

A Bialgebra Framework for Positivity in Schubert Calculus

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Schubert polynomials

- Basis of the polynomial ring $\mathbb{Z}[x_1, x_2, \dots]$, sharing their structure constants with the Schubert classes in the cohomology of the complete flag variety.
- Indexed by permutations, with the Schubert polynomial $\mathfrak{S}_w$ corresponding to a permutation w .
- Generalize the Schur polynomials, so their structure constants generalize the Littlewood-Richardson coefficients.
- Structure constants are nonnegative integers, but no positive combinatorial formula is known for them in general.

$$\mathfrak{S}_{41523}(x_1, x_2, x_3) = x_1^3 x_2^2 + x_1^3 x_2 x_3 + x_1^3 x_3^2$$

The “Schubert bialgebra”

Define

$$\mathbf{S} = \bigoplus_{n \geq 0} \mathbf{S}_n, \quad \mathbf{S}_n = \mathbb{Q}[x_1, \dots, x_n].$$

Multiplication in $\mathbf{S}$ is componentwise:

- if $f, g \in \mathbf{S}_n$, then $f \cdot g$ is the usual product in $\mathbb{Q}[x_1, \dots, x_n]$;
- if $f \in \mathbf{S}_m$, $g \in \mathbf{S}_n$ with $m \neq n$ and $m, n > 0$, then $f \cdot g = 0$.

A monomial in n variables corresponds to a weak composition a

$$x_a = x_1^{a_1} \cdots x_n^{a_n}.$$

The coproduct is deconcatenation

$$\Delta(x_c) = \sum_{ab=c} x_a \otimes x_b$$

$$\Delta(x_1^2 x_3 x_5) = 1 \otimes x_1^2 x_3 x_5 + x_1^2 \otimes x_3 x_5 + x_1^2 x_3 \otimes x_5 + x_1^2 x_3 x_5 \otimes 1.$$

Dual algebra

We define the dual algebra $\mathcal{DS}$ to be the graded dual of $\mathbf{S}$. For the monomial basis, which is indexed by weak compositions, we define the dual basis $[a]$ for weak compositions a by

$$\langle x_a, [b] \rangle = \delta_{a,b}$$

We have that the dual product is concatenation:

$$[a] \cdot [b] = [a b].$$

Hence $\mathcal{DS}$ is a free associative algebra (tensor algebra on the generators $[k]$, $k \geq 0$). The coproduct is

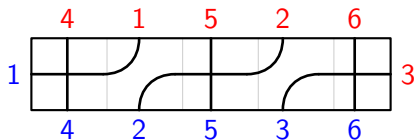
$$\Delta([c]) = \sum_{a+b=c} [a] \otimes [b].$$

What do the duals of Schubs look like?

Crossing diagrams

Think of a one-row strip of $n - 1$ boxes.

- Label the left edge by 1.
- Label the bottom by a permutation of the numbers $2, 3, \dots, n$ (left to right).
- In each box, place either a *cross* $\boxplus$ or a *bump* $\curvearrowright$.



Follow the strands through the row. The admissibility condition is that it cannot undo inversions, only create new ones.

A nice combinatorial basis

Let Ξ_w^k be the basis of $\mathcal{DS}_k$ indexed by permutations w such that $w(p) < w(p+1)$ for all $p > k$ satisfying the following:

- $\Xi_w^1 = [w(1) - 1]$
- For a nonnegative integer $[p]$, we have that

$$[p] \cdot \Xi_w^k = \sum_{w'} \Xi_{w'}^{k+1}$$

where w' is obtained from w through a valid crossing diagram with p crosses after adding 1 to each element of w and adding a new first element 1.

I can provide an explicit (but mildly cancellative) formula for Ξ_w^k in the word basis, as well as a positive formula for the product of Ξ_u^p and Ξ_v^q in terms of crossing diagrams.

The coproduct structure constants are also nonnegative integers, but I do not have a positive combinatorial formula for them yet...

Taking off the mask: The dual Schubert basis

The basis satisfies

$$\langle \mathfrak{S}_u(x_1, \dots, x_n), \Xi_v^n \rangle = \delta_{u,v}$$

where $\mathfrak{S}_u(x_1, \dots, x_n)$ is the Schubert polynomial corresponding to the permutation u .

The coefficients $c_{u,v}^w$ of the coproduct formula

$$\Delta \Xi_w^n = \sum_{u,v} c_{u,v}^w \Xi_u^n \otimes \Xi_v^n$$

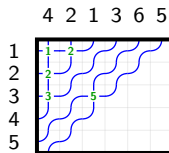
are the elusive Littlewood-Richardson coefficients for Schubert polynomials!

Solution of the classical LR problem

- One solution of the classical LR problem (positive, combinatorial formula for multiplying Schur polynomials) is the associative algebra governed by the Knuth relations on words of tableaux (Lascoux and Schützenberger, 1981 [6]).
- This combinatorial construction creates an algebra with clearly nonnegative structure constants
- The subalgebra generated by sums of tableaux of a given shape behave precisely like the Schur polynomials, giving the solution
- I will do this here for dual Schubert polynomials!

Bounded pipe dreams

Pipe dreams are well-studied objects introduced in the 1990 by Bergeron and Billey to represent monomials of Schubert polynomials. We associate a fixed bound $\text{length}(R)$ that is $\geq$ the length of the Lehmer code



Theorem (Bergeron, Billey, 1993 [1])

The Schubert polynomial $\mathfrak{S}_w$ is equal to

$$\sum_{R \in \text{Pipe}(w)} x_{\text{wt}(R)}$$

Transition algorithm on bounded pipe dreams

There is an algorithm that takes a bounded pipe dream of length n with row n empty to a unique bounded pipe dream of length $n - 1$ by a method that adheres to the Schubert transition formula in a way that preserves important combinatorial invariants and structures

Input: A bounded pipe dream R with $\text{length}(R) = n$ and row n empty.

Output: A bounded pipe dream R' with $\text{length}(R') = n - 1$, $|R'| = |R|$ (same number of crossing in each row), and $w_R \xrightarrow{n} w_{R'}$.

if R with last row removed is a valid bounded pipe dream **then**

return R with last row removed

else

 Locate crossings corresponding to maximal right descents of $w(R)$ until the maximal right descent is $n - 1$, say at rows $i_1 < i_2 < \dots < i_p$ with corresponding columns $j_1, j_2, \dots, j_p$.

$R^- \leftarrow R \setminus \{(i_1, j_1), \dots, (i_p, j_p)\}$.

 Let $I \leftarrow (i_1, i_2, \dots, i_p)$.

$D \leftarrow R^- \cup \{(n, 1), (n, 2), \dots, (n, p)\}$.

$D' \leftarrow I \overset{n-1}{\rightsquigarrow} D$ (an insertion algorithm similar to existing ones for pipe dreams)

$R' \leftarrow \{(i, j) \in D' \mid i < n\}$ as a bounded pipe dream with $n - 1$ rows.

return R' .

end if

Bounded pipe dream ring product

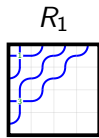
Given bounded pipe dreams R_1, R_2 , define $R_1 \star R_2$ by taking the sum of all R obtained by

- *untransitioning* R_1 in all possible valid ways to add as many empty rows as the length of $\text{perm}(R_2)$
- shifting R_2 downward by $\text{length}(R_1)$
- placing the shifted R_2 into the new empty region, unchanged, if doing so yields a valid pipe dream (meaning we can connect the pipes consistently)

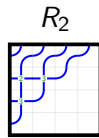
Theorem (S, 2026+)

The product $\star$ is associative and gives a well-defined ring structure on the vector space Pipe spanned by bounded pipe dreams, graded by length and degree

Product example



$\star$



Untransitioning of R_1

A



B



C



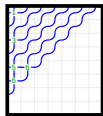
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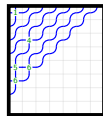
E



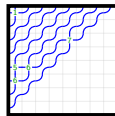
$$R_1 \star R_2 =$$



+



+



Pipe dream ring as a lift of $\mathcal{DS}$

There are two surjective linear maps from the pipe dream ring Pipe to the dual algebra $\mathcal{DS}$:

$$\pi_{\text{Sch}} : \text{Pipe} \twoheadrightarrow \mathcal{DS}, \quad \pi_{\text{Sch}}(G) = \Xi_{\text{perm}(G)}^{(n)},$$

$$\pi_{\text{wt}} : \text{Pipe} \twoheadrightarrow \mathcal{DS}, \quad \pi_{\text{wt}}(G) = [\text{wt}(G)].$$

By virtue of the fact that $\mathcal{DS}$ is free, there is a section $\iota : \mathcal{DS} \hookrightarrow \text{Pipe}$ of π_{wt} that sends $[i]$ for an integer i to the unique pipe dream r_i with $\text{wt}(r_i) = (i)$. It turns out that these are all ring homomorphisms and ι is also a section of π_{Sch} .

Theorem (S, 2026+)

We have that π_{Sch} and π_{wt} are surjective ring homomorphisms to $\mathcal{DS}$ and ι is an injective ring homomorphism from $\mathcal{DS}$ to Pipe such that

$$\pi_{\text{Sch}} \circ \iota = \pi_{\text{wt}} \circ \iota = \text{id}_{\mathcal{DS}}.$$

Littlewood-Richardson rule for (dual) Schubert polynomials

We immediately obtain a formula for the product of dual Schubert elements.

Theorem (S, 2026+)

For dual Schubert elements $\Xi_u^{n_1}, \Xi_v^{n_2}$, we have

$$\Xi_u^{n_1} \cdot \Xi_v^{n_2} = \sum_w d_{u,v}^w(n_1, n_2) \Xi_w^{n_1+n_2},$$

where, fixing pipe dreams R_u for u with $\text{length}(R_u) = n_1$ and R_v for v with $\text{length}(R_v) = n_2$, $d_{u,v}^w(n_1, n_2)$ is the number of bounded pipe dreams R that occur in the product $R_u \star R_v$ such that $\text{perm}(R) = w$.

Equivalently,

$$\mathfrak{S}_w(x_1, \dots, x_{n_1+n_2}) = \sum_{u,v} d_{u,v}^w(n_1, n_2) \mathfrak{S}_u(x_1, \dots, x_{n_1}) \mathfrak{S}_v(x_{n_1+1}, \dots, x_{n_1+n_2}).$$

A warm-up: dual key polynomials

Key polynomials κ_α are a well-known basis of $\mathbb{Q}[x_1, \dots, x_n]$ (Demazure characters) closely related to Schubert polynomials. We can deduce an LR rule for dual keys as follows:

- The *Coxeter-Knuth equivalence* on reduced words [7] (Morse-Schilling 2016) as well as the Q tableau [5] (Hamaker-Young 2014) are preserved by $\star$. Using this, we define $R \sim A$ iff $Q(R) = Q(A)$, $\text{wt}(R) = \text{wt}(A)$, and $\text{word}(R)$, $\text{word}(A)$ are Coxeter-Knuth equivalent.
- Quotient the ring by this equivalence relation to obtain κPipe with $[R] \mapsto \kappa_{\text{exwt}(R)}^*$.

Theorem (S, 2026+)

$\pi_\kappa : \kappa\text{Pipe} \rightarrow \mathcal{DS}$, $[R] \mapsto \kappa_{\text{wt}(R)}^*$ is a surjective ring homomorphism. The structure constants $k_{\alpha,\beta}^\gamma$ in $\kappa_\alpha^* \kappa_\beta^* = \sum_\gamma k_{\alpha,\beta}^\gamma \kappa_\gamma^*$ count classes $[R_\gamma]$ in $[R_\alpha] \star [R_\beta]$, and in particular are nonnegative.

Other combinatorial interpretations of these coefficients are known.

Forest polynomials

Nearly the same construction works for the recently discovered *forest polynomials*, whose products expand nonnegatively in **S**, unlike key polynomials

- Forest polynomials (Nadeau and Tewari 2024 [9], +Spink [8]): $\mathfrak{P}_F$ are polynomials indexed by “indexed forests” F (bijectively related to weak compositions), which can be characterized by an equivalence relation on words, or quasisymmetric divided difference operators based on the Bergeron-Sottile map [2]
- Using the equivalence relation, we can define a “forest invariant” $\Omega(R)$ for each pipe dream R , and obtain

Theorem (S, 2026+)

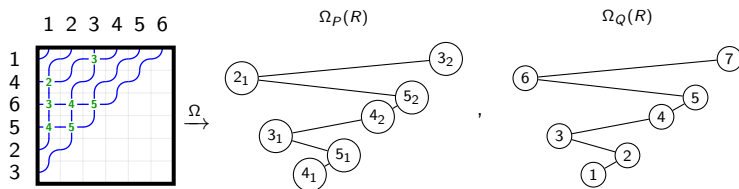
Let F be an indexed forest and let $R \in \text{Pipe}$ be such that $\text{code}(\Omega(R)) = F$. Then

$$\mathfrak{P}_F = \sum_{A \in \text{Pipe}, \Omega(A) = \Omega(R)} \text{wt}(A)$$

In Guo and Woodruff 2026 [4] there is also a pipe dream formula

Forest equivalence on Pipe

- We can actually do better, algebraically speaking. The equivalence relation is as strong as Edelman-Greene insertion (Edelman-Greene 1987 [3]) and uniquely encodes a word and compatible sequences as a pair of labeled indexed forests
- Let $\sim$ be the equivalence relation on Pipe such that $R \sim A$ if $\text{wt}(R) = \text{wt}(A)$, $\text{code}(\Omega(R)) = \text{code}(\Omega(A))$ and $\Omega_Q(R) = \Omega_Q(A)$, where Ω_Q is the Q -tableau of the insertion algorithm. Let ΩPipe be the set of equivalence classes of Pipe under $\sim$.



Algebra of forest pipe dreams

Theorem (S, 2026+)

ΩPipe inherits the product $\star$ from Pipe via the function $R \mapsto [R]$ from the subalgebra of Pipe generated by the pipe dreams satisfying $\text{code}(\Omega(R)) = \text{code}(\text{perm}(R))$, which is a surjective homomorphism of rings onto ΩPipe . The function

$$\pi_{\mathfrak{P}} : \Omega\text{Pipe} \rightarrow \mathcal{DS}$$

such that

$$[R] \mapsto \mathfrak{P}_{\text{code}(\Omega(R))}^*$$

is a surjective homomorphism of rings onto $\mathcal{DS}$.

Note: Unlike for key polynomials, this fails without restricting to this particular subalgebra. ΩPipe is not a quotient of Pipe by this equivalence relation.

Dual forest LR rule/branching formula

Theorem (S, 2026+)

Suppose

$$\mathfrak{P}_F^* \mathfrak{P}_G^* = \sum_H f_{F,G}^H \mathfrak{P}_H^*$$

where $f_{F,G}^H$ are the structure constants for the product in $\mathcal{DS}$. Then $f_{F,G}^H$ counts the sum of the coefficients of all $[R_H]$ such that $\text{code}(\Omega(R_H)) = H$ in a product $[R_F] \star [R_G] \in \Omega\text{Pipe}$ with $\text{code}(\Omega(R_F)) = F$ and $\text{code}(\Omega(R_G)) = G$, and in particular is nonnegative.

Dual forest polynomial coproduct?

It is known (combinatorially, but not with a formula) that forest polynomials have nonnegative structure constants, so that dual forest polynomials $\mathfrak{P}_F^*$ have nonnegative structure constants for the coproduct in $\mathcal{DS}$.

For example,

$$\begin{aligned}\Delta(\mathfrak{P}_{1,2}^*) &= \mathfrak{P}_{0,0}^* \otimes \mathfrak{P}_{1,2}^* + \mathfrak{P}_{0,1}^* \otimes \mathfrak{P}_{0,2}^* + \mathfrak{P}_{0,1}^* \otimes \mathfrak{P}_{1,1}^* + \mathfrak{P}_{0,2}^* \otimes \mathfrak{P}_{0,1}^* \\ &\quad + \mathfrak{P}_{0,2}^* \otimes \mathfrak{P}_{1,0}^* + \mathfrak{P}_{1,0}^* \otimes \mathfrak{P}_{0,2}^* + \mathfrak{P}_{1,1}^* \otimes \mathfrak{P}_{0,1}^* + \mathfrak{P}_{1,2}^* \otimes \mathfrak{P}_{0,0}^*\end{aligned}$$

- It is distinctly possible that a positive coproduct formula for $\mathfrak{P}_F^*$ in $\mathcal{DS}$ can be obtained by analyzing the $\star$ product of pipe dreams with $\text{code}(\Omega(R)) = F$.

Summary

- The pipe dream ring Pipe with product $\star$ lifts the algebra $\mathcal{DS}$ and gives positive LR rules for dual Schubert, key, and forest polynomials by taking appropriate quotients of subalgebras.
- Positive formulas for the forest polynomial coproduct remain open and are potentially accessible via this framework.

Thank you!

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