is isomorphic as a free module to  $\mathcal{A}$ . We identify the dual basis element  $x_a^*$  with the sequence of nonnegative integers  $a$ . That is,

$$\langle a, x_b \rangle = \delta_{ab}.$$

Inspection of the coproduct reveals that the product of  $a$  and  $b$  in  $\mathcal{D}$  is simply the concatenation  $ab$ . Hence  $\mathcal{D}$  is isomorphic to the free associative algebra on a countable set indexed by nonnegative integers.

$\mathcal{D}$  also has a coproduct compatible with its product, namely

$$c \mapsto \sum_{a+b=c} a \otimes b$$

Thus  $\mathcal{A}$  and  $\mathcal{D}$  are dual bialgebras.

Calling  $\mathcal{A}$  the “Schubert bialgebra” may seem unnecessarily grandiose; however, the reason will become clear below.

**2.3. Quasisym.** Define

$$R_i^- f = f(x_1, \dots, x_{i-1}, t_i, x_i, \dots)$$

and

$$R_i^+ f = f(x_1, \dots, x_i, t_i, x_{i+1}, \dots)$$

By (REFERENCE) above, we have

$$R_i^- \mathfrak{S}_v(x; t) = \sum_{v \searrow_i v'} \mathfrak{S}_{v'}(x; t) \prod_{b \in Q_i(v', v)} (t_i - t_b)$$

and

$$\begin{aligned} R_i^+ \mathfrak{S}_v(x; t) &= \sum_{v \searrow_{i+1} v''} \mathfrak{S}_{v''}(x; t) \prod_{b \in Q_{i+1}(v'', v)} (t_i - t_b) \\ R_i^- \mathfrak{S}_v(x; t) &= \sum_{v \searrow_i v' \searrow_i \tilde{v}} \mathfrak{S}_{\tilde{v}}(x^{(i)}; t) \prod_{a \in Q_i(\tilde{v}, v')} (x_i - t_a) \prod_{b \in Q_i(v', v)} (t_i - t_b) \\ R_i^+ \mathfrak{S}_v(x; t) &= \sum_{v \searrow_{i+1} v'' \searrow_i \tilde{v}} \mathfrak{S}_{\tilde{v}}(x^{(i)}; t) \prod_{a \in Q_i(\tilde{v}, v'')} (x_i - t_a) \prod_{b \in Q_{i+1}(v'', v)} (t_i - t_b) \\ R_i^- &= \text{varop}_i(t_i) R_i \end{aligned}$$

We have

$$E_i \mathfrak{S}_w(x; t) = R_i^- \partial^i = \prod_{a \in Q_i(v', v s_i)} (t_i - t_a) \mathfrak{S}_{w s_i}(x; t)$$

Vine marked forest

Double forest polynomial: sum over pairs of RC graphs  $R_1, R_2$  where  $s(i, j) > 1$  for all  $(i, j) \in R_2$  and the paired row word is a forest reduced word for  $c$ .

**2.4. The Schubert bialgebra over the symmetric functions.** Consider the algebra  $\mathcal{A}$  as a left module over  $\Lambda$ .  $\Lambda$  has a homomorphism  $\pi_n : \Lambda \rightarrow \mathbb{Z}[x_1, \dots, x_n]$  for each  $n$ , and we may define a module structure on  $\mathcal{A}$  by

$$f \cdot g = \pi_n(f)g$$

Consider the algebra  $\text{hom}_\Lambda(\mathcal{A}, \Lambda)$  of graded linear functionals  $\mathcal{A}_n \rightarrow \Lambda_n$ .

**2.5. The double Schubert basis.** In  $\mathcal{A}(y)$ , each  $\mathcal{A}_n$  has a basis of double Schubert polynomials  $\mathfrak{S}_u^{(n)}(y)$ , where the largest right descent of  $u$  is at most  $n$ , ensuring that  $\mathfrak{S}_u(x)$  has at most  $n$  variables. Schubert polynomials limited to a specific number of variables have well-defined structure constants  $c_{u,v}^w$ , independent of  $n$ , given by

$$\mathfrak{S}_u^{(n)}(x; y) \mathfrak{S}_v^{(n)}(x; y) = \sum_w c_{u,v}^w(y; y) \mathfrak{S}_w^{(n)}(x; y)$$

These are known to be Graham-nonnegative by [?]. The coproduct of  $\mathfrak{S}_w^{(n)}(x; y)$  is given by

$$\Delta(\mathfrak{S}_w^{(n)}(x; y)) = \sum_k \sum_{u,v} c_{u\mu_{[k]}, v\mu_{[k]^c}}^{w\mu}(-y; -y) \mathfrak{S}_u^{(k)}(x; y) \otimes \mathfrak{S}_v^{(n-k)}(x; y)$$

where  $\mu$  is a suitably chosen dominant permutation.

We have the following formula.

**Lemma 2.5.1.** *Let  $w$  be a permutation. Then*

$$\mathfrak{S}_w^{(n)} = \sum_{u \cdot v = w} \mathfrak{S}_v^{(n)}(x; y) \mathfrak{S}_u(y)$$

**Corollary 2.5.2.**

$$\langle \mathfrak{S}_w^{(n)}, \Xi_v^n(y) \rangle = \begin{cases} \mathfrak{S}_{wv^{-1}}(y) & \text{if } \ell(wv^{-1}) = \ell(w) - \ell(v) \\ 0 & \text{otherwise} \end{cases}$$

**2.6. The generic elementary basis.** Recall the top factorial elementary symmetric polynomial in  $x$  and  $t_1, \dots, t_n$  is defined by

$$E_{n,n}(x; t) = \prod_{i=1}^n (x_i - t_1)$$

Lower degrees are given by

$$(-1)^{n-k} \partial_t^{s_{n-k} \cdots s_1} (E_{n,n}(x; t)) = E_{k,n}(x; t)$$

where  $\partial_t^{s_{n-k} \cdots s_1}$  is the divided difference operator in the  $t$  variables. We may introduce a new doubly-indexed set of indeterminates  $t_{i,j}$  for  $i \geq 1$  and  $1 \leq i \leq j$  and define the generic factorial elementary symmetric polynomial  $E_i^j$  by

$$E_i^j = E_{i,j}(x; t_{1,j}, \dots, t_{j+1-i,j})$$

An  $n$ -elementary weak composition is a finite sequence of nonnegative integers  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)$  such that  $\alpha_i \leq i$  for all  $i < n$  and  $\alpha_i \leq n$  for all  $i \geq n$ , satisfying the additional restriction that  $\alpha_i \geq \alpha_{i+1}$  if  $i \geq n$ .

An  $n$ -elementary monomial is an element of  $\mathcal{A}_n$  of the following form:

$$e_{\alpha_1}^1 \cdots e_{\alpha_n}^n e_{\alpha_{n+1}}^n \cdots e_{\alpha_m}^n$$

where  $\alpha$  is an  $n$ -elementary weak composition. Define  $\mathcal{E}_n$  to be the set of  $n$ -elementary monomials.

**Proposition 2.6.1.**  $\mathcal{E}_n$  forms a  $\mathbb{Z}$ -basis for  $\mathcal{A}_n$ , and  $\bigcup_n \mathcal{E}_n$  forms a  $\mathbb{Z}$ -basis for  $\mathcal{A}$ .

*Proof.* It is a theorem of Macdonald that the polynomial ring  $\mathbb{Z}[x_1, \dots, x_n]$  is a free module over  $\Lambda_n$  with basis the Schubert polynomials  $\mathfrak{S}_u(x)$  such that  $u \in S_n$ . These Schubert polynomials are uniquely expressed in terms of strict elementary symmetric monomials with fewer than  $n$  variables. Since  $\Lambda_n$  is a polynomial ring in the  $e_{i,n}$  by the standard characterization of  $\Lambda_n$  as a polynomial ring in  $e_1, \dots, e_n$ ,  $\Lambda_n$  has a basis of monomials  $e_\lambda^n$  as  $\lambda$  ranges over all partitions with parts bounded by  $n$ . The multiplicative combination of these two bases is therefore a  $\mathbb{Z}$ -basis of  $\mathcal{A}_n$  by freeness of the module. This is exactly the description of the monomials  $e_\alpha^n$  for  $\alpha$  an  $n$ -elementary weak composition.  $\square$

We can ask for transition formulas between the Schubert and the elementary basis. From elementary to Schubert, we may use Sottile's Pieri formula iteratively [8], which provides a positive formula for multiplying a Schubert polynomial by an elementary symmetric polynomial. That is,

$$e_\alpha^n = \sum_{1 \xrightarrow{1,2,\dots,n} \alpha u} \mathfrak{S}_u^n$$

For the reverse transition, we need a lemma. Recall the skew divided difference operators  $\partial_u^w$  for  $u, w \in S_\infty$  defined originally in [5], though we use the notation of [6], by the Leibniz formula

$$\partial^w(pq) = \sum_u \partial^u(p) \partial_u^w(q)$$

where  $p$  and  $q$  are arbitrary polynomials and  $u$  is a permutation such that  $u \leq w$  in Bruhat order. The formula below provides a formula for application of a skew divided difference operator of degree  $-n$  to a polynomial of degree  $n$ , which yields an integer for integer-coefficient polynomials. Specifically, it is a formula for application of such an operator with  $\ell(u, w) = n$  specifically to  $x_p^n$  for some  $p > 0$ .

**Definition 2.6.2.** For  $u, w \in S_\infty$  such that  $u \leq w$ , we define  $\mathcal{C}^{(p)}(u, w)$  to be the set of saturated chains  $C$  in the Bruhat interval  $[u, w]$  of the form

$$u = u_0 \prec u_1 \prec \cdots \prec u_n = w$$

such that  $u_{i-1}(p) \neq u_i(p)$  for all  $i$ . We also define  $\sigma(C)$  to be the number of indices  $i$  for which the index  $j \neq p$  such that  $u_{i-1}(j) \neq u_i(j)$  satisfies  $j < p$ .

**Lemma 2.6.3.** Let  $n \geq 0$  be an integer, let  $p > 0$  be an integer, and let  $u, w \in S_\infty$  be permutations such that  $\ell(u, w) = n$ . Then we have that

$$\partial_u^w(x_p^n) = \sum_{C \in \mathcal{C}^{(p)}(u, w)} (-1)^{\sigma(C)}.$$

*Proof.* Consider the case where  $\ell(u, w) = 1$ . Then there is a transposition  $t_{ij}$  such that  $w = ut_{ij}$ . Assume that  $i < j$ . If  $p \notin \{i, j\}$ , then  $\partial_u^w(x_p) = 0$ . If  $p = i$ , then  $\partial_u^w(x_p) = \partial^{ij}(x_p) = 1$ . If instead  $j = p$ , then  $\partial_u^w(x_p) = \partial^{ij}(x_p) = -1$ . In all cases, the result holds.

Now, consider the case where  $\ell(u, w) = n > 1$ . We have that

$$x_p^n = x_p \cdot x_p^{n-1}$$

Applying the Leibniz formula, we have

$$\partial_u^w(x_p^n) = \sum_{u \prec u'} \partial_u^{u'}(x_p) \partial_{u'}^w(x_p^{n-1})$$

By the above,  $\partial_u^{u'}(x_p)$  is nonzero if and only if  $u \prec u'$  is a cover in the Bruhat order such that  $u(p) \neq u'(p)$ . In this case,  $\partial_u^{u'}(x_p)$  is either 1 or  $-1$  depending on the number of indices  $j < p$  such that  $u(j) \neq u'(j)$ , which is either 0 or 1, respectively. By induction,

$$\partial_{u'}^w(x_p^{n-1}) = \sum_{C' \in \mathcal{C}^{(p)}(u', w)} (-1)^{\sigma(C')}.$$

Hence,

$$\partial_u^{u'}(x_p) \partial_{u'}^w(x_p^{n-1}) = \begin{cases} - \sum_{C' \in \mathcal{C}^{(p)}(u', w)} (-1)^{\sigma(C')} & \text{if } u(j) \neq u'(j) \text{ for some } j < p \\ \sum_{C' \in \mathcal{C}^{(p)}(u', w)} (-1)^{\sigma(C')} & \text{otherwise,} \end{cases}$$

which, by definition, is

$$\sum_{C = C' \cup \{u\}, C' \in \mathcal{C}^{(p)}(u', w)} (-1)^{\sigma(C)}$$

Summing over all  $u'$  such that  $u \prec u' \leq w$ , we obtain the result.  $\square$

Using a minor sleight of hand, this affords us an explicit formula for transition of a Schubert polynomial into the elementary basis.

**Definition 2.6.4.** Let  $\delta^u$  be the divided difference operator associated to  $u$  on the  $y$  variables for a given  $u \in S_\infty$ , and let

$$E_k(x; y_p) = \prod_{i=1}^k (x_i - y_p)$$

which is a factorial elementary symmetric polynomial, a special case of a double Schubert polynomial.

**Lemma 2.6.5.** *Let  $u, w \in S_\infty$  satisfy  $u \leq w$  and let  $p > 0, k > 0$  be integers. Then we have*

$$\partial_u^w(E_k(x; -y_p)) = \sum_{C \in \mathcal{C}^{(p)}(u, w)} (-1)^{\sigma(C)} e_{k-\ell(u, w)}^k + \mathcal{O}(y_p)$$

*Proof.* This follows from the observation that

$$E_k(x; -y_p) = \sum_{i=0}^k e_{k-i}^k y_p^i$$

and Lemma 2.6.3. □

**Definition 2.6.6.** Suppose we want to express  $\mathfrak{S}_w^n$  in the elementary basis. Let  $w_\lambda$  be a dominant permutation such that

$$\mathfrak{c}^*(w_\lambda) = \lambda = (n, n, \dots, n, n-1, \dots, 1)$$

and consider the double Schubert polynomial

$$\mathfrak{S}_{w_\lambda}(x; -y) = \prod_i E_{\lambda_i}(x; -y_i)$$

(this equation for  $\mathfrak{S}_{w_\lambda}$  follows from [5, (6.14)]).

We have that

$$\mathfrak{S}_w(x; y) = \delta^{w_\lambda^{-1}} \mathfrak{S}_{w_\lambda}(x; y)$$

We may apply the divided difference  $\delta^{w_\lambda^{-1}}$  to the product  $\prod_i E_{\lambda_i}(x; -y_i)$  using the Leibniz formula and set  $y_i = 0$  for all  $i$ , which gives us, by Lemma 2.6.5,

$$\sum_{\substack{P=(C_1, \dots, C_{\ell(\lambda)}) \\ C_i \in \mathcal{C}^{(i)}(u_{i-1}, u_i)}} (-1)^{\sigma(C_1) + \dots + \sigma(C_{\ell(\lambda)})} e_{\lambda_1 - |C_1|}^{\lambda_1} e_{\lambda_2 - |C_2|}^{\lambda_2} \cdots e_{\lambda_{\ell(\lambda)} - |C_{\ell(\lambda)}|}^{\lambda_{\ell(\lambda)}}$$

where  $u_0 \leq u_1 \leq \dots \leq u_{\ell(\lambda)}$  is a saturated chain. Define, for convenience,

$$\text{Path}_\lambda(w) = \{P = (C_1, \dots, C_{\ell(\lambda)}) \mid C_i \in \mathcal{C}^{(i)}(u_{i-1}, u_i), u_0 = 1, u_{\ell(\lambda)} = w w_\lambda^{-1}\}$$

and for  $P \in \text{Path}_\lambda(w)$  define an integer

$$\sigma(P) = \sum_{i=1}^{\ell(\lambda)} \sigma(C_i)$$

Furthermore, define a weak composition  $\alpha(P)$  by

$$\alpha(P) = (\lambda_{\ell(\lambda)+1-i} - |C_{\ell(\lambda)+1-i}|)_{i=1}^{\ell(\lambda)}$$

Then we define

$$E_w^{\alpha; n} = \sum_{\substack{P \in \text{Path}_\lambda(w) \\ \alpha(P) = \alpha}} (-1)^{|\lambda| - \ell(w) + \sigma(P)}$$

**Theorem 2.2** (Transition to the elementary basis). *For each  $w$ , we have*

$$\mathfrak{S}_w^n = \sum_{\alpha} E_w^{\alpha; n} e_{\alpha}^n$$

These numbers are stable for fixed  $w$  as soon as the number of variables is at least as large as the value of  $n$  such that  $w \in S_n$ , and are well-studied but poorly understood.

**Example 2.6.7.** Let  $n = 5$  and let  $w$  be the permutation such that  $\mathfrak{c}(w) = (0, 2, 0, 2, 3)$ . Then

$$\begin{aligned} \mathfrak{S}_w^n &= e_{(0,0,0,0,5,1,1)} - e_{(0,0,0,0,4,2,1)} + e_{(0,0,0,0,4,3)} - e_{(0,1,0,0,4,1,1)} + e_{(0,0,1,0,3,2,1)} \\ &\quad + e_{(0,1,0,0,5,1)} + e_{(0,1,0,0,3,3)} \end{aligned}$$

*Remark.* This transition formula from the Schubert basis to the elementary basis can be used to obtain an extremely efficient formula for multiplication of Schubert polynomials. If we want to compute  $\mathfrak{S}_u^n \mathfrak{S}_v^n$ , we can first express  $\mathfrak{S}_v^n$  in the elementary basis using Theorem 2.2. Then we can multiply by  $\mathfrak{S}_u^n$  by iterating the Pieri formula. This is the method used in the python package `schubmult` written by the author [7]. In fact, there is also a version for double Schubert polynomials, quantum Schubert polynomials, and quantum double Schubert polynomials, which we will not go into here but are implemented in the same package.

### 3. THE DUAL ALGEBRA

We examine the graded dual algebra  $\mathcal{D}$  more closely now. This is a graded ring generated by countably many elements that we denote by  $[i]$ , where  $i$  is a nonnegative integer. The product, as mentioned previously, is concatenation of sequences. Thus

$$[a_1 \cdots a_p][b_1 \cdots b_q] = [a_1 \cdots a_p b_1 \cdots b_q]$$

It is not hard to see that  $\mathcal{D}$  is a free algebra on these generators. Thus the set of sequences  $[a_1 \cdots a_n]$  forms a  $\mathbb{Z}$ -basis for  $\mathcal{D}$ . We may realize this as the dual of  $\mathcal{A}$  by declaring that

$$\langle [a_1 \cdots a_n], x_1^{b_1} \cdots x_n^{b_n} \rangle = \prod_i \delta_{a_i, b_i}$$

We have a  $\mathbb{Z} \times \mathbb{Z}$  grading such that

$$\deg([a_1 \cdots a_n]) = (n, -a_1 - a_2 \cdots - a_n)$$

The set of elements such that  $\deg(a) = (n, -)$  is  $\mathcal{D}_n$ , dual as a graded module to  $\mathcal{A}_n$ .

**3.1. The dual Schubert basis.** There is a basis  $\Xi_u^n$  dual to the Schubert basis for  $\mathcal{D}_n$ . Specifically, with the unique pairing  $\langle -, - \rangle : \mathcal{D} \times \mathcal{A} \rightarrow \mathbb{Z}$  such that

$$\langle \alpha, x_\beta \rangle = \delta_{\alpha\beta}$$

we define  $\Xi_u^n$  to be the unique basis of  $\mathcal{D}_n$  such that

$$\langle \Xi_u^n, \mathfrak{S}_v^n \rangle = \delta_{uv}$$

We characterize it with an explicit formula.

**Definition 3.1.1.** A permutation  $\mu$  is said to be a *strict dominant permutation* if  $\mathbf{c}(\mu)$  is a strict partition (that is, a strictly decreasing sequence of positive integers, padded at the end with 0s in our convention).

**Theorem 3.1.** For each permutation  $u$  and integer  $n$  with  $\mathbf{m}(u) \leq n$  we have the equation

$$\Xi_u^n = \sum_{\ell(\alpha)=n} E_{u\mu^{-1}}^{\mathbf{c}(\mu)-\alpha; n} \alpha$$

where  $\mu$  is any strict dominant permutation such that  $0 \neq \mathbf{c}_n(\mu) \geq \ell(u)$  and  $\mathbf{c}_{n+1}(\mu) = 0$ .

*Proof.* By definition, the coefficient of  $\alpha$  in  $\Xi_u^n$  is the coefficient of  $\mathfrak{S}_u^n$  in  $x_\alpha$ . The result can be derived from the Cauchy formula for double Schubert polynomials. Recall that for any permutation  $w$ , we have the formula

$$\mathfrak{S}_w(x; -y) = \sum_{u: \ell(u) + \ell(wu^{-1}) = \ell(w)} \mathfrak{S}_u(x) \mathfrak{S}_{wu^{-1}}(y)$$

Note that for any permutation  $\mu$  as laid out in the statement of the theorem,  $\ell(u\mu^{-1}) = \ell(\mu) - \ell(u)$ . This is due to the lattice property of dominant permutations in weak order [4].

Thus,

$$\delta^{u\mu^{-1}} \mathfrak{S}_\mu(x; -y) = \mathfrak{S}_u(x; -y)$$

We have that

$$\mathfrak{S}_\mu(x; -y) = \sum_u \mathfrak{S}_u(x) \mathfrak{S}_{u\mu^{-1}}(y)$$

Expressing the second factor in the  $e_{\alpha, \lambda}(y)$  basis, we have

$$\mathfrak{S}_\mu(x; -y) = \sum_{u, \alpha} \mathfrak{S}_u(x) E_{u\mu^{-1}}^{\mathbf{c}(\mu)-\alpha; n} e_{\mathbf{c}(\mu)-\alpha, \mathbf{c}(\mu)}(y)$$

An alternative expression for  $\mathfrak{S}_\mu(x; -y)$  is

$$\mathfrak{S}_\mu(x; -y) = \sum_{\alpha} x_{\alpha} e_{\mathfrak{c}(\mu) - \alpha, \mathfrak{c}(\mu)}(y)$$

from which we see that the coefficient is correct.  $\square$

This is not stable for fixed  $u$  as  $n$  increases, and this is expected.

**Example 3.1.2.** Suppose  $w = [1, 4, 5, 2, 3]$ . Then

$$\Xi_w^3 = [022] - [013]$$

If we chose  $[1, 3, 5, 7, 2, 4, 6]$  instead, we have

$$\Xi_{1357246} = [0015] - [0033] - [0114] + [0123]$$

For  $w = [4, 2, 7, 1, 3, 5, 6]$ ,

$$\Xi_{4271356}^3 = -[026] + [035] + [125] - [134] + [206] - [215] - [305] + [314]$$

Thus the product structure of  $\mathcal{D}$  encodes splitting of the Schubert polynomial into two sets of variables. In particular,

**Lemma 3.1.3.** *Let  $a \geq 0$  be an integer and  $w \in S_\infty$ . Then we have*

$$[a] \cdot \Xi_w^n = \sum_{\substack{w' \xrightarrow{1} w \\ \ell(w, w') = a}} \Xi_{w'}^{n+1}$$

*Proof.* This follows from Proposition 2.1.5 with  $i = 1$ .  $\square$

**3.2. The coproduct.** Consider the coproduct dual to the product on  $\mathcal{A}$ . We will denote it by

$$\Delta^* : \mathcal{D} \rightarrow \mathcal{D} \otimes \mathcal{D},$$

and we recall that it is characterized by the pairing identity

$$\langle \Delta^*(\alpha), x \otimes y \rangle = \langle \alpha, xy \rangle$$

for all  $\alpha \in \mathcal{D}$  and  $x, y \in \mathcal{A}$ . Since the product on  $\mathcal{A}$  preserves the length grading (multiplication of polynomials within a fixed  $\mathcal{A}_n$ , zero across  $\mathcal{A}_p \otimes \mathcal{A}_q$  for  $p \neq q$  with  $p, q > 0$ ),  $\Delta^*$  sends  $\mathcal{D}_n$  into  $\mathcal{D}_n \otimes \mathcal{D}_n$  for each  $n$ .

The Schubert structure constants now reappear as the coefficients of  $\Delta^*$  in the dual Schubert basis.

**Theorem 3.2** (Schubert LR coefficients as dual Schubert coproduct). *Let  $w \in S_\infty$  with  $\mathfrak{m}(w) \leq n$ . Then in  $\mathcal{D}_n \otimes \mathcal{D}_n$ ,*

$$\Delta^*(\Xi_w^n) = \sum_{u, v} c_{u, v}^w \Xi_u^n \otimes \Xi_v^n,$$

where  $c_{u, v}^w$  are the Schubert structure constants  $\mathfrak{S}_u^n \mathfrak{S}_v^n = \sum_w c_{u, v}^w \mathfrak{S}_w^n$ , and the sum runs over pairs  $u, v \in S_\infty$  with  $\mathfrak{m}(u), \mathfrak{m}(v) \leq n$ .

*Proof.* Expand  $\Delta^*(\Xi_w^n) = \sum_{u, v} \lambda_{u, v} \Xi_u^n \otimes \Xi_v^n$  in the basis  $\{\Xi_u^n \otimes \Xi_v^n\}$  of  $\mathcal{D}_n \otimes \mathcal{D}_n$ . Pairing with  $\mathfrak{S}_u^n \otimes \mathfrak{S}_v^n$  and using the defining identity of  $\Delta^*$ ,

$$\lambda_{u, v} = \langle \Delta^*(\Xi_w^n), \mathfrak{S}_u^n \otimes \mathfrak{S}_v^n \rangle = \langle \Xi_w^n, \mathfrak{S}_u^n \mathfrak{S}_v^n \rangle = \langle \Xi_w^n, \sum_z c_{u, v}^z \mathfrak{S}_z^n \rangle = c_{u, v}^w.$$

$\square$

*Remark.* In particular, the Schubert Littlewood–Richardson coefficients are the structure constants of the coproduct  $\Delta^*$  on  $\mathcal{D}$  in the dual Schubert basis. Thus the otherwise mysterious nonnegativity of  $c_{u, v}^w$  has the bialgebra-theoretic content that  $\Delta^*$  has nonnegative integer coefficients in the basis  $\{\Xi_u^n\}$ . We will see in Corollary ?? that the same phenomenon recurs for the dual forest basis, with structure constants  $\beta_{ab}^c$  of Theorem ??.

**Example 3.2.1.** Take  $w = 24135 \in S_5$  (so  $\mathbf{m}(w) = 2$ ) and  $n = 2$ . Then in  $\mathcal{D}_2 \otimes \mathcal{D}_2$ ,

$$\begin{aligned} \Delta^*(\Xi_{24135}^2) &= 1 \otimes \Xi_{2413}^2 + \Xi_{2413}^2 \otimes 1 \\ &\quad + \Xi_{132}^2 \otimes \Xi_{1423}^2 + \Xi_{1423}^2 \otimes \Xi_{132}^2 \\ &\quad + \Xi_{132}^2 \otimes \Xi_{231}^2 + \Xi_{231}^2 \otimes \Xi_{132}^2 \\ &\quad + \Xi_{21}^2 \otimes \Xi_{1423}^2 + \Xi_{1423}^2 \otimes \Xi_{21}^2. \end{aligned}$$

By Theorem 3.2, the coefficient of  $\Xi_u^2 \otimes \Xi_v^2$  is the Schubert structure constant  $c_{u,v}^{24135}$ ; for instance,  $c_{132,1423}^{24135} = c_{21,1423}^{24135} = 1$ , while  $c_{u,v}^{24135} = 0$  for every pair  $(u, v)$  not appearing above.

#### 4. THE RING OF BOUNDED RC GRAPHS

##### 4.1. The module of bounded RC graphs.

**Definition 4.1.1.** Let  $R \subseteq \mathbb{P} \times \mathbb{P}$  be a finite set. To each such set we associate an element  $w_R$  of  $S_\infty$  as follows. Given a pair  $(i, j)$  where  $i, j > 0$  are integers, define  $s(i, j) = s_{i+j-1}$ . Totally order the grid such that  $(i, j) < (a, b)$  if and only if  $i < a$  or  $i = a$  and  $j > b$  (in other words, lexicographical order, except that the order on the second coordinate is reversed). By this ordering, index  $R$  as  $r_1, r_2, \dots, r_m$  in increasing order. Then

$$w_R = s_{r_1} \cdots s_{r_m}$$

If  $\ell(w_R) = m$ , then we say that  $R$  is an *RC-graph*.

A *bounded RC-graph* is an RC-graph  $R$  together with a specified number of rows  $n \geq \max\{i : (i, j) \in R \text{ for some } j\}$ , with the added condition that  $\mathbf{m}(w_R) \leq n$ . The definition is as an ordered pair  $(R, n)$ , but it will be far more convenient to abuse notation and consider the number of rows a property of  $R$  itself. To denote the number of rows, we define the *height* of  $R$  to be  $\mathbf{ht}(R) = n$ . A bounded RC graph has an associated weight vector  $\mathbf{wt}(R)$  where  $\mathbf{wt}_i(R)$  is the number of elements in row  $i$ .

To a bounded RC graph  $R$  with  $\mathbf{ht}(R) \geq 1$ , we define the *trim* operation  $\mathbf{trim}(R)$  to be the bounded RC graph with height  $\mathbf{ht}(R) - 1$  and underlying set  $\{(i-1, j-1) : (i, j) \in R, i \geq 2\}$ . We also define

$$\uparrow^m R = \{(i+m, j) : (i, j) \in R\}$$

Let  $\mathcal{BRC}$  be the free abelian group spanned by all bounded RC graphs. If  $\mathcal{BRC}_n$  is the subgroup spanned by all bounded RC graphs  $R$  with  $\mathbf{ht}(R) = n$ , then we have a grading

$$\mathcal{BRC} = \bigoplus_{n=0}^{\infty} \mathcal{BRC}_n$$

There is an evident evaluation map  $\text{ev} : \mathcal{BRC} \rightarrow \mathcal{A}$  defined on basis elements as

$$\text{ev}(R) = x_{\mathbf{wt}(R)}$$

which is a surjective homomorphism of graded modules. There is also a map  $\alpha : \mathcal{BRC} \rightarrow \mathcal{D}$  defined by

$$\alpha(R) = \Xi_{w_R}^{\mathbf{ht}(R)}$$

which is also a surjective homomorphism of graded modules. In addition, there is  $\omega : \mathcal{BRC} \rightarrow \mathcal{D}$  defined by

$$\omega(R) = [\mathbf{wt}(R)]$$

which is similarly surjective.

We can define elements  $\mathcal{S}_w(n)$  as

$$\mathcal{S}_w(n) = \sum_{\substack{w_R = w \\ \mathbf{ht}(R) = n}} R$$

For a generating element  $[a]$  of  $\mathcal{D}$  and a bounded RC graph  $R$ , we define

$$[a] \cdot R = \sum_{\substack{\mathbf{trim}(R') = R \\ \mathbf{wt}_1(R') = a}} R'$$

which is an element of  $\mathcal{BRC}$ . By virtue of the fact that  $\mathcal{D}$  is a free algebra generated by these elements, it is nearly a trivial observation that this is a left module action on  $\mathcal{BRC}$ . The consequences of this, however, are not at all trivial.

**Lemma 4.1.2** ([1, Corollary 3.11]). *We have that the set  $\{w' \mid w \xrightarrow{1} w'\}$  is equal to the set of permutations  $w'$  such that there exists a permutation  $v = s_{a_1} \cdots s_{a_m}$  for some integers  $a_1 > a_2 > \cdots > a_m \geq 1$  such that  $w = v \uparrow w'$ .*

**Lemma 4.1.3.** *Let  $v$  be a permutation. If  $v \xrightarrow{1} v'$ , let  $a_1, \dots, a_k$  be the elements of  $Q_1(v', v)$  in decreasing order. Then*

$$v = s_{a_1} \cdots s_{a_k} \uparrow v'$$

*Proof.* Suppose  $v \in S_n$ . We have by definition that there is a sequence of integers  $b_1, \dots, b_p$ , all distinct and greater than 1, such that

$$vt_{1,b_1} \cdots t_{1,b_p}(1) = n + 1$$

and

$$vt_{1,b_1} \cdots t_{1,b_p}(i + 1) = v'(i)$$

for all  $i < n - 1$ . This means that

$$vt_{1,b_1} \cdots t_{1,b_p} = s_n s_{n-1} \cdots s_1 \uparrow v'$$

This is because if  $v'' = \uparrow v'$ , then

$$v''(1) = 1$$

and

$$v''(i) = v(i - 1) + 1$$

The cycle  $s_n \cdots s_1$  sends  $1 \mapsto n + 1$  and  $i \mapsto i - 1$  if  $1 < i \leq n + 1$ , hence

$$vt_{1,b_1} \cdots t_{1,b_p} = s_n s_{n-1} \cdots s_1 \uparrow v'$$

In particular,

$$v = s_n s_{n-1} \cdots s_1 \uparrow v' t_{1,b_p} \cdots t_{1,b_1}$$

This results in the factorization

$$v = s_n \cdots s_1 t_{1,v'(b_p-1)+1} \cdots t_{1,v'(b_1-1)+1} \uparrow v'$$

The value  $v'(b_j - 1)$  necessarily decreases as  $j$  decreases, since applying the corresponding reflection in the reverse order strictly increases the length with each application. Consequently, multiplying  $s_n \cdots s_1$  on the right by these reflections removes the simple reflections  $s_{v'(b_p-1)}, \dots, s_{v'(b_1-1)}$ . The indices removed are precisely the complement of the elements of  $Q_1(v', v)$ , hence the elements of  $Q_1(v', v)$  in decreasing order are what remain, as desired.  $\square$

**Definition 4.1.4.** For a set of positive integers  $A = \{a_1, \dots, a_m\}$  with  $a_1 > a_2 > \cdots > a_m \geq 1$  and a positive integer  $i$ , define

$$\text{row}_i(A) = \{(i, a_j) \mid a_j \in A\}$$

**Theorem 4.1.** *Let  $a \geq 0$  be an integer and let  $R \in \mathcal{BRC}$ . Then*

$$[a] \cdot R = \sum_{\substack{w' \xrightarrow{1} w_R \\ \ell(w_R, w') = a}} (\text{row}_1(Q_1(w_R, w')) \cup \uparrow R)$$

where the right side denotes bounded RC graphs with  $\text{ht}(R) + 1$  rows.

*Proof.* If  $\text{trim}(R') = R$ , then by definition  $w_{R'} = s_{a_1} \cdots s_{a_m} \uparrow w_R$ . It follows by Lemma 4.1.2 then  $w_{R'} \xrightarrow{1} w_R$ . By Lemma 4.1.3, the integers  $a_1, \dots, a_m$  are precisely the elements of  $Q_1(w_R, w_{R'})$  in decreasing order. This establishes the result.  $\square$

**Lemma 4.1.5.** *Let  $a \geq 0$  be an integer and let  $w \in S_\infty$ . For any valid  $n$ , we have*

$$[a] \cdot \mathcal{S}_w(n) = \sum_{\substack{w \xrightarrow{1} w' \\ \ell(w, w') = a}} \mathcal{S}_{w'}(n + 1)$$

*Proof.* Theorem 4.1 establishes an explicit bijection between  $[a] \cdot R$  and the set of bounded RC graphs  $R'$  such that  $w_{R'} \xrightarrow{1} w_R$  and  $\ell(w_R, w_{R'}) = a$ . The result follows by summing over all  $R$  such that  $w_R = w$  and  $\text{ht}(R) = n$ .  $\square$

Let  $t$  be an indeterminate commuting with all elements of  $\mathcal{D}$ . Write

$$\mathfrak{S}_{\mathcal{D}}(t) = \sum_{a=0}^{\infty} [a] t^a$$

Define

$$\mathfrak{S}_{\mathcal{D}}(x_1, x_2, \dots, x_n) = \mathfrak{S}_{\mathcal{D}}(x_1) \mathfrak{S}_{\mathcal{D}}(x_2) \cdots \mathfrak{S}_{\mathcal{D}}(x_n)$$

**Theorem 4.2.** *Let  $\emptyset \in \mathcal{BRC}$  denote the empty bounded RC graph with  $\text{ht}(\emptyset) = 0$ . Then*

$$\mathfrak{S}_{\mathcal{D}}(x_1, \dots, x_n) \cdot \emptyset = \sum_{\substack{w \in S_{\infty} \\ \text{mw} \leq n}} \mathfrak{S}_w(x_1, \dots, x_n) \mathcal{S}_w(n)$$

*Proof.* The result for  $n = 1$  is Lemma 4.1.5 together with Theorem 4.1. The general result follows by induction on  $n$ . Consider the inductive hypothesis

$$\mathfrak{S}_{\mathcal{D}}(x_2, \dots, x_n) \cdot \emptyset = \sum_{w \in S_{\infty}} \mathfrak{S}_w(x_2, \dots, x_n) \mathcal{S}_w(n-1)$$

Then applying  $\mathfrak{S}_{\mathcal{D}}(x_1)$  to both sides, we have

$$\mathfrak{S}_{\mathcal{D}}(x_1) \mathfrak{S}_{\mathcal{D}}(x_2, \dots, x_n) \cdot \emptyset = \sum_{w \in S_{\infty}} \mathfrak{S}_w(x_2, \dots, x_n) \mathfrak{S}_{\mathcal{D}}(x_1) \mathcal{S}_w(n-1)$$

which is equal to

$$\sum_{a=0}^{\infty} \sum_{w \in S_{\infty}} x_1^a \mathfrak{S}_w(x_2, \dots, x_n) [a] \mathcal{S}_w(n-1)$$

Applying Lemma 3.1.3 to each term, we have

$$\sum_{a=0}^{\infty} \sum_{w \in S_{\infty}} x_1^a \mathfrak{S}_w(x_2, \dots, x_n) \sum_{\substack{w' \xrightarrow{1} w \\ \ell(w, w') = a}} \mathcal{S}_{w'}(n)$$

Bringing the polynomial inside the inner sum, this is

$$\sum_{w' \in S_{\infty}} \left( \sum_{w: w' \xrightarrow{1} w} x_1^{\ell(w, w')} \mathfrak{S}_w(x_2, \dots, x_n) \right) \mathcal{S}_{w'}(n)$$

which simplifies to

$$\sum_{w' \in S_{\infty}} \mathfrak{S}_{w'}(x_1, \dots, x_n) \mathcal{S}_{w'}(n)$$

as desired.  $\square$

**Corollary 4.1.6.** *For each  $w \in S_{\infty}$  and valid  $n$ , we have*

$$\text{ev}(\mathcal{S}_w(n)) = \mathfrak{S}_w^n(x_1, \dots, x_n)$$

**4.2. The pipe dream visualization and roots.** Given a pair of positive integers  $i, j$  and an RC graph  $R$ , define

$$R[i, j] = \{(a, b) \in R \mid (a, b) > (i, j)\}$$

Given an RC graph  $R$  and a pair of positive integers  $i, j$ , we define an ordered pair

$$\mathbf{rt}_R(i, j) = (w_{R[i, j]}^{-1}(i + j - 1), w_{R[i, j]}^{-1}(i + j))$$

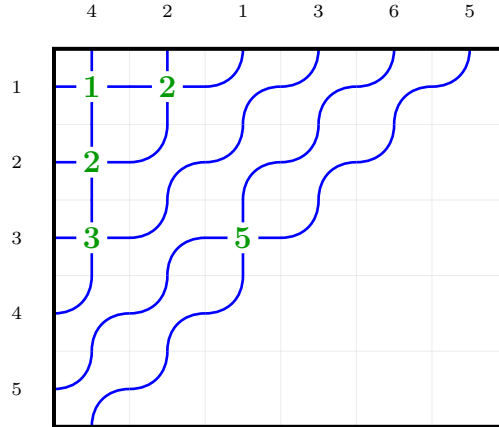
There is a common visualization of RC graphs as pipe dreams. In this visualization, we draw an infinite grid in the first quadrant, and in each position  $(i, j)$  we draw either a crossing (if  $(i, j) \in R$ ) or an elbow (if  $(i, j) \notin R$ ). Then we draw pipes entering from the left edge of the grid, with the pipe entering at row  $i$  labeled  $i$ . The pipes travel through the grid, turning at elbows and crossing at crossings, and exit at the top of the grid, which is labeled with the same number.

A modification to this common visualization that we use has the following additional features:

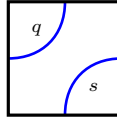
- We write the index of the simple reflections corresponding to the crossings in the grid area itself. Thus, at position  $(i, j)$  we write the number  $i + j - 1$  if there is a crossing at that position.
- For a bounded RC graph  $R$  with  $\mathbf{ht}(R) = n$ , we only draw the first  $n$  pipes entering from the left side of the grid and clip features outside of the first  $n$  rows. The pipes exiting at the top are still labeled since the width is not limited.
- *Note:* The labels at the top are not of the permutation  $w_R$ , but rather of the permutation  $w_R^{-1}$ .

See Figure 1 for an example of this visualization.

FIGURE 1. The pipe dream visualization of the bounded RC graph  $R$  with  $\mathbf{ht}(R) = 5$  where  $R = \{(1, 1), (1, 2), (2, 1), (3, 1), (3, 3)\}$



The main benefit of this is that visualizing  $\mathbf{rt}_R(i, j)$  is easy. The pipes that pass through position  $(i, j)$  are labeled  $s$  and  $q$  for some  $s, q > 0$ . For a positive root in an unoccupied square, we will have the following labeling, where  $q < s$ :

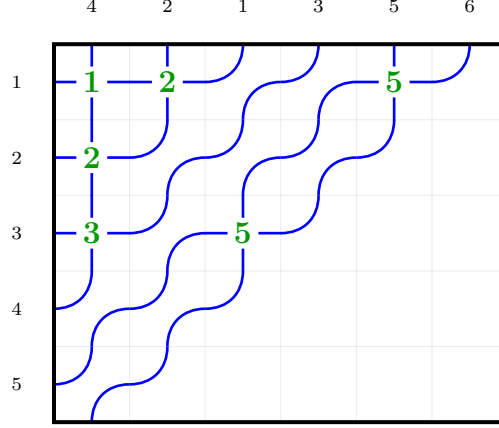


We observe that in the above RC graph, the position  $(1, 5)$  does not have this configuration. Placing a crossing there would create a negative root, and the pipes would cross twice. *Note that this results in a collection of ordered pairs that is not an RC graph, if such a crossing exists.* For a valid RC graphs, only positive roots occur as crossings, and this happens if and only if pipes cross at most once.

**4.3. Zeroing out the last row.** We proceed now to define a product on  $\mathcal{BRC}$  turning it into a ring. To do this, we need to be able to define a function  $\mathcal{Z} : \mathcal{BRC} \rightarrow \mathcal{BRC}$  trimming empty rows from the bottom instead of from the top. This is far more complicated.

See Figure 3 for an example of the zeroing operation (Algorithm 3).

FIGURE 2. An invalid set of crossings causing pipes to cross more than once, caused by inserting a crossing at a negative root




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**Algorithm 1** Basic monk insertion  $i \xrightarrow{k} R$

---

- 1: **Input:** An RC graph  $R$ , parameter  $k$ , and an integer  $i$  with  $1 \leq i \leq k$ .
  - 2: **Output:** A modified RC graph  $R'$  with  $\text{wt}_j(R') = \text{wt}_j(R)$  for  $j \neq i$  and  $\text{wt}_i(R') = \text{wt}_i(R) + 1$  such that  $w_R \xrightarrow{k} w_{R'}$ .
  - 3: Find leftmost position  $(i, j) \notin R$  in row  $i$  such that if  $\text{rt}_R(i, j) = (a, b)$ , then  $a \leq k < b$ .
  - 4:  $R \leftarrow R \cup \{(i, j)\}$
  - 5: **if** there exists  $(i', j') \in R$  with  $\text{rt}_R(i', j') \in \Phi^-$  **then**
  - 6:      $R \leftarrow R \setminus \{(i', j')\}$
  - 7:     Return to step 3 substituting  $i = i'$ .
  - 8: **end if**
  - 9: **return**  $R$
- 

Note that Algorithm 2 is *not* a realization of Sottile's Pieri formula for complete symmetric polynomials, despite preserving the relevant Bruhat relation. This is because the map

$$\left( \binom{k}{p} \right) \times \mathcal{RC}(w) \rightarrow \mathcal{RC}(n)$$

given by

$$(I, R) \mapsto I \xrightarrow{k} R$$

need not be injective.

**Definition 4.3.1.** Let  $R$  be an RC graph with  $\text{ht}(R) = n$  such that row  $n$  is empty. We define  $\mathcal{Z}(R)$  to be the output of Algorithm 3, an RC graph with  $n - 1$  rows, on input  $R$ .

**Lemma 4.3.2.** Given an RC graph  $R$ , an integer  $k$ , and  $k \geq i_1 \geq \dots \geq i_m \geq 1$ , Algorithm 2 produces a valid RC graph  $R'$  satisfying

$$\text{wt}_i(R') = \text{wt}_i(R) + \#\{j \mid i_j = i\}$$

and

$$w_R \xrightarrow{k} w_{R'}$$

*Proof.* Set  $R_0 = R$  to be the initial RC graph. We claim that given  $R$  and  $L$  at the start of iteration  $p$  of the main loop, we have that  $R$  is a valid RC graph satisfying

$$\text{wt}_i(R) = \text{wt}_i(R_0) + \#\{j \leq p - 1 \mid i_j = i\}$$

and

$$w_R = w_{R_0} t_{a_1 b_1} \cdots t_{a_{p-1} b_{p-1}}$$

**Algorithm 2** Pseudo-Pieri insertion  $I \xrightarrow{k} R$ 


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1: Input: An RC graph  $R$ , parameter  $k$ , and sequence  $I = \{i_1, \dots, i_m\}$  with  $k \geq i_1 \geq i_2 \geq \dots \geq i_m \geq 1$ .
2: Output: A modified RC graph  $R'$  with  $\text{wt}_i(R') = \text{wt}_i(R)$  for  $i \notin I$  and  $\text{wt}_i(R') = \text{wt}_i(R) + \#\{j \mid i_j = i\}$ 
   for  $i \in I$  such that  $w_R \xrightarrow{k} w_{R'}$ .
3: Initialize: Let  $L \leftarrow []$  (empty list of pairs  $(a, b)$  where  $a \leq k < b$ ), and  $U \leftarrow []$  (empty list of positions).
4: for each  $i \in (i_1, \dots, i_m)$  do
5:   Find leftmost position  $(i, j) \notin R$  in row  $i$  satisfying  $j > j'$  for all  $(i, j') \in U$  such that either:
6:   a)  $\text{rt}_R(i, j) = (s, q)$  where  $s \leq k < q$  and  $q \notin \{b \mid (a, b) \in L\}$ .
7:   b)  $\text{rt}_R(i, j) = (b_r, q)$  where  $q > k$ ,  $b_r < q$ , and  $q \notin \{b \mid (a, b) \in L\}$ .
8:    $R \leftarrow R \cup \{(i, j)\}$  and  $U \leftarrow U \cup \{(i, j)\}$ 
9:   Update  $L$  by adding  $(s, q)$  or replacing  $(a_r, b_r)$  with  $(a_r, q)$  and  $(a_r, b_r)$ .
10:  if there exists  $(i', j') \in R$  with  $\text{rt}_R(i', j') \in \Phi^-$  then
11:     $R \leftarrow R \setminus \{(i', j')\}$ 
12:    Let  $\text{rt}_R(i', j') = (q, s)$ 
13:    if  $(s, q) \in L$  then
14:      Remove  $(s, q)$  from  $L$ 
15:    else
16:      Remove  $(a_r, q)$  from  $L$ , where  $(a_r, q), (a_r, s) \in L$ 
17:    end if
18:    Return to step 5 to insert  $i'$ .
19:  end if
20: end for
21: return  $R$ 

```

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**Algorithm 3** Map  $\mathcal{Z}(R)$  zeroing out last row

---

```

1: Input: A bounded RC graph  $R$  with  $\text{ht}(R) = n$  and row  $n$  empty.
2: Output: A bounded RC graph  $R'$  with  $\text{ht}(R') = n - 1$ ,  $|R'| = |R|$  (same number of crossings), and
    $w_R \xrightarrow{n} w_{R'}$ .
3: if  $R$  with last row removed is a valid bounded RC graph then
4:   return  $R$  with last row removed
5: else
6:   Let  $R_0 \leftarrow R$ .
7:   Find the maximal  $p$  and sequence  $\{(i_m, j_m)\}_{m=1}^p$  such that:
8:    $\text{rt}_{R_{m-1}}(i_m, j_m) = (n + m - 1, n + m)$  for each  $m$ , where  $R_m = R_{m-1} \setminus \{(i_m, j_m)\}$  for each  $m \geq 1$ .
9:    $R^- \leftarrow R \setminus \{(i_1, j_1), \dots, (i_p, j_p)\}$ .
10:  Let  $I \leftarrow (i_1, i_2, \dots, i_p)$ .
11:   $D \leftarrow R^- \cup \{(n, 1), (n, 2), \dots, (n, p)\}$ .
12:   $D' \leftarrow I \xrightarrow{n-1} D$ 
13:   $R' \leftarrow \{(i, j) \in D' \mid i < n\}$  as a bounded RC graph with  $n - 1$  rows.
14:  return  $R'$ .
15: end if

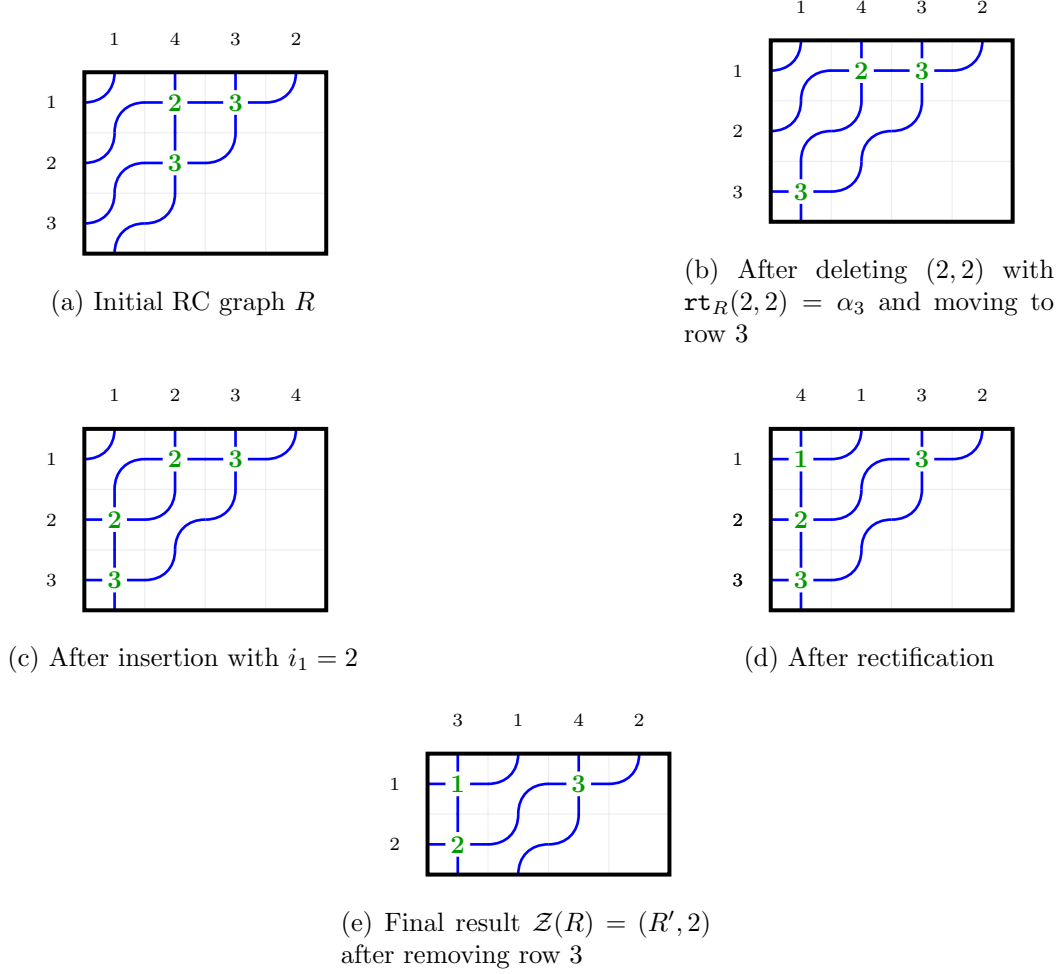
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where  $L = [(a_1, b_1), \dots, (a_p, b_p)]$ . This is clear for  $p = 1$ . For larger  $p$ , suppose we are inserting at row  $i_p$ . If we are in case (a) of Step 5, then adding  $(i_p, j)$  to  $R$  adds the root  $t_s - t_q$  to the inversion set of  $w_R$ , and multiplying by  $t_{sq}$  gives the desired result. If we are in case (b) of Step 5, then adding  $(i_p, j)$  to  $R$  results in multiplication by  $t_{b_r, q}$ . This commutes past the other reflections to meet  $t_{a_r, b_r}$ . We thus have the situation

$$t_{a_r, b_r} t_{b_r, q} = t_{a_r, q} t_{a_r, b_r}$$

This ensures that the product remains as desired if the RC graph is valid. However, the condition that the RC graph  $R$  be valid is not guaranteed after this step, so we must rectify. Suppose we must rectify at row  $i - 1$ . We find a negative root  $(i - 1, j')$  which must have been created by the insertion, and hence by the

FIGURE 3. Step-by-step computation of  $\mathcal{Z}(R)$  for  $R = \{(1,2), (1,3), (2,2)\}$ .

strong exchange property equal to  $\mathbf{rt}_R(i, j)$  that was inserted. Removing this root (from both the graph and from  $L$ ) returns us to the original permutation at the beginning of the iteration, and performing the insert at row  $i - 1$  adds a new positive root, giving the desired result on the permutation. The weight is unchanged since we removed and added one crossing in row  $i - 1$ . If the algorithm ends here, we have instead multiplied by the new reflection obtained in the ultimate insert step. Otherwise, repeating this process for all necessary rectifications completes the proof of the claim. The claim iterated to  $p = m + 1$  yields the desired result, and the method of updating  $L$  ensures that the  $b_j$  are all distinct, so that  $w_R \xrightarrow{k} w_{R'}$ .  $\square$

**Lemma 4.3.3.** *Given a bounded RC graph  $R$  with  $\mathbf{ht}(R) = n$  such that row  $n$  is empty, Algorithm 3 produces a valid bounded RC graph  $R'$  with  $\mathbf{ht}(R') = n - 1$  satisfying*

$$\mathbf{wt}(R') = \mathbf{wt}(R)$$

*Proof.* Stripping out the crossings at roots  $(n, n + 1), (n + 1, n + 2), \dots, (n + p - 1, n + p)$  removes exactly one crossing from each of rows  $i_1, i_2, \dots, i_p$ , and moving them to the last row to obtain  $R_1$  ensures that  $w_R = w_{R_1}$ . By construction, the portion of the RC graph in rows with index less than  $n$  is valid (meaning its max descent is at most  $n - 1$ ). Applying Algorithm 2 to insert at rows  $i_1, i_2, \dots, i_p$  adds different crossings to preserve the number of elements in each row without adding descents past index  $n$  in the permutation, by Lemma 4.3.2. Removing row  $n$  then yields a valid bounded RC graph  $R'$  with  $|R'| = n - 1$  and the desired properties.  $\square$

**Theorem 4.3.** *Let  $w$  be a permutation with last descent at most  $n$ . Let  $\mathcal{RC}(w, n)^0$  be the set of bounded RC graphs  $R$  with  $\text{ht}(R) = n$  such that  $w_R = w$  and row  $n$  is empty. Then*

$$\mathcal{Z} : \mathcal{RC}(w, n)^0 \rightarrow \bigcup_{w \xrightarrow{n} w'} \mathcal{RC}(w', n-1)$$

*is a weight-preserving bijection.*

To prove Theorem 4.3 for  $n = 2$ , we may characterize the bounded RC graphs  $R$  with  $|R| = 2$  such that row 2 is empty and  $\mathbf{m}(w_R) = 2$  as those for permutations  $w = s_k s_{k-1} \cdots s_2$  for some  $k > 1$ . Algorithm 3 in this case moves the entirety of the crossings in row 1 to row 2. The procedure then inserts  $k - 1$  crossings into row 1 in order from left to right, which must have roots  $t_q - t_s$  such that  $q = 1$ . Thus  $R'$  is precisely  $\{(1, 1), (1, 2), (1, 3), \dots, (1, k - 1)\}$ , which is the unique bounded RC graph for the permutation  $w' = s_{k-1} s_{k-2} \cdots s_1$  with one row. This establishes the base case.

For the induction step, we require the following lemmas.

**Lemma 4.3.4.** *Let  $R$  be a bounded RC graph with  $\text{ht}(R) = n \geq 2$  such that row  $n$  is empty. Then if  $\mathcal{Z}(R) = R'$ , we have*

$$w_R \xrightarrow{n} w_{R'}$$

*Proof.* Let  $w = w_R$  and suppose  $\mathbf{c}_n(w) = m$ . By construction, the second to last step of Algorithm 3 yields a bounded RC graph  $\tilde{R}$  with  $|\tilde{R}| = n$  such that if  $\tilde{w} = w_{\tilde{R}}$ , then

$$\tilde{w}(n) > \tilde{w}(n+1) < \tilde{w}(n+2) < \cdots < \tilde{w}(n+m)$$

and

$$w \xrightarrow{n-1} \tilde{w}$$

via reflections  $t_{ab}$  such that  $n \leq b \leq n+m$ . Let  $w' = w_{R'}$ . We also have

$$\tilde{w} \xrightarrow{n} \varphi_{n,N}(w')$$

via only reflections  $t_{nb}$  such that  $b > n+m$ . We claim that

$$w \xrightarrow{n} \varphi_{n,N}(w')$$

This can only fail if  $\tilde{w}(n) \neq w(n)$  by the observations above. However, the pipe labeled  $n$ , by virtue of the fact that we have  $(n, 1), \dots, (n, m)$  populated, will always lie to the right of pipes  $n+1, n+2, \dots, n+m$  in the wiring diagram for  $\tilde{w}$  or any intermediate stage. Inserting into any row that contained a crossing  $(n, n+p)$ , there must be a valid empty space to the left of the pipe labeled  $n$  since we have deleted these crossings. By virtue of the fact that the leftmost is always chosen, no reflection  $(i, n)$  will ever be inserted. This ensures that  $\tilde{w}(n) = w(n)$ , as desired.  $\square$

**Lemma 4.3.5.** *Let  $(R, n)$  be a bounded RC graph with  $n \geq 2$  such that row  $n$  is empty. Then*

$$\mathcal{Z}(\text{trim}(R)) = \text{trim}(\mathcal{Z}(R))$$

*Proof.* This is almost a trivial observation. In terms of the word of  $R$ ,  $\text{trim}(R, n)$  preserves a suffix. Removing all of the initial roots before trimming is therefore the same as removing the initial roots that still remain after trimming, by the exchange property. Afterwards, in performing the insertion algorithm, there is no dependency on rows with a lower index, the modification only proceeds downward in row number. Therefore, trimming the first row at any point only stops the process earlier, and does not change rows with higher index than 1 in the outcome.  $\square$

**Lemma 4.3.6.** *Let  $w$  be a permutation with last descent at most  $n$ . Let  $\mathcal{RC}(w, n)^0$  be the set of bounded RC graphs  $R$  with  $\text{ht}(R) = n$  such that  $w_R = w$  and row  $n$  is empty. Then*

$$\mathcal{Z} : \mathcal{RC}(w, n)^0 \rightarrow \mathcal{RC}(n-1)$$

*is injective.*

*Proof.* If  $n = 2$  this is clear. Suppose now that  $n > 2$  and that the result holds for  $n - 1$ . Let  $(R, n) \in \mathcal{RC}(w, n)^0$ . If  $(R', n) \in \mathcal{RC}(w, n)^0$  is another bounded RC graph such that

$$\mathcal{Z}(R) = \mathcal{Z}(R')$$

then by Lemma 4.3.5, we have

$$\mathcal{Z}(\text{trim}(R)) = \mathcal{Z}(\text{trim}(R'))$$

hence  $R$  and  $R'$  agree on all rows except possibly row 1 by the inductive hypothesis. Since  $w_R = w_{R'}$ , it follows that  $R = R'$  from Theorem 4.1, establishing injectivity.  $\square$

*Proof of Theorem 4.3.* By the previous lemma, it suffices to show that  $\mathcal{Z}$  is surjective. We have by the transition formula that if  $\mathbf{m}(w) \leq n$ , then

$$\mathfrak{S}_w(x_1, \dots, x_{n-1}, 0) = \sum_{\substack{w \searrow_n w' \\ \ell(w) = \ell(w')}} \mathfrak{S}_{w'}(x_1, \dots, x_{n-1})$$

Note that this sum is without multiplicity. The right hand side is equal to

$$\sum_{\substack{w \searrow_n w' \\ \ell(w) = \ell(w')}} \sum_{R' \in \mathcal{RC}_{n-1}(w')} x^{\mathbf{wt}(R')}$$

and the left hand side is equal to

$$\sum_{R \in \mathcal{RC}(w, n)^0} x^{\mathbf{wt}(R)}$$

since the only RC graphs that contribute to the left hand side are those with row  $n$  empty. This implies that

$$|\mathcal{RC}_n^0(w)| = \sum_{\substack{w \searrow_n w' \\ \ell(w) = \ell(w')}} |\mathcal{RC}_{n-1}(w')|$$

Now, the set

$$\{\mathcal{Z}(R) \mid R \in \mathcal{RC}_n(w)^0\}$$

has the same cardinality as  $\mathcal{RC}_n(w)^0$  by injectivity (Lemma 4.3.6), and is a subset of  $\bigcup_{w \searrow_n w'} \mathcal{RC}_{n-1}(w')$ . Since the right hand side has the same cardinality as  $\mathcal{RC}_n(w)^0$ , we conclude that  $\mathcal{Z}$  is surjective, as desired.  $\square$

We may extend  $\mathcal{Z}$  to an endomorphism  $\mathcal{BRC} \rightarrow \mathcal{BRC}$  by defining  $\mathcal{Z}(R) = 0$  if the last row of  $R$  is not empty. Then we have the following result.

**Corollary 4.3.7** (Combinatorial Lascoux-Scützenberger transition formula). *Let  $w$  be a permutation with last descent at most  $n$ . Then*

$$\mathcal{Z}(\mathcal{S}_w(n)) = \sum_{\substack{w \searrow_n w' \\ \ell(w) = \ell(w')}} \mathcal{S}_{w'}(n-1)$$

#### 4.4. Definition of the ring product.

**Definition 4.4.1.** Suppose we have two bounded RC graphs  $R_1$  with  $\text{ht}(R_1) = m$  and  $R_2$  with  $\text{ht}(R_2) = n$ . The product of these is defined by

$$R_1 \boxplus R_2 = \sum_{\substack{\text{clip}^m(R') = R_1 \\ \text{trim}^n(R') = R_2}} R'$$

where  $R'$  ranges over bounded RC graphs with  $\text{ht}(R') = m + n$ .

Associativity of  $\boxplus$  rests on two structural identities relating  $\text{clip}^p$  and  $\text{trim}^p$ : a functoriality of  $\text{clip}$  under iteration, and a commutativity of  $\text{clip}$  with  $\text{trim}$ .

**Lemma 4.4.2.** *Let  $R$  be a bounded RC graph with  $\text{ht}(R) = n$ , and let  $0 \leq p \leq p + q \leq n$ . Then*

$$\text{clip}^p(\text{clip}^{p+q}(R)) = \text{clip}^p(R).$$

*Proof.* By definition,  $\text{clip}^k(R')$  is computed by iterating the row-zeroing map  $\mathcal{Z}$  on the bottom row, one row at a time, until the height drops to  $k$ . Set  $r = n - (p + q)$ , so that  $\text{clip}^{p+q}(R)$  is obtained from  $R$  by  $r$  successive applications of  $\mathcal{Z}$  to the bottom row, and  $\text{clip}^p(R)$  is obtained from  $R$  by  $r + q$  such applications. The first  $r$  applications of  $\mathcal{Z}$  in the computation of  $\text{clip}^p(R)$  produce exactly  $\text{clip}^{p+q}(R)$ ; the remaining  $q$  applications then produce  $\text{clip}^p(\text{clip}^{p+q}(R))$ . Hence the two compositions agree.  $\square$

**Lemma 4.4.3** (Commutativity of  $\text{clip}$  and  $\text{trim}$ ). *Let  $R$  be a bounded RC graph with  $\text{ht}(R) = p + q + r$ . Then*

$$\text{trim}^p(\text{clip}^{p+q}(R)) = \text{clip}^q(\text{trim}^p(R)).$$

*Proof.* Both sides have height  $q$ . Lemma 4.3.5 asserts  $\mathcal{Z}(\text{trim}(R')) = \text{trim}(\mathcal{Z}(R'))$  for any  $R'$  whose last row is empty. Since  $\text{clip}^{p+q}$  on a graph of height  $p + q + r$  and  $\text{clip}^q$  on a graph of height  $q + r$  are both equal to the  $r$ -fold iterate of  $\mathcal{Z}$  on the bottom row (cf. Lemma 4.4.2), iterating Lemma 4.3.5  $r$  times gives

$$\text{trim}^p(\text{clip}^{p+q}(R)) = \text{trim}^p(\mathcal{Z}^r(R)) = \mathcal{Z}^r(\text{trim}^p(R)) = \text{clip}^q(\text{trim}^p(R)),$$

as required.  $\square$

**Theorem 4.4.** *The product  $\sqcup$  is associative and  $\mathcal{BRC}$  is a ring when equipped with  $\sqcup$ .*

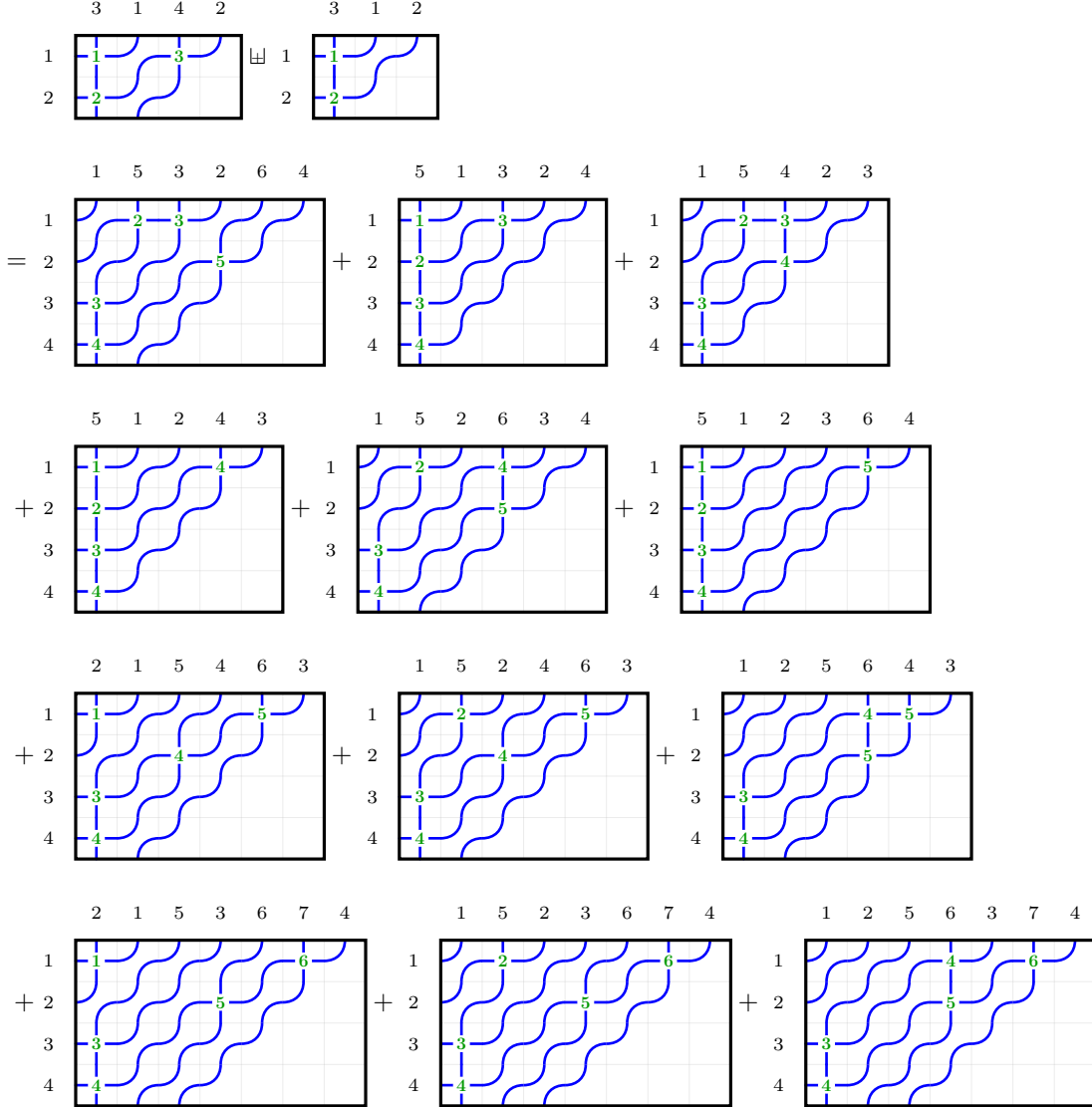
*Proof.* Let  $R_1, R_2, R_3$  be bounded RC graphs of heights  $p, q, r$  respectively, and let  $R$  be a bounded RC graph with  $\text{ht}(R) = p + q + r$ . We claim that  $R$  appears as a term in  $(R_1 \sqcup R_2) \sqcup R_3$  if and only if it appears as a term in  $R_1 \sqcup (R_2 \sqcup R_3)$ , and in both cases with multiplicity 1. Unwinding the definition of  $\sqcup$  in each bracketing yields the following conditions on  $R$ :

|       | Left bracketing $(R_1 \sqcup R_2) \sqcup R_3$ | Right bracketing $R_1 \sqcup (R_2 \sqcup R_3)$ |
|-------|-----------------------------------------------|------------------------------------------------|
| (i)   | $\text{clip}^p(\text{clip}^{p+q}(R)) = R_1$   | $\text{clip}^p(R) = R_1$                       |
| (ii)  | $\text{trim}^p(\text{clip}^{p+q}(R)) = R_2$   | $\text{clip}^q(\text{trim}^p(R)) = R_2$        |
| (iii) | $\text{trim}^{p+q}(R) = R_3$                  | $\text{trim}^q(\text{trim}^p(R)) = R_3$        |

Row (i) is equivalent on both sides by Lemma 4.4.2. Row (ii) is equivalent on both sides by Lemma 4.4.3. Row (iii) is equivalent on both sides because  $\text{trim}^{p+q} = \text{trim}^q \circ \text{trim}^p$  by definition of  $\text{trim}$ . Thus the term  $R$  contributes to the same bracketed product on both sides. Since the membership conditions are equalities (not weighted by any combinatorial multiplicity) and  $R$  is determined to lie in either product by these three conditions, the multiplicity is 1 in each case. We conclude that  $(R_1 \sqcup R_2) \sqcup R_3 = R_1 \sqcup (R_2 \sqcup R_3)$ .

To tie up the claim that  $\mathcal{BRC}$  is a ring, distributivity is directly imposed by the definition of the product as extending from the definition bilinearly, and the identity element is the empty RC graph of height 0.  $\square$

**Example 4.4.4.** We compute the product of two bounded RC graphs  $(R_1, 2) \in \mathcal{BRC}(2413, 2)$  and  $(R_2, 2) \in \mathcal{BRC}(231, 2)$ :



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