

A Bialgebra Framework for Positivity in Schubert Calculus

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Schubert polynomials

- ▶ Multivariate polynomials representing cohomology classes of Schubert varieties in the complete flag variety.
- ▶ Indexed by permutations, with the Schubert polynomial $\mathfrak{S}_w$ corresponding to a permutation w .
- ▶ Generalize the Schur polynomials, so their structure constants generalize the Littlewood-Richardson coefficients.
- ▶ Structure constants are nonnegative integers, but no positive combinatorial formula is known for them in general.

$$\mathfrak{S}_{41523}(x_1, x_2, x_3) = x_1^3 x_2^2 + x_1^3 x_2 x_3 + x_1^3 x_3^2$$

The “Schubert bialgebra”

Define

$$\mathbf{S} = \bigoplus_{n \geq 0} \mathbf{S}_n, \quad \mathbf{S}_n = \mathbb{Q}[x_1, \dots, x_n].$$

Multiplication in $\mathbf{S}$ is componentwise:

- ▶ if $f, g \in \mathbf{S}_n$, then $f \cdot g$ is the usual product in $\mathbb{Q}[x_1, \dots, x_n]$;
- ▶ if $f \in \mathbf{S}_m$, $g \in \mathbf{S}_n$ with $m \neq n$ and $m, n > 0$, then $f \cdot g = 0$.

A monomial in n variables corresponds to a weak composition a

$$x_a = x_1^{a_1} \cdots x_n^{a_n}.$$

The coproduct is deconcatenation

$$\Delta(x_c) = \sum_{ab=c} x_a \otimes x_b$$

$$\Delta(x_1^2 x_3 x_5) = 1 \otimes x_1^2 x_3 x_5 + x_1^2 \otimes x_3 x_5 + x_1^2 x_3 \otimes x_5 + x_1^2 x_3 x_5 \otimes 1.$$

Dual algebra

We define the dual algebra $\mathcal{DS}$ to be the graded dual of $\mathbf{S}$. For the monomial basis, which is indexed by weak compositions, we define the dual basis $[\alpha]$ for weak compositions α by

$$\langle x_\alpha, [\beta] \rangle = \delta_{\alpha, \beta}$$

We have that the dual product is concatenation:

$$[\alpha] \cdot [\beta] = [\alpha \beta].$$

Hence $\mathcal{DS}$ is a free associative algebra (tensor algebra on the generators $[k]$, $k \geq 0$). The coproduct is

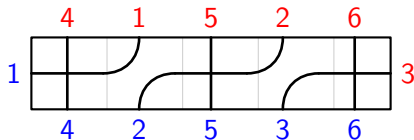
$$\Delta([\gamma]) = \sum_{\alpha + \beta = \gamma} [\alpha] \otimes [\beta].$$

What do the duals of Schubs look like?

Crossing diagrams

Think of a one-row strip of $n - 1$ boxes.

- ▶ Label the left edge by 1.
- ▶ Label the bottom by a permutation of the numbers $2, 3, \dots, n$ (left to right).
- ▶ In each box, place either a *cross* $\boxplus$ or a *bump* $\boxminus$.



Follow the strands through the row. The admissibility condition is that it cannot undo inversions, only create new ones.

A nice combinatorial basis

Let Ξ_w^k be the basis indexed by permutations w such that $w(p) < w(p+1)$ for all $p > k$ satisfying the following:

- ▶ $\Xi_w^1 = [w(1) - 1]$
- ▶ For a nonnegative integer $[p]$, we have that

$$[p] \cdot \Xi_w^k = \sum_{w'} \Xi_{w'}^{k+1}$$

where w' is obtained from w through a valid crossing diagram with p crosses after adding 1 to each element of w and adding a new first element 1.

I can provide an explicit (but mildly cancellative) formula for Ξ_w^k in the word basis, as well as a positive formula for the product of Ξ_u^p and Ξ_v^q in terms of crossing diagrams. The coproduct structure constants are also nonnegative integers, but I do not have a positive combinatorial formula for them yet. . .

Taking off the mask: The dual Schubert basis

The basis satisfies

$$\langle \mathfrak{S}_u(x_1, \dots, x_n), \Xi_v^n \rangle = \delta_{u,v}$$

where $\mathfrak{S}_u(x_1, \dots, x_n)$ is the Schubert polynomial corresponding to the permutation u .
The coefficients $c_{u,v}^w$ of the coproduct formula

$$\Delta \Xi_w^n = \sum_{u,v} c_{u,v}^w \Xi_u^n \otimes \Xi_v^n$$

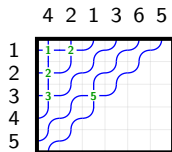
are the Littlewood-Richardson coefficients for Schubert polynomials!

Solution of the classical LR problem

- ▶ One solution of the classical LR problem (positive, combinatorial formula for multiplying Schur polynomials) is the associative algebra governed by the Knuth relations on words of tableaux (Lascoux and Schützenberger, 1981 [4]).
- ▶ This combinatorial construction creates an algebra with clearly nonnegative structure constants
- ▶ The subalgebra generated by sums of tableaux of a given shape behave precisely like the Schur polynomials, giving the solution
- ▶ I will do this here for Schubert polynomials! With a catch

Bounded pipe dreams

Pipe dreams are well-studied objects introduced in the 1990s by Bergeron and Billey to give a nicer combinatorial formula for Schubert polynomials equivalent to an earlier one by Billey, Jockusch, and Stanley involving reduced words/compatible sequences.



Theorem (Bergeron, Billey, 1993 [1])

The Schubert polynomial $\mathfrak{S}_w$ is equal to

$$\sum_{R \in \text{Pipe}(w)} x_{\text{wt}(R)}$$

where $\text{Pipe}(w)$ is the set of pipe dreams with permutation w and $\text{wt}(R)$ is the weak composition whose i -th part is the number of crosses in row i of R .

Transition algorithm on bounded pipe dreams

There is an algorithm that takes a bounded pipe dream of length n with row n empty to a unique pipe dream of length $n - 1$ in a way that matches the Schubert transition formula in a strong, weight-preserving, structure-preserving form.

Input: A bounded pipe dream R with $\text{length}(R) = n$ and row n empty.

Output: A bounded pipe dream R' with $\text{length}(R') = n - 1$, $|R'| = |R|$ (same number of crossing in each row), and $w_R \xrightarrow{n} w_{R'}$.

if R with last row removed is a valid bounded pipe dream **then**

return R with last row removed

else

 Locate crossings corresponding to maximal right descents of $w(R)$ until the maximal right descent is $n - 1$, say at rows $i_1 < i_2 < \dots < i_p$ with corresponding columns $j_1, j_2, \dots, j_p$.

$R^- \leftarrow R \setminus \{(i_1, j_1), \dots, (i_p, j_p)\}$.

 Let $I \leftarrow (i_1, i_2, \dots, i_p)$.

$D \leftarrow R^- \cup \{(n, 1), (n, 2), \dots, (n, p)\}$.

$D' \leftarrow I \overset{n-1}{\rightsquigarrow} D$ (an insertion algorithm similar to existing ones for pipe dreams)

$R' \leftarrow \{(i, j) \in D' \mid i < n\}$ as a bounded pipe dream with $n - 1$ rows.

return R' .

end if

Bounded pipe dream ring product

Given bounded pipe dreams R_1, R_2 , define $R_1 \star R_2$ by taking the sum of all R obtained by

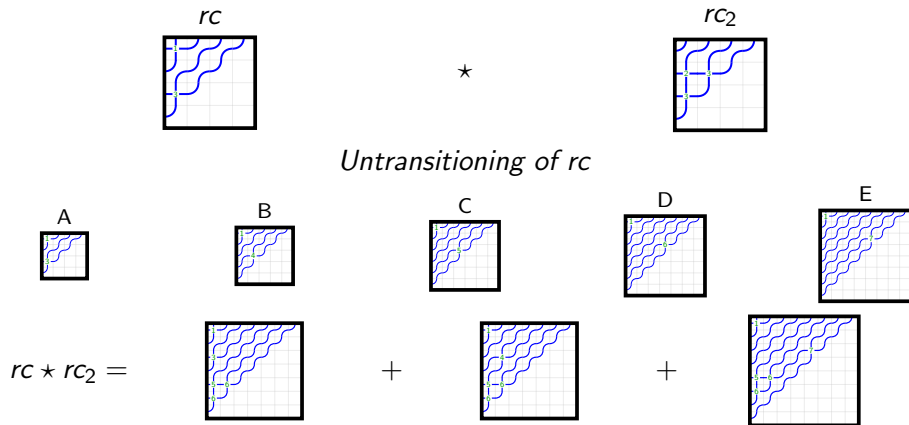
- ▶ *untransitioning* R_1 in all possible valid ways to create space while remaining a pipe dream,
- ▶ shifting R_2 downward,
- ▶ placing the shifted R_2 into the new empty region, unchanged, if doing so yields a valid pipe dream.

This allows us to lift the product of the dual algebra $\mathcal{DS}$ to a product on formal linear combinations of bounded pipe dreams.

Theorem (S, 2026+)

The product $\star$ is associative and gives a well-defined ring structure on the vector space Pipe spanned by bounded pipe dreams.

Product example



Pipe dream ring as a lift of $\mathcal{DS}$

There are two surjective linear maps from the pipe dream ring Pipe to the dual algebra $\mathcal{DS}$:

$$\begin{aligned}\pi_{\text{Sch}} : \text{Pipe} &\twoheadrightarrow \mathcal{DS}, & \pi_{\text{Sch}}(G) &= \Xi_{\text{perm}(G)}^{(n)}, \\ \pi_{\text{wt}} : \text{Pipe} &\twoheadrightarrow \mathcal{DS}, & \pi_{\text{wt}}(G) &= [\text{wt}(G)].\end{aligned}$$

By virtue of the fact that $\mathcal{DS}$ is free, there is a section $\iota : \mathcal{DS} \hookrightarrow \text{Pipe}$ of π_{wt} that sends $[i]$ for an integer i to the unique pipe dream r_i with $\text{wt}(r_i) = (i)$. It turns out that these are all ring homomorphisms and ι is also a section of π_{Sch} .

Theorem (S, 2026+)

We have that π_{Sch} and π_{wt} are surjective ring homomorphisms to $\mathcal{DS}$ and ι is an injective ring homomorphism from $\mathcal{DS}$ to Pipe such that

$$\pi_{\text{Sch}} \circ \iota = \pi_{\text{wt}} \circ \iota = \text{id}_{\mathcal{DS}}.$$

Littlewood-Richardson rule for (dual) Schubert polynomials

We immediately obtain a formula for the product of dual Schubert elements.

Theorem (S, 2026+)

For dual Schubert elements $\Xi_u^{n_1}, \Xi_v^{n_2}$, we have

$$\Xi_u^{n_1} \cdot \Xi_v^{n_2} = \sum_w d_{u,v}^w(n_1, n_2) \Xi_w^{n_1+n_2},$$

where, fixing pipe dreams R_u for u with $\text{length}(R_u) = n_1$ and R_v for v with $\text{length}(R_v) = n_2$, $d_{u,v}^w(n_1, n_2)$ is the number of bounded pipe dreams $R_w \in \text{Pipe}(w)$ such that R_w has a coefficient of 1 in the product $R_u \star R_v$.

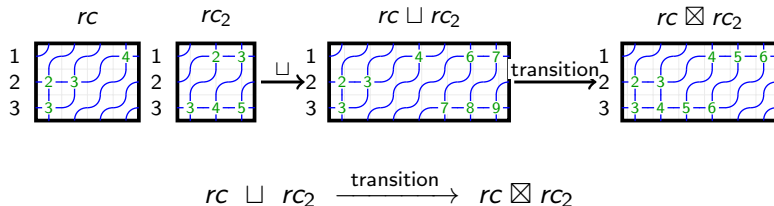
Equivalently,

$$\mathfrak{S}_w(x_1, \dots, x_{n_1+n_2}) = \sum_{u,v} d_{u,v}^w(n_1, n_2) \mathfrak{S}_u(x_1, \dots, x_{n_1}) \mathfrak{S}_v(x_{n_1+1}, \dots, x_{n_1+n_2}).$$

What about a coproduct?

- ▶ What we want now is to construct a coproduct on this algebra of pipe dreams that lifts the coproduct on $\mathcal{DS}$.
- ▶ I don't know how to do this. However, there is a specialized product that can be derived from the $\star$ product that does act like the dual of a hypothetical coproduct on Pipe

The squash product



- ▶ Start with disjoint union after shifting R_2 to the right by the length of $\text{perm}(R_1)$.
- ▶ Apply combinatorial transition to return to an n -row pipe dream.
- ▶ This is well-defined for any pipe dream, but restricting to R_2 n -Grassmannian it results in a right module action of GrassPipe_n (which becomes a ring under this product isomorphic to the opposite of the plactic algebra)

RC factor algebra

Consider the algebra $\mathcal{B}_n$ that is given by the quotient of the tensor algebra

$$T \left(\bigoplus_{k=1}^n \text{GrassPipe}_k \right)$$

by the relations

- ▶ Pipe dreams of different lengths commute,
- ▶ The product of two pipe dreams of the same length is given by the squash product

There is a linear map $\text{Pipe} : \mathcal{B}_n \rightarrow \text{Pipe}_n$ given by sending a pure tensor of Grassmannian pipe dreams to the $\boxtimes$ product of those pipe dreams in increasing order of length. This is a surjective homomorphism of right GrassPipe_n -modules.

Unfortunately, the kernel is not an ideal of $\mathcal{B}_n$. **There may not be a product that “checks all the boxes” on pipe dreams at all.**

Schubert elements

Theorem (S, 2026+)

There are canonical elements $\mathfrak{S}_w^{(n)}$ of $\mathcal{B}_n$ such that

$$\text{Pipe}(\mathfrak{S}_w^{(n)}) = \sum_{P \in \text{Pipe}_n(w)} P$$

and

$$\text{Pipe}(\mathfrak{S}_u^{(n)} \cdot \mathfrak{S}_v^{(n)}) = \sum_w c_{u,v}^w \text{Pipe}(\mathfrak{S}_w^{(n)})$$

where $c_{u,v}^w$ is the Littlewood-Richardson coefficient of the complete flag variety.

These $\mathfrak{S}_w^{(n)}$ do not form a basis of the subring they generate, but there may be additional relations that can make this happen.

Caveat

Caveat. The elements $\mathfrak{S}_w^{(n)}$ are not usually nonnegative in the Grassmannian pipe dream basis of $\mathcal{B}_n$.

$$\mathfrak{S}_{1432}^{(3)} = - \begin{array}{|c|c|c|} \hline 1 & & \\ \hline 2 & & \\ \hline 3 & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & \\ \hline 2 & & \\ \hline & & \\ \hline \end{array} \otimes \begin{array}{|c|c|c|} \hline & 2 & \\ \hline & & \\ \hline 3 & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & \\ \hline 2 & & \\ \hline & & \\ \hline \end{array} \otimes \begin{array}{|c|c|c|} \hline & 2 & \\ \hline & & \\ \hline 3 & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & 2 & \\ \hline & & \\ \hline & & \\ \hline \end{array} \otimes \begin{array}{|c|c|c|} \hline & 2 & \\ \hline & & \\ \hline 3 & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & \\ \hline 2 & & \\ \hline & & \\ \hline \end{array} \otimes \begin{array}{|c|c|c|} \hline & 2 & \\ \hline & & \\ \hline 3 & & \\ \hline \end{array}$$

$$+ \begin{array}{|c|c|c|} \hline & 2 & \\ \hline & & \\ \hline & & \\ \hline \end{array} \otimes \begin{array}{|c|c|c|} \hline & 2 & \\ \hline & & \\ \hline 3 & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & \\ \hline 2 & & \\ \hline & & \\ \hline \end{array} \otimes \begin{array}{|c|c|c|} \hline & 2 & \\ \hline & & \\ \hline 3 & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & \\ \hline 2 & & \\ \hline & & \\ \hline \end{array} \otimes \begin{array}{|c|c|c|} \hline & 2 & \\ \hline & & \\ \hline 3 & & \\ \hline \end{array}$$

However, with some work, we can show the following:

Theorem (S, 2026+)

This formula is nonnegative when both u and v avoid 1432, 3142, and 4132.

More is likely possible.

Other bases of the dual algebra

Any basis of the polynomial ring in n variables results in a dual basis of $\mathcal{DS}_n$. If it can be treated as a series of stable bases in n , we can analyze bases related to Schubert polynomials in this way.

Examples

- ▶ Key polynomials (Demazure 1974 [2]): κ_α , have a formula in terms of pipe dreams. The transition algorithm respects the Demazure crystal structure, so we can obtain related results.
- ▶ Forest polynomials (Nadeau and Tewari 2024 [5]): $\mathfrak{P}_F$, can be defined by a certain equivalence relation on reduced words/compatible sequences. A paper was posted on arXiv with a pipe dream formula for $\mathfrak{P}_F$ (Guo and Woodruff 2026 [3]). The transition algorithm respects the equivalence relation, so there is a relationship with the $\star$ product.

Results

Theorem (S, 2026+)

- ▶ *Key polynomial branching formula (known but not with this particular combinatorial interpretation), highest weight quotient of Pipe lifts the dual algebra.*
- ▶ *Forest polynomial branching formula (not previously known).*

References I

- [1] Nantel Bergeron and Sara Billey. RC-graphs and Schubert polynomials. *Experimental Math.*, 2(4):257–269, 1993.
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- [3] Annie Guo and Dora Woodruff. Forest polynomials and pattern avoidance, 2026. arXiv:2602.04036 [math.CO].
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- [5] Philippe Nadeau and Vasu Tewari. Forest polynomials and the class of the permutahedral variety. *Advances in Mathematics*, 453:109834, 2024.