

A LITTLEWOOD–RICHARDSON RULE FOR FOREST POLYNOMIALS VIA OPERATIONS ON RC GRAPHS

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ABSTRACT.

1. INTRODUCTION

2. NIL-COXETER ACTION

Let R be an RC graph and let $i > 0$ be an integer. We define $\mathfrak{d}^i R$, an element of $\mathcal{RC}$, as follows. Define

$$\mathfrak{d}^i R = 0$$

unless s_i is a right descent of w_R and there exists $(i, j) \in R$ such that $\mathbf{rt}_R(i, j) = (i, i + 1)$. If these latter two conditions are satisfied, define

$$R' = R \setminus \{(i, j)\}$$

where $\mathbf{rt}_R(i, j) = (i, i + 1)$. If $e_i(R') \neq \emptyset$, define $\mathfrak{d}^i R = 0$. Otherwise, define

$$\mathfrak{d}^i R = \sum_{p=0}^{\varphi_i(R')} f_i^p R'$$

Lemma 2.0.1. *Let R be an RC graph and let $i > 0$ be an integer. If s_i is not a right descent of w_R , then there is a unique R' such that the coefficient of R in $\mathfrak{d}^i R'$ is 1, and for other R' the coefficient is 0.*

Proof. Suppose s_i is not a right descent of w_R . Assume without loss of generality that $e_i R = \emptyset$. If $(i + 1, 1) \notin R$, then $(i, 1) \notin R$ because s_i is not a right descent of w_R . In that case, let $R' = R \cup \{(i, 1)\}$. Then

$$\mathfrak{d}^i R' = R + \text{other terms}$$

If instead $(i + 1, 1) \in R$, since $e_i(R) = \emptyset$ it follows that $(i, 1) \in R$, and if j is the maximum value such that $(i + 1, j') \in R$ for all $j' < j$, it follows that $(i, j') \in R$ for all $j' < j$. We must have that $(i, j + 1) \notin R$, because $\mathbf{rt}_R(i, j + 1) = (i, i + 1)$. Setting $R' = R \cup \{(i, j + 1)\}$ gives us the result. $\square$

Theorem 2.1. *Suppose $w \in S_\infty$ and $i > 0$. If i is not a right descent of w , then*

$$\mathfrak{d}^i \mathcal{S}_w(n) = 0$$

Otherwise,

$$\mathfrak{d}^i \mathcal{S}_w(n) = \mathcal{S}_{ws_i}(n)$$

Proof. The result is trivial if i is not a right descent of w . Suppose i is a right descent of w . By Lemma 2.0.1, for each RC graph R such that $w_R = ws_i$, there is a unique RC graph R' such that the coefficient of R in $\mathfrak{d}^i R'$ is 1. Since $w_{R'} = w$, the result follows. $\square$

Corollary 2.0.2. *Suppose R is an RC graph such that $w_R = w$ is dominant. Then for any reduced word $(s_{i_1}, s_{i_2}, \dots, s_{i_m})$ for a permutation u such that*

$$\ell(wu^{-1}) = \ell(w) - \ell(u)$$

we have that

$$\mathfrak{d}^{i_1} \mathfrak{d}^{i_2} \dots \mathfrak{d}^{i_m} R = \mathcal{S}_{wu^{-1}}(n)$$

for sufficiently large n .

Theorem 2.2. *The crystal divided difference operators satisfy the relations:*

$$\begin{aligned} \mathfrak{d}^i \mathfrak{d}^i &= 0 \\ \mathfrak{d}^i \mathfrak{d}^j &= \mathfrak{d}^j \mathfrak{d}^i \quad \text{for } |i - j| > 1 \\ \mathfrak{d}^i \mathfrak{d}^{i+1} \mathfrak{d}^i &= \mathfrak{d}^{i+1} \mathfrak{d}^i \mathfrak{d}^{i+1} \end{aligned}$$

Proof. The first and second relations are not difficult. Otherwise, assume without loss of generality that $i = 1$. For an RC graph with precisely three rows, let us consider applying $\mathfrak{d}^1 \mathfrak{d}^2 \mathfrak{d}^1$. Suppose $\mathbf{c}(w) = (a, b, c)$. It is well known that if $m = \min(a, b, c)$, then the first m columns of R are completely full, hence without loss of generality let us assume that $m = 0$. The only way the braid relation can apply to yield a nonzero result, therefore, is if $a > b > c = 0$. Since this yields a dominant permutation, it follows that R is the unique RC graph for w . Applying the braid relation in either direction yields the complete sum of RC graphs for the permutation with code $(0, a - 1, b - 2)$ by application of Corollary 2.0.2. By transfer, this result also applies to the last two or three rows of any RC graph with respect to the code.

Considering a general RC graph, we may similarly assume that $\mathbf{c}_3(w) = 0$ by removing full columns, since only ledges will yield nonzero results. If there are extra elements in the first three rows that are not in the implied dominant block, then they all satisfy $r + c - 1 > \mathbf{c}_r(w) + 1$, where $r \in \{1, 2, 3\}$ and the extra element is at row (r, c) , since they have come down from a row past row 4 (with respect to the principal RC graph). Elements with $r + c - 1 > \mathbf{c}_1(w) + 1$ can be seen to have no essential interaction with rows 1, 2, and 3 and can be disregarded. We are left with the configuration in Figure 2.

In this situation, without ambiguity, application of the braid triple in either direction will delete the immobile $(2, 1)$, $(1, \mathbf{c}_2(w) + 1)$, and $(1, 1)$. Without application of any lowering operators, simply deleting these elements yields the highest weight of the crystal component containing R with respect to the first three rows. Since we may apply $\mathbf{clip}^3(R)$ to obtain a Demazure crystal, the lowering operators transitively generate the sum. To see that we indeed get all of them, we note that for this highest weight $R' = \mathbf{clip}^3(R)$, application of $\mathfrak{d}^1 \mathfrak{d}^2 \mathfrak{d}^1$ should yield elements of the form

$$f_1^{a_1} f_2^{a_2} f_1^{a_3} R'$$

and for the other direction,

$$f_2^{b_1} f_1^{b_2} f_2^{b_3} R'$$

These are two distinct parametrizations of the Demazure crystal $B_{s_1 s_2 s_1}(\lambda) = B_{s_2 s_1 s_2}(\lambda)$ with highest weight λ , yielding the same elements. $\square$

		○	○	○			○
○		○	○	○	○	○	○

FIGURE 2. Configuration in the final steps of the proof of the braid relation.

3. THOMPSON MONOID

Definition 3.0.1 (Pulling out an empty row). Let R be a bounded RC graph with $\mathbf{ht}(R) = n$ and let $p \leq n$ be such that row p of R is empty. Let $R_{\leq p}$ denote the RC-graph consisting of the first p rows and $R_{> p}$ the RC-graph consisting of rows $p + 1$ through n . Now, $\mathbf{m}(R_{\leq p})$ could be greater than p . If it is less than or equal to p , define

$$\mathfrak{R}_p(R) = \mathcal{Z}(R_{\leq p}) \cup \downarrow R_{> p}$$

as a bounded RC graph with $\mathbf{ht}(R) - 1$ rows. Otherwise, define $R' = R_{\leq p}$, let $d_1 = \mathbf{m}(w_{R'})$ and set $R_1 = R'$. While $\mathbf{m}(R_i) > p$, do:

Set $R' = R_i$. While $\mathbf{m}(R') \geq d_i$, delete the element $(a, b) \in R'$ such that $\mathbf{rt}_{R'}(a, b) = (\mathbf{m}(R'), \mathbf{m}(R') + 1)$ from R' and store a in the list (multiset) A_i . After enough iterations such that $\mathbf{m}(R') < d_i$, set $R_{i+1} = R'$ and set $d_{i+1} = \mathbf{m}(R_{i+1})$, then repeat.

At termination, say at step k , set

$$\mathfrak{R}_p(R) = (A_1 \xrightarrow{d_1-1} \cdots A_k \xrightarrow{d_k-1} \mathcal{Z}(\overline{R})) \cup \downarrow R_{>p}$$

be the desired bounded RC graph with $\text{ht}(R) - 1$ rows.

We show an example in Figure 3.

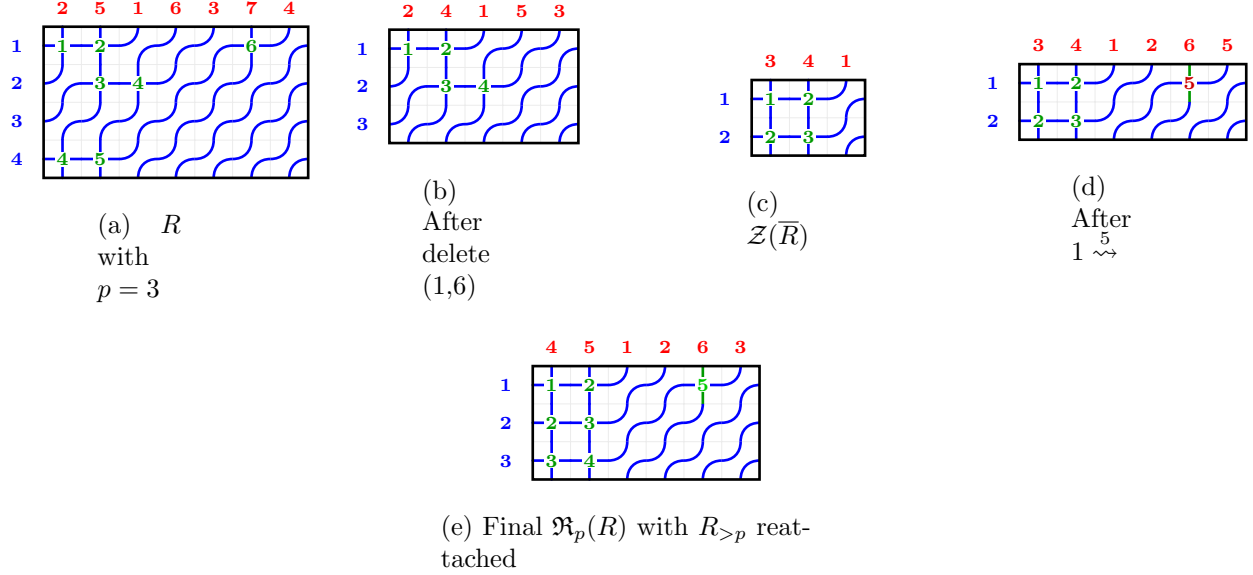


FIGURE 3. Example of $\mathfrak{R}_p$ operation with $p = 3$. Starting with (a) R where row 3 is empty, we identify element $(1, 6)$ creates $\max d > p$. (b) After deleting it (storing row 1 in A_1). (c) Apply $\mathcal{Z}$ to get 2-row graph. (d) Insert back at row 1 with $k = d_1 - 1 = 5$ (shown in red). (e) Reattach $R_{>p}$ shifted down to get final result.

Definition 3.0.2. Let R be an RC graph. We define an RC graph $\overline{R}$ and a sequence $\text{strip}(R)$ as follows. Let $\mathbf{m}(R) = n$, set $R_0 = R$, and set $A_0 = ()$. While $\mathbf{m}(R_i) > n - 1$, delete the element $(a, b) \in R_i$ such that $\mathbf{rt}_{R_i}(a, b) = (\mathbf{m}(R_i), \mathbf{m}(R_i) + 1)$ to obtain R_{i+1} , and add a to A_i to obtain A_{i+1} . When the algorithm terminates at step k , set

$$\text{strip}(R) = A_k$$

and set $\overline{R} = \overline{R}$.

Lemma 3.0.3. Let R be an RC graph. Then

$$R = \text{strip}(R) \xrightarrow{\mathbf{m}(R)} \overline{R}$$

Proof. Let $n = \mathbf{m}(R)$. We prove this by induction on $\mathbf{c}_n(R)$. Let (i, j) be the location of the descent n in R . If $\mathbf{c}_n(R) = 1$, then we know that insertion of i into $\overline{R}$ returns an RC graph with the same weight and permutation as R , the latter assertion because the only n -reflection that increases the length of $w_{\overline{R}}$ by exactly 1 is s_n . By the exchange property, there is precisely one location in row i that this n -reflection can be inserted, and we have the result when $\mathbf{c}_n(R) = 1$.

Assume the inductive hypothesis that we have the result for $\mathbf{c}_n(R) = k$ for some $k \geq 1$ and additionally that insertion order of reflections has been $(n, n + 1), \dots, (n, n + k)$. Assume $\text{strip}(R) = (r_1, \dots, r_{k+1})$ in increasing order and we have inserted all of $r_2, \dots, r_{k+1}$ in Algorithm ???. By the induction hypothesis, the RC graph R' we have at this point has permutation $\mathbf{c}_i R' = \mathbf{c}_i R$ for $i < n$ and $\mathbf{c}_n(R') = k$, and we have equality of R' with R except deletion of the root $(n, n + k + 1)$. Since $w_{R'}(n) < w_{R'}(n + k + 1)$, the only n -reflection that can be inserted to increase the length of $w_{R'}$ by 1 is $(n, n + k + 1)$. By pipe considerations, this must be inserted in column $c = n + k + 1 - r_1$. Since this is the only possible insertion location and the permutation is the same as w_R , we have the result by induction. $\square$

Theorem 3.1. *Let $R \in \mathcal{BRC}(w, n)^{(0,p)}$ be the set of all bounded RC graphs with $\mathbf{ht}(R) = n$ such that row p of R is empty. Then we have that the restriction of $\mathfrak{R}_p$ is a bijection*

$$\mathfrak{R}_p : \mathcal{BRC}(w, n)^{(0,p)} \rightarrow \bigcup_{\substack{w \xrightarrow{p} w' \\ \ell(w) = \ell(w')}} \mathcal{BRC}(w', n-1)$$

satisfying $\mathbf{wt}(\mathfrak{R}_p(R)) = (i_1, \dots, i_{p-1}, i_{p+1}, \dots, i_n)$, where $\mathbf{wt}(R) = (i_1, \dots, i_n)$.

Proof. Consider first the special case that $R_{>p}$ is empty. Let $w_0 = w_{\mathcal{Z}(\overline{R})}$. Then we know that

$$\overline{R} \xrightarrow{p} \mathcal{Z}(\overline{R})$$

Since every element of A_k is less than p and $d_k > p$, the insertion $A_k \xrightarrow{d_k-1} \mathcal{Z}(\overline{R})$ only adds right roots (a, b) with $a < p$ and $b > p$. Thus

$$A_k \xrightarrow{d_k} \overline{R} \xrightarrow{p} A_k \xrightarrow{d_k-1} \mathcal{Z}(\overline{R})$$

We assert the (descending) inductive hypothesis that for some index j we have

$$A_j \xrightarrow{d_j} \dots A_k \xrightarrow{d_k} \overline{R} \xrightarrow{p} A_j \xrightarrow{d_j-1} \dots A_k \xrightarrow{d_k-1} \mathcal{Z}(\overline{R})$$

Then the induction step follows with precisely the same reasoning; since every element of A_{j-1} is less than p and $d_{j-1} > p$, the insertion $A_{j-1} \xrightarrow{d_{j-1}-1}$ only adds right roots (a, b) with $a < p$ and $b > p$. Thus

$$A_{j-1} \xrightarrow{d_{j-1}} \dots A_k \xrightarrow{d_k} \overline{R} \xrightarrow{p} A_{j-1} \xrightarrow{d_{j-1}-1} \dots A_k \xrightarrow{d_k-1} \mathcal{Z}(\overline{R})$$

and the result follows by induction in the special case that $R_{>p}$ is empty, where we invoke Lemma 3.0.3 to assert that the left-hand side is equal to R .

Since the row was initially empty, we have at our disposal the exchange property on $R_{>p}$, shifting the reflections down a row to obtain the general case. $\square$

Definition 3.0.4. Let $\text{lift} : \mathcal{BRC}_n \rightarrow \mathcal{M}_n$ be the lift map, and let $\mathcal{L} < \mathcal{M}_n$ be the subring generated by the image. Let $Z \leq \mathcal{L}$ be the ideal that is the kernel of $\mathcal{RC} : \mathcal{L} \rightarrow \mathcal{BRC}_n$. Then Z is the two sided ideal generated by the elements

$$\{D - \text{lift}(\mathcal{RC}(D)) \mid D \in \mathcal{L}\}$$

for all $D \in \mathcal{L}$. Consider the two-sided ideal $\overline{Z} \leq \mathcal{M}_n$ generated by the elements

$$R - \text{lift}(\mathcal{RC}(R))$$

for all $R \in \mathcal{BRC}_n$.

[1]

REFERENCES

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