

A LITTLEWOOD–RICHARDSON RULE FOR FOREST POLYNOMIALS VIA THE SCHUBERT BIALGEBRA

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ABSTRACT. The forest polynomials $\mathfrak{P}_a$ of Nadeau–Tewari form a $\mathbb{Z}$ -basis of $\mathbb{Z}[x_1, x_2, \dots]$ whose role for the cohomology of the quasisymmetric flag variety parallels that of Schubert polynomials for the classical flag variety. Nonnegativity of the structure constants $\beta_{a,b}^c$ in $\mathfrak{P}_a \mathfrak{P}_b = \sum_c \beta_{a,b}^c \mathfrak{P}_c$ is known, but no Littlewood–Richardson-style enumerative rule has been available. We give such a rule: $\beta_{a,b}^c$ counts pairs of forest RC graphs of forest-codes a and b whose lift product lands on a forest RC graph of forest-code and weight both equal to c . The same rule descends to the cup product on $H^\bullet(\mathrm{QFl}_n)$. The proof introduces a *Schubert bialgebra* $\mathcal{A}$ and lifts the multiplication on its graded dual $\mathcal{D}$ to a product on a free abelian group $\mathcal{BRC}$ of bounded RC graphs; the same machinery yields enumerative LR rules for the dual Schubert, dual key, dual forest, and dual slide bases of $\mathcal{D}$.

1. INTRODUCTION

The forest polynomial Littlewood–Richardson rule. The *forest polynomials* $\mathfrak{P}_a$ of Nadeau and Tewari [31], indexed by weak compositions a , form a $\mathbb{Z}$ -basis of the polynomial ring whose role for the cohomology of the quasisymmetric flag variety QFl_n [6] parallels that of Schubert polynomials for the classical flag variety Fl_n . The structure constants $\beta_{a,b}^c$ of multiplication of forest polynomials,

$$\mathfrak{P}_a \mathfrak{P}_b = \sum_c \beta_{a,b}^c \mathfrak{P}_c,$$

are nonnegative integers; this was first established by Nadeau and Tewari [31] via the forest equivalence class structure on monomials, and was given a constructive straightening algorithm by Bergeron–Gagnon–Nadeau–Spink–Tewari [5]. A combinatorial Littlewood–Richardson-style rule realizing each $\beta_{a,b}^c$ as the cardinality of an explicit finite set — in the spirit of the classical LR rule for Schur functions — has been raised as a natural question in this circle of work [31, 30, 6], but no such rule has been available.

The combinatorial centerpiece of this paper is such a rule. The forest-code map $\mathfrak{c}_{\mathfrak{P}}$ partitions $\mathcal{RC}_n$ into Nadeau–Tewari forest equivalence classes whose generating polynomials are the $\mathfrak{P}_a$. Within each such class we single out a distinguished representative: call an RC graph R a *forest RC graph* if its forest-code equals its weight, $\mathfrak{c}_{\mathfrak{P}}(R) = \mathbf{wt}(R)$. Each forest equivalence class contains a unique forest RC graph, but — in contrast to the rigid correspondence between RC graphs and Schubert polynomials — several distinct forest equivalence classes can share the same forest code, so a given weak composition a may be realized by more than one forest RC graph. This nonuniqueness is genuine and visible in computations with the rule below: a forest LR coefficient of 2 already arises from two distinct forest RC graphs of the same composition c (see Example 8.4.6).

Theorem 1.1 (Forest polynomial LR rule). *Let a, b be weak compositions of length n . Then*

$$\mathfrak{P}_a \mathfrak{P}_b = \sum_c \beta_{a,b}^c \mathfrak{P}_c,$$

where $\beta_{a,b}^c$ is the number of pairs (A, B) of RC graphs such that $\mathfrak{c}(w_A) = \mathfrak{c}_{\mathfrak{P}}(A) = a$, $\mathfrak{c}(w_B) = \mathfrak{c}_{\mathfrak{P}}(B) = b$, and $A * B = R$ for some forest RC graph R with $\mathfrak{c}_{\mathfrak{P}}(R) = \mathbf{wt}(R) = c$.

Here $A * B$ is the *lift product* (§8.2), a binary operation on RC graphs that we introduce for this rule.

Under the Borel-type surjection $\mathcal{A}_n \rightarrow \mathcal{A}_n / \mathrm{QSym}_n^+ \cdot \mathcal{A}_n \cong H^\bullet(\mathrm{QFl}_n)$ of [6], the forest polynomials descend to a basis of $H^\bullet(\mathrm{QFl}_n)$, so the same rule computes the cup product there.

The proof requires substantial machinery, which we develop in its own right because it is of independent interest. Beyond the above, we also define the “squash product” $\boxtimes$, which gives rise to a multiplactic algebra $\mathcal{M}_n$, which is a tensor product of plactic algebras that correctly models Schubert polynomial multiplication.

In the process of proving the rule we invoke the quasi-crystal framework of Cain–Guilherme–Malheiro [11] as it applies to RC graphs.

The Schubert bialgebra and its dual. In initial sections, we introduce a bialgebra $\mathcal{A}$ that we call the Schubert bialgebra. This bialgebra has a basis of Schubert polynomials $\mathfrak{S}_u^n$ for $u \in S_\infty$, where the last descent of u is not past index n . The structure constants for multiplication in this basis are the same as the structure constants for multiplication of Schubert polynomials. The difference from the usual realm of study of Schubert polynomials, which is the polynomial ring in infinitely many variables, is that this algebra is graded into length components that interact trivially. That is,

$$\mathcal{A} = \bigoplus_{n=0}^{\infty} \mathbb{Z}[x_1, \dots, x_n].$$

This allows us to consistently define a “splitting” coproduct that is compatible with the usual product of polynomials but actively requires the triviality of products between length components to achieve a bialgebra structure.

We focus primarily on the dual $\mathcal{D}$ of this algebra, where the product is derived from this splitting (deconcatenation) coproduct on $\mathcal{A}$. Specifically, let Comp_n be the set of weak compositions of length n . Then, additively,

$$\mathcal{D} = \bigoplus_{n=0}^{\infty} \mathbb{Z}^{\oplus \text{Comp}_n}$$

and multiplicatively the product of two basis elements indexed by $a \in \text{Comp}_p$ and $b \in \text{Comp}_q$ is the basis element indexed by the concatenation of a and b , which is an element of Comp_{p+q} . We will denote this product by $\sqcup$, writing $a \sqcup b$ for the concatenation, and we wish to flag the choice of symbol: the product on $\mathcal{D}$ is *not* the dual of polynomial multiplication on $\mathcal{A}$ in any sense one might naively expect, and using a distinguished symbol for it serves as a constant reminder. The coproduct on $\mathcal{D}$ is given by splitting sums of exponents, that is to say

$$\Delta(c) = \sum_{a+b=c} a \otimes b$$

where a and b are weak compositions of lengths p and q respectively, and the equation $a+b=c$ indicates that the pointwise sum of a and b is equal to c . The counit is given by $\varepsilon(c) = 0$ unless c is the empty composition, in which case $\varepsilon(c) = 1$. With these definitions, $\mathcal{A}$ and $\mathcal{D}$ are dual bialgebras over $\mathbb{Z}$, a fact that is straightforward to verify. Neither $\mathcal{A}$ nor $\mathcal{D}$ is connected, and neither is a Hopf algebra. A “Littlewood–Richardson rule” for a basis of the dual algebra is equivalent to a branching/splitting formula for the corresponding polynomial basis under the partition of variables into two blocks $\{x_1, \dots, x_p\}$ and $\{x_{p+1}, \dots, x_{p+q}\}$.

The algebraic structure of $\mathcal{D}$ is not itself new; what is new is the product on the dual Schubert basis Ξ_u^n of $\mathcal{D}$ indexed by $u \in S_\infty$ and n at least as large as the last descent of u , defined by

$$\langle \Xi_u^m, \mathfrak{S}_v^n \rangle = \delta_{u,v} \delta_{m,n}$$

where we take $\langle -, - \rangle$ to be the pairing where for each composition $c \in \text{Comp}_n$,

$$\langle c, x^a \rangle = \delta_{c,a}$$

where a is any composition. (It is worth pointing out that c and c padded with an additional 0 are considered different here.) We present an explicit formula for the dual Schubert basis in $\mathcal{D}$ with integer coefficients.

A dual basis to the Schubert polynomials is not new either: Postnikov and Stanley [34] introduced a family of dual Schubert polynomials $\mathfrak{D}_w$ in the polynomial ring with respect to a particular integral pairing, characterized by a formula that mimics the action of divided difference operators on Schubert polynomials. The relationship between the two bases is that the coefficients of $\mathfrak{D}_u$ in the divided power algebra are, up to a scalar factor, the same as the coefficients of Ξ_u^n in the weak composition basis.

In addition, the dual forest basis $\{\mathfrak{P}_a^*\}$ of $\mathcal{D}$ is defined by $\langle \mathfrak{P}_a^*, \mathfrak{P}_b \rangle = \delta_{a,b}$. As we record in Proposition 6.2.2, this dual basis coincides, up to a divided-power normalization, with the volume polynomials of Nadeau–Spink–Tewari [30] and, geometrically, with the Kronecker dual of the homology basis $\{[X(F)]\} \subset H_\bullet(\text{QFl}_n)$ of Bergeron–Gagnon–Nadeau–Spink–Tewari [6]. We also consider the dual bases of key polynomials and slide polynomials.

Littlewood–Richardson rules for dual bases. Each of these dual bases in fact has nonnegative structure constants with $\sqcup$, for which we also give rules in terms of RC graphs. For an RC graph R with n rows and an integer $0 \leq p \leq n$, we associate to R two RC graphs $\text{clip}^p(R) \in \mathcal{RC}_p$ and $\text{trim}^p(R) \in \mathcal{RC}_{n-p}$, which the reader should picture as a decomposition of R across the horizontal cut between rows p and $p+1$. The map trim^p is genuinely simple: it discards the top p rows and keeps the remaining $n-p$ rows (suitably reindexed). The map clip^p is the subtler partner. It is built by iterating an auxiliary map $\mathcal{Z} : \mathcal{RC}_n^0 \rightarrow \mathcal{RC}_{n-1}$ (Definition 4.3.1), where $\mathcal{RC}_n^0$ denotes the set of RC graphs whose last row is empty. The map $\mathcal{Z}$ realizes, on the level of RC graphs, the Lascoux–Schützenberger transition formula for Schubert polynomials (see [26]), and ends up being a generalization of the main result of [25], though this was not intentional. The whole construction preserves a wealth of additional structure (crystal operators, forest invariants) that we exploit later. Indeed, the product of $\mathcal{D}$ is lifted to a product $\sqcup$ on the free abelian group $\mathcal{BRC}$ of bounded RC graphs using clip^p and trim^p , for which $\mathcal{D}$ is a quotient (§4). The nonnegativity of the structure constants of all of these bases of $\mathcal{D}$ is a consequence of this quotient structure.

For the dual Schubert basis $\{\Xi_u^n\}$ for permutations u with last descent at most n , we have the following result.

Theorem 1.2 (LR rule for dual Schubert polynomials). *Suppose $u, v \in S_\infty$ and $p, q \geq 0$ are integers such that the last descent of u is at most p and the last descent of v is at most q . Define integers $d_{u,v}^w(p, q)$ by the equation*

$$\Xi_u^p \sqcup \Xi_v^q = \sum_w d_{u,v}^w(p, q) \Xi_w^{p+q}.$$

Then for any fixed choices of RC graphs $U \in \mathcal{RC}_p(u)$, $V \in \mathcal{RC}_q(v)$, and a permutation $w \in S_\infty$, we have that $d_{u,v}^w(p, q)$ is the number of RC graphs $R \in \mathcal{RC}_{p+q}(w)$ such that $\text{clip}^p(R) = U$ and $\text{trim}^p(R) = V$, and is in particular nonnegative.

Cast as a splitting rule for $\mathfrak{S}_w$, Theorem 1.2 fits the framework of *branching Schubert calculus* of Ressayre–Richmond [36] for the Levi inclusion $\text{GL}_p \times \text{GL}_q \hookrightarrow \text{GL}_{p+q}$ (with $P = B$): the Ressayre–Richmond branching coefficients $d_w^{\bar{w}}$ in this case are exactly our $d_{u,v}^w(p, q)$, and Theorem 1.2 supplies for them an explicit positive cardinality formula. The associativity of $\sqcup$ encodes the compatibility of these splittings across all $p, q \geq 0$ simultaneously, packaging the entire family of Levi comorphisms into the single algebra $(\mathcal{D}, \sqcup)$.

The dual forest basis has nonnegative structure constants for the same structural reason. The coefficient $f_{a,b}^c$ in $\mathfrak{P}_a^* \sqcup \mathfrak{P}_b^* = \sum_c f_{a,b}^c \mathfrak{P}_c^*$ counts, within the forest equivalence class of composition c , the RC graphs whose row-splitting lands in forest-classes of compositions a and b ; the count is well-defined on equivalence classes and is nonnegative (Theorem 6.4).

The key polynomials κ_a (Demazure characters in type A) form a $\mathbb{Z}$ -basis of $\mathbb{Z}[x_1, x_2, \dots]$ indexed by weak compositions; we defer definitions to §5. The dual key basis $\{\kappa_a^*\}$ of $\mathcal{D}$ is defined by $\langle \kappa_a^*, \kappa_b \rangle = \delta_{a,b}$; its structure as a bialgebra basis appears to be new. The dual key rule (Theorem 5.4) has the same shape as the dual Schubert and dual forest rules but replaces forest equivalence classes with Demazure crystal components: $k_{a,b}^c$ counts, within the crystal component of highest weight c , those RC graphs whose row-splitting produces highest-weight pieces of compositions a and b . We later found that nonnegativity of the $k_{a,b}^c$ also follows from Mathieu’s existence theorem for excellent filtrations on tensor products of Demazure modules [28]; an explicit positive combinatorial rule had not previously appeared.

In a similar vein, the dual slide basis $\{\mathfrak{F}_a^*\}$ of $\mathcal{D}$ is defined by $\langle \mathfrak{F}_a^*, \mathfrak{F}_b \rangle = \delta_{a,b}$, where $\mathfrak{F}_b$ is the slide polynomial of Assaf–Searles [2]. We give a Littlewood–Richardson rule for this basis as well.

These formulas bear a striking similarity to each other, which is no coincidence: they are all consequences of the same underlying machinery, and in particular of the same maps clip^p and trim^p on RC graphs. A general “dual LR rule factory” is given in Theorem 7.3, which gives a recipe for producing such rules whenever one has a family of polynomials with a positive combinatorial formula in terms of RC graphs that is compatible with the maps clip^p and trim^p . The dual Schubert, dual key, dual forest, and dual slide bases all fit into this framework, and thus yield the four rules above as special cases.

Further directions. Beyond this, there is potentially much to explore. For example, the coproduct of $\mathcal{D}$ has as its structure constants the structure constants of Schubert polynomials, which have evaded expression in terms of positive formulas for decades, and at the time of this writing no positive, combinatorial formula is known for the structure constants of Schubert polynomials that applies in all cases. At the very least, this

coproduct is a new potential method of attack on this and related problems, since, to our knowledge, it was previously unknown. An expression for Ξ_w^n in the composition basis leads to a simple, purely mechanical method for computing all $c_{u,v}^w$ (generalized Littlewood-Richardson coefficients for Schubert polynomials) by leveraging the extremely simple form the coproduct takes on the weak composition basis, as well as a neat positive formula for the expansion of a weak composition $[a]$ in the Schubert basis.

For the reader familiar with bumpless pipe dreams [22], we state as a conjecture that our apparently complex transition algorithm for RC graphs can be realized almost trivially for bumpless pipe dreams. Specifically, while (reduced) bumpless pipe dreams are typically defined in such a way that they cannot deviate from a square shape, there is indeed a consistent way to remove rows from a bumpless pipe dream unambiguously. This was observed by Yu in [44, Lemma 3.18] in the definition of a k -row BPD. Weigandt in [42] defines a transition formula $\mathbf{t} : \text{BPD}_n \rightarrow \text{BPD}_{n-1}$, and while the image is realized as square, it agrees with Yu's notion of an $n - 1$ -row BPD.

Gao and Huang in [15] defined a natural bijection between RC graphs and bumpless pipe dreams: given a bumpless pipe dream B with n rows, we define $\phi(B)$ to be the corresponding RC graph under this bijection.

Conjecture 1.3. *Let B be a k -row reduced bumpless pipe dream such that row k has no blanks (“weighty tiles”). Then*

$$\mathcal{Z}(\phi(B)) = \phi(\mathbf{t}(B))$$

This simplicity does not come for free, however; the $\mathbf{trim}^p$ operation on BPDs is no longer trivial.

Outline of the paper. Sections 2 and 3 establish the algebraic foundations: §2 (The Schubert Bialgebra and its dual) constructs $\mathcal{A}$, identifies its Schubert basis $\mathfrak{S}_u^n$, and exhibits the elementary basis $\mathcal{E}_n$ of $\mathcal{A}_n$; §3 (The dual algebra) builds the graded dual $\mathcal{D}$, identifies the dual Schubert basis Ξ_u^n , and records the coproduct.

Section 4 (The ring of bounded RC graphs) defines the module $\mathcal{BRC}$ of bounded RC graphs, the row-decomposition maps $\mathcal{Z}$, $\mathbf{clip}^p$, $\mathbf{trim}^p$, and the associative product $\boxplus$ on $\mathcal{BRC}$; this culminates in Theorem 1.2, the Littlewood–Richardson rule for the dual Schubert basis.

Sections 5–7 produce three further LR rules by quotienting $\mathcal{BRC}$ by successively coarser equivalence relations. Section 5 (The ring of RC graph crystals) imports the Demazure crystal structure on RC graphs from [1] and identifies an ideal $\Theta \subset \mathcal{BRC}$ whose quotient yields Theorem 5.4, the LR rule for the dual key basis. Section 6 (The ring of forest classes of RC graphs) does the analogous construction for the forest invariant of [31], producing an ideal $\Gamma \subset \mathcal{BRC}$ whose quotient yields Theorem 6.4, the LR rule for the dual forest basis. Section 7 (Product rule for dual slide polynomials) introduces the quasi-Yamanouchi map $\mathbb{QY}$ and the corresponding ideal $\Sigma \subset \mathcal{BRC}$, yielding Theorem 7.2, a positive product rule for the dual slide basis; the section closes with Theorem 7.3, a general ideal-quotient Littlewood–Richardson template that unifies all four rules under a single set of axioms on an equivalence relation $\sim$ on $\mathcal{BRC}$.

Section 8 (The Littlewood–Richardson rule for forest polynomials) proves what we regard as the main theorem of the paper: Theorem 1.1, a positive, manifestly combinatorial Littlewood–Richardson rule for the forest polynomials *themselves* (not merely their duals). The proof leverages the squash product $\boxtimes$, an auxiliary lift product $*$ on a multiplactic algebra $\mathcal{M}_n$ of elementary RC factors, and the slide quotient of Section 7.

The appendix is devoted to the proof of Proposition 2.1.5, which would otherwise be distracting.

2. THE SCHUBERT BIALGEBRA AND ITS DUAL

2.1. Some preliminaries.

Definition 2.1.1 (Two Pieri relations $\xrightarrow{k}$ and $\xRightarrow{k}$). The relations $\xrightarrow{k}$ and $\xRightarrow{k}$ are defined on S_∞ as follows, with different notation, in [40]. For $v, w \in S_\infty$, we have $v \xrightarrow{k} w$ if there is a sequence of transpositions $t_{a_1 b_1}, \dots, t_{a_m b_m}$ such that the following hold:

- (1) $a_i \leq k < b_i$ for all i .
- (2) $\ell(ut_{a_1 b_1} \cdots t_{a_i b_i}) = \ell(u) + i$ for all i .
- (3) $w = ut_{a_1 b_1} \cdots t_{a_m b_m}$.
- (4) The integers $a_1, \dots, a_m$ are pairwise distinct.

For $\xRightarrow{k}$, only (4) changes:

- (4') The integers $b_1, \dots, b_m$ are pairwise distinct.

Notice that necessarily $m \leq k$ for $\xrightarrow{k}$, but there is no upper bound on m for $\xRightarrow{k}$. This reflects the fact that the following theorem holds:

Theorem 2.1 ([40, Theorem 1]). *Let $p \geq 0$, $k > 0$ be integers and let u be a permutation.*

I. *We have the formula*

$$\mathfrak{S}_u(x) h_{p,k}(x) = \sum_{\substack{u \xrightarrow{k} w \\ \ell(u,w)=p}} \mathfrak{S}_w(x)$$

where $h_{p,k}(x)$ is the complete homogeneous symmetric polynomial of degree p in the first k variables.

II. *We have the formula*

$$\mathfrak{S}_u(x) e_{p,k}(x) = \sum_{\substack{u \xrightarrow{k} w \\ \ell(u,w)=p}} \mathfrak{S}_w(x)$$

where $e_{p,k}(x)$ is the elementary symmetric polynomial of degree p in the first k variables.

Definition 2.1.2 (Lehmer code, dual code, dominant permutations). For a permutation $w \in S_\infty$, the (right) *Lehmer code* of w is the sequence $\mathfrak{c}(w) = (c_1, c_2, \dots)$ defined by

$$c_i = \#\{j > i : w(j) < w(i)\}.$$

This is a weak composition with $\sum_i c_i = \ell(w)$, and the map $w \mapsto \mathfrak{c}(w)$ is a bijection from S_∞ onto the set of weak compositions of finite support. We will sometimes write $\mathfrak{c}_i(w) := c_i$. The *dual code* of w is

$$\mathfrak{c}^*(w) := \mathfrak{c}(w^{-1}).$$

A permutation $\mu \in S_\infty$ is called *dominant* if $\mathfrak{c}(\mu)$ is a partition (a weakly decreasing sequence of nonnegative integers, padded by 0s). For a partition λ , we write $\mathfrak{d}_\lambda$ for the dominant permutation with

$$\mathfrak{c}^*(\mathfrak{d}_\lambda) = \lambda.$$

Definition 2.1.3 (Pulling out variables from double Schubert polynomials, the relation $\xrightarrow{i}$). For a permutation $v' \in S_n$, define

$$\varphi_{i,n}(v')(j) = \begin{cases} v'(j) & \text{if } j < i \\ n+1 & \text{if } j = i \\ v'(j-1) & \text{if } i < j \leq n+1 \\ j & \text{if } j > n+1 \end{cases}$$

Now fix $v \in S_n$ and $i \geq 1$ an integer, we define a relation $\xrightarrow{i}$ by declaring that $v \xrightarrow{i} v'$ if

$$v \xRightarrow{i} \varphi_{i,n}(v')$$

Equivalently, $v \searrow^i v'$ if whenever $v \in S_n$ and n is minimal, we have that v satisfies the relation $\xRightarrow{i}$ with respect to the permutation obtained from v' by inserting $n+1$ at position i . We note that this concept was introduced by Bergeron and Sottile in [8].

If $v \searrow^i v'$, we define a set of integers $Q_i(v', v)$ by

$$Q_i(v', v) = \{v(j) \mid j > i \text{ and } v'(j-1) = v(j)\}$$

Given the fact that any element of S_∞ fixes all but finitely many positive integers, it follows that $Q_i(v', v)$ is a finite set.

The permutations v' such that $v \searrow^i v'$ arise when pulling a variable out of a Schubert polynomial and expressing the coefficients of the powers of this variable as Schubert polynomials in the remaining variables, as is done in [7], from which this definition essentially comes.

Definition 2.1.4 (Upward shifting of permutations, maximum right descent). For a permutation $v \in S_\infty$, we define $\uparrow v$ to be the permutation defined by

$$\uparrow v(i) = \begin{cases} v(i-1) + 1 & \text{if } i > 1 \\ 1 & \text{if } i = 1 \end{cases}$$

Recursively, we define $\uparrow^k(v)$ to be $\uparrow(\uparrow^{k-1}(v))$. In the literature, this is sometimes written as

$$\uparrow^k v = 1^k \times v$$

For a permutation w , we define $\mathfrak{m}(w)$ to be the maximum right descent of w . That is

$$\mathfrak{m}(w) = \max\{i \mid w(i) > w(i+1)\}$$

The next proposition specializes, in the case of ordinary Schubert polynomial $\mathfrak{S}_v(x)$, to a formula that isolates a chosen index i : one can express $\mathfrak{S}_v(x)$ as a sum of terms of the form $x_i^p \mathfrak{S}_{v'}(x^{(i)})$, thereby effectively extracting the variable x_i and leaving Schubert polynomials in the remaining variables [8, Theorem 5.1]. Proposition 2.1.5 is the double Schubert polynomial version of this, which, as far as we know, is new.

Proposition 2.1.5. *Let $v \in S_\infty$ and let $i > 0$ be an integer. Then we have*

$$\mathfrak{S}_v(x; y) = \sum_{v \searrow^i v'} \mathfrak{S}_{v'}(x^{(i)}; y) \prod_{b \in Q_i(v', v)} (x_i - y_b)$$

See the Appendix for a proof.

2.2. Definition. We define a commutative algebra $\mathcal{A}$ over the integers as follows. For each n , define $\mathcal{A}_n$ to be the polynomial ring over $\mathbb{Z}$ in the variables $x_1, \dots, x_n$. Then define

$$\mathcal{A} = \bigoplus_{n=0}^{\infty} \mathcal{A}_n$$

The multiplication within $\mathcal{A}_n$ is as usually defined for the polynomial ring. However, if $a \in \mathcal{A}_m$ and $b \in \mathcal{A}_n$ with $m \neq n$ and $m, n > 0$, then

$$ab = 0$$

Otherwise, the component for $n = 0$ is identified with the coefficient ring.

Remark (Fat identities). The constant polynomial $1 \in \mathcal{A}_n$ for $n > 0$ acts as a unit on $\mathcal{A}_n$ itself but is *not* a unit of $\mathcal{A}$ as a whole: by the rule above it annihilates every $\mathcal{A}_m$ with $m \neq n$ and $m > 0$. We will refer to these length- n units informally as “fat identities”. This is the conceptual hinge that makes $\mathcal{A}$ a bialgebra rather than a Hopf algebra. The triviality of cross-length products is precisely what allows the deconcatenation coproduct Δ defined below to be a homomorphism of rings; conversely, it forces $\mathcal{A}$ to fail to be connected, so $\mathcal{A}$ admits no antipode. Every theorem in this article should be read in light of this length-grading: the length component never mixes under multiplication, only under the coproduct.

Each $\mathcal{A}_n$ has a basis consisting of elements x_a , where a is a weak composition of length n , and the notation indicates that

$$x_a = x_1^{a_1} \cdots x_n^{a_n}$$

The direct sum therefore has a basis that can canonically be identified with the union of these.

We define a coproduct $\Delta : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ on the basis $x_c^{(n)}$ by

$$\Delta(x_c) = \sum_{ab=c} x_a \otimes x_b$$

where the equation $ab = c$ indicates that a concatenated with b is equal to c . We also define a counit $\varepsilon : \mathcal{A} \rightarrow \mathbb{Z}$ by $\varepsilon(x_a) = 0$ unless a is the empty composition.

Lemma 2.2.1. *With Δ and ε , $\mathcal{A}$ is a coassociative, counital coalgebra.*

Proof. We have

$$\Delta(x_d^{(n)}) = \sum x_a^{(p)} \otimes x_c^{(q)}$$

Applying Δ to either tensor factor results in

$$\sum x_a^{(p)} \otimes x_b^{(q)} \otimes x_c^{(r)}$$

The symmetry of this is exactly the coassociativity condition. Seeing that we may choose $p = n$ or $q = n$, the definition of the counit gives us the result that $\mathcal{A}$ is counital as well under Δ and ε . $\square$

Lemma 2.2.2. *$\Delta : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ is a homomorphism of rings.*

Proof. This is where the condition that $x_a^{(p)} x_b^{(q)} = 0$ unless $p = q$ when both $p, q > 0$ comes in. It ensures that only monomials of the same length have nonzero products and preserves the structure of the coproduct as a homomorphism of rings. $\square$

Lemma 2.2.3. *$\nabla : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$ is a homomorphism of coalgebras.*

Proof. Consider the basis element $x_a \otimes x_b$ of $\mathcal{A} \otimes \mathcal{A}$. The product is

$$x_{a+b}$$

Applying Δ results in

$$\sum_{a'b'=a+b} x_{a'} \otimes x_{b'}$$

Conversely, applying $\Delta \otimes \Delta$ yields

$$\sum_{a_1 a_2 = a, b_1 b_2 = b} x_{a_1} \otimes x_{a_2} \otimes x_{b_1} \otimes x_{b_2}$$

and applying $\nabla \otimes \nabla$ yields

$$\sum x_{a_1+b_1} \otimes x_{a_2+b_2}$$

The definition of ∇ ensures that a_1 and b_1 are the same length for a nonzero term, and similarly for a_2 and b_2 . Setting $a_1 + b_1 = a'$, $a_2 + b_2 = b'$ yields the result. $\square$

Corollary 2.2.4. *$\mathcal{A}$ is a bialgebra over $\mathbb{Z}$.*

$\mathcal{A}$ is afforded a grading into finite dimensional components by observing that each $\mathcal{A}_n$ itself is a graded ring with each homogeneous component being a finitely generated free module. Considering the pair (n, d) , where n is the number of variables and d is the degree, as a $\mathbb{Z}^2$ grading, we may take the graded dual module $\mathcal{D}$, which, by virtue of the grading, is isomorphic as a free module to $\mathcal{A}$. We identify the dual basis element x_a^* with the sequence of nonnegative integers a . That is,

$$\langle a, x_b \rangle = \delta_{ab}.$$

Inspection of the coproduct reveals that the product of a and b in $\mathcal{D}$ is simply the concatenation ab . Hence $\mathcal{D}$ is isomorphic to the free associative algebra on a countable set indexed by nonnegative integers.

$\mathcal{D}$ also has a coproduct compatible with its product, namely

$$c \mapsto \sum_{a+b=c} a \otimes b$$

Thus $\mathcal{A}$ and $\mathcal{D}$ are dual bialgebras.

Calling $\mathcal{A}$ the “Schubert bialgebra” may seem unnecessarily grandiose; however, the reason will become clear below.

2.3. The Schubert basis. In $\mathcal{A}$, each $\mathcal{A}_n$ has a basis of Schubert polynomials $\mathfrak{S}_u^{(n)}$, where the largest right descent of u is at most n , ensuring that $\mathfrak{S}_u(x)$ has at most n variables. Schubert polynomials limited to a specific number of variables have well-defined structure constants $c_{u,v}^w$, independent of n , given by

$$\mathfrak{S}_u^{(n)} \mathfrak{S}_v^{(n)} = \sum_w c_{u,v}^w \mathfrak{S}_w^{(n)}$$

These are known to be nonnegative integers by virtue of the fact that they count points of transverse intersections of generic translates of Schubert varieties; however, except in special cases no positive combinatorial formula is known.

2.4. The elementary basis. An n -elementary weak composition is a finite sequence of nonnegative integers $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)$ such that $\alpha_i \leq i$ for all $i < n$ and $\alpha_i \leq n$ for all $i \geq n$, satisfying the additional restriction that $\alpha_i \geq \alpha_{i+1}$ if $i \geq n$.

An n -elementary monomial is an element of $\mathcal{A}_n$ of the following form:

$$e_{\alpha_1}^1 \cdots e_{\alpha_n}^n e_{\alpha_{n+1}}^n \cdots e_{\alpha_m}^n$$

where α is an n -elementary weak composition. Define $\mathcal{E}_n$ to be the set of n -elementary monomials.

Proposition 2.4.1. $\mathcal{E}_n$ forms a $\mathbb{Z}$ -basis for $\mathcal{A}_n$, and $\bigcup_n \mathcal{E}_n$ forms a $\mathbb{Z}$ -basis for $\mathcal{A}$.

Proof. It is a theorem of Macdonald that the polynomial ring $\mathbb{Z}[x_1, \dots, x_n]$ is a free module over Λ_n with basis the Schubert polynomials $\mathfrak{S}_u(x)$ such that $u \in S_n$. These Schubert polynomials are uniquely expressed in terms of strict elementary symmetric monomials with fewer than n variables. Since Λ_n is a polynomial ring in the $e_{i,n}$ by the standard characterization of Λ_n as a polynomial ring in $e_1, \dots, e_n$, Λ_n has a basis of monomials e_λ^n as λ ranges over all partitions with parts bounded by n . The multiplicative combination of these two bases is therefore a $\mathbb{Z}$ -basis of $\mathcal{A}_n$ by freeness of the module. This is exactly the description of the monomials e_α^n for α an n -elementary weak composition. $\square$

We can ask for transition formulas between the Schubert and the elementary basis. From elementary to Schubert, we may use Sottile’s Pieri formula iteratively [40], which provides a positive formula for multiplying a Schubert polynomial by an elementary symmetric polynomial. That is,

$$e_\alpha^n = \sum_{1 \xrightarrow{1,2,\dots,n} \alpha u} \mathfrak{S}_u^n$$

For the reverse transition, we need a lemma. Recall the skew divided difference operators ∂_u^w for $u, w \in S_\infty$ defined originally in [26], though we use the notation of [37], by the Leibniz formula

$$\partial^w(pq) = \sum_u \partial^u(p) \partial_u^w(q)$$

where p and q are arbitrary polynomials and u is a permutation such that $u \leq w$ in Bruhat order. The formula below provides a formula for application of a skew divided difference operator of degree $-n$ to a polynomial of degree n , which yields an integer for integer-coefficient polynomials. Specifically, it is a formula for application of such an operator with $\ell(u, w) = n$ specifically to x_p^n for some $p > 0$.

Definition 2.4.2. For $u, w \in S_\infty$ such that $u \leq w$, we define $\mathcal{C}^{(p)}(u, w)$ to be the set of saturated chains C in the Bruhat interval $[u, w]$ of the form

$$u = u_0 \prec u_1 \prec \cdots \prec u_n = w$$

such that $u_{i-1}(p) \neq u_i(p)$ for all i . We also define $\sigma(C)$ to be the number of indices i for which the index $j \neq p$ such that $u_{i-1}(j) \neq u_i(j)$ satisfies $j < p$.

Lemma 2.4.3. *Let $n \geq 0$ be an integer, let $p > 0$ be an integer, and let $u, w \in S_\infty$ be permutations such that $\ell(u, w) = n$. Then we have that*

$$\partial_u^w(x_p^n) = \sum_{C \in \mathcal{C}^{(p)}(u, w)} (-1)^{\sigma(C)}.$$

Proof. Consider the case where $\ell(u, w) = 1$. Then there is a transposition t_{ij} such that $w = ut_{ij}$. Assume that $i < j$. If $p \notin \{i, j\}$, then $\partial_u^w(x_p) = 0$. If $p = i$, then $\partial_u^w(x_p) = \partial^{ij}(x_p) = 1$. If instead $j = p$, then $\partial_u^w(x_p) = \partial^{ij}(x_p) = -1$. In all cases, the result holds.

Now, consider the case where $\ell(u, w) = n > 1$. We have that

$$x_p^n = x_p \cdot x_p^{n-1}$$

Applying the Leibniz formula, we have

$$\partial_u^w(x_p^n) = \sum_{u \prec u'} \partial_u^{u'}(x_p) \partial_{u'}^w(x_p^{n-1})$$

By the above, $\partial_u^{u'}(x_p)$ is nonzero if and only if $u \prec u'$ is a cover in the Bruhat order such that $u(p) \neq u'(p)$. In this case, $\partial_u^{u'}(x_p)$ is either 1 or -1 depending on the number of indices $j < p$ such that $u(j) \neq u'(j)$, which is either 0 or 1, respectively. By induction,

$$\partial_{u'}^w(x_p^{n-1}) = \sum_{C' \in \mathcal{C}^{(p)}(u', w)} (-1)^{\sigma(C')}.$$

Hence,

$$\partial_u^{u'}(x_p) \partial_{u'}^w(x_p^{n-1}) = \begin{cases} - \sum_{C' \in \mathcal{C}^{(p)}(u', w)} (-1)^{\sigma(C')} & \text{if } u(j) \neq u'(j) \text{ for some } j < p \\ \sum_{C' \in \mathcal{C}^{(p)}(u', w)} (-1)^{\sigma(C')} & \text{otherwise,} \end{cases}$$

which, by definition, is

$$\sum_{C = C' \cup \{u\}, C' \in \mathcal{C}^{(p)}(u', w)} (-1)^{\sigma(C)}$$

Summing over all u' such that $u \prec u' \leq w$, we obtain the result. $\square$

Using a minor sleight of hand, this affords us an explicit formula for transition of a Schubert polynomial into the elementary basis.

Definition 2.4.4. Let δ^u be the divided difference operator associated to u on the y variables for a given $u \in S_\infty$, and let

$$E_k(x; y_p) = \prod_{i=1}^k (x_i - y_p)$$

which is a factorial elementary symmetric polynomial, a special case of a double Schubert polynomial.

Lemma 2.4.5. *Let $u, w \in S_\infty$ satisfy $u \leq w$ and let $p > 0, k > 0$ be integers. Then we have*

$$\partial_u^w(E_k(x; -y_p)) = \sum_{C \in \mathcal{C}^{(p)}(u, w)} (-1)^{\sigma(C)} e_{k-\ell(u, w)}^k + \mathcal{O}(y_p)$$

Proof. This follows from the observation that

$$E_k(x; -y_p) = \sum_{i=0}^k e_{k-i}^k y_p^i$$

and Lemma 2.4.3. $\square$

Definition 2.4.6. Suppose we want to express $\mathfrak{S}_w^n$ in the elementary basis. Let w_λ be a dominant permutation such that

$$\mathbf{c}^*(w_\lambda) = \lambda = (n, n, \dots, n, n-1, \dots, 1)$$

and consider the double Schubert polynomial

$$\mathfrak{S}_{w_\lambda}(x; -y) = \prod_i E_{\lambda_i}(x; -y_i)$$

(this equation for $\mathfrak{S}_{w_\lambda}$ follows from [26, (6.14)]).

We have that

$$\mathfrak{S}_w(x; y) = \delta^{w\lambda^{-1}} \mathfrak{S}_{w_\lambda}(x; y)$$

We may apply the divided difference $\delta^{w\lambda^{-1}}$ to the product $\prod_i E_{\lambda_i}(x; -y_i)$ using the Leibniz formula and set $y_i = 0$ for all i , which gives us, by Lemma 2.4.5,

$$\sum_{\substack{P=(C_1, \dots, C_{\ell(\lambda)}) \\ C_i \in \mathcal{C}^{(i)}(u_{i-1}, u_i)}} (-1)^{\sigma(C_1) + \dots + \sigma(C_{\ell(\lambda)})} e_{\lambda_1 - |C_1|}^{\lambda_1} e_{\lambda_2 - |C_2|}^{\lambda_2} \cdots e_{\lambda_{\ell(\lambda)} - |C_{\ell(\lambda)}|}^{\lambda_{\ell(\lambda)}}$$

where $u_0 \leq u_1 \leq \dots \leq u_{\ell(\lambda)}$ is a saturated chain. Define, for convenience,

$$\text{Path}_\lambda(w) = \{P = (C_1, \dots, C_{\ell(\lambda)}) \mid C_i \in \mathcal{C}^{(i)}(u_{i-1}, u_i), u_0 = 1, u_{\ell(\lambda)} = w\lambda^{-1}\}$$

and for $P \in \text{Path}_\lambda(w)$ define an integer

$$\sigma(P) = \sum_{i=1}^{\ell(\lambda)} \sigma(C_i)$$

Furthermore, define a weak composition $\alpha(P)$ by

$$\alpha(P) = (\lambda_{\ell(\lambda)+1-i} - |C_{\ell(\lambda)+1-i}|)_{i=1}^{\ell(\lambda)}$$

Then we define

$$E_w^{\alpha; n} = \sum_{\substack{P \in \text{Path}_\lambda(w) \\ \alpha(P) = \alpha}} (-1)^{|\lambda| - \ell(w) + \sigma(P)}$$

Theorem 2.2 (Transition to the elementary basis). *For each w , we have*

$$\mathfrak{S}_w^n = \sum_{\alpha} E_w^{\alpha; n} e_{\alpha}^n$$

These numbers are stable for fixed w as soon as the number of variables is at least as large as the value of n such that $w \in S_n$, and are well-studied but poorly understood.

Example 2.4.7. Let $n = 5$ and let w be the permutation such that $\mathfrak{c}(w) = (0, 2, 0, 2, 3)$. Then

$$\begin{aligned} \mathfrak{S}_w^n &= e_{(0,0,0,0,5,1,1)} - e_{(0,0,0,0,4,2,1)} + e_{(0,0,0,0,4,3)} - e_{(0,1,0,0,4,1,1)} + e_{(0,0,1,0,3,2,1)} \\ &\quad + e_{(0,1,0,0,5,1)} + e_{(0,1,0,0,3,3)} \end{aligned}$$

Remark. This transition formula from the Schubert basis to the elementary basis can be used to obtain an extremely efficient formula for multiplication of Schubert polynomials. If we want to compute $\mathfrak{S}_u^n \mathfrak{S}_v^n$, we can first express $\mathfrak{S}_v^n$ in the elementary basis using Theorem 2.2. Then we can multiply by $\mathfrak{S}_u^n$ by iterating the Pieri formula. This is the method used in the python package `schubmult` written by the author [39]. In fact, there is also a version for double Schubert polynomials, quantum Schubert polynomials, and quantum double Schubert polynomials, which we will not go into here but are implemented in the same package.

3. THE DUAL ALGEBRA

We examine the graded dual algebra $\mathcal{D}$ more closely now. This is a graded ring generated by countably many elements that we denote by $[i]$, where i is a nonnegative integer. The product, as mentioned previously, is concatenation of sequences. Thus

$$[a_1 \cdots a_p][b_1 \cdots b_q] = [a_1 \cdots a_p b_1 \cdots b_q]$$

It is not hard to see that $\mathcal{D}$ is a free algebra on these generators. Thus the set of sequences $[a_1 \cdots a_n]$ forms a $\mathbb{Z}$ -basis for $\mathcal{D}$. We may realize this as the dual of $\mathcal{A}$ by declaring that

$$\langle [a_1 \cdots a_n], x_1^{b_1} \cdots x_n^{b_n} \rangle = \prod_i \delta_{a_i, b_i}$$

We have a $\mathbb{Z} \times \mathbb{Z}$ grading such that

$$\deg([a_1 \cdots a_n]) = (n, -a_1 - a_2 \cdots - a_n)$$

The set of elements such that $\deg(a) = (n, -)$ is $\mathcal{D}_n$, dual as a graded module to $\mathcal{A}_n$.

3.1. The dual Schubert basis. There is a basis Ξ_u^n dual to the Schubert basis for $\mathcal{D}_n$. Specifically, with the unique pairing $\langle -, - \rangle : \mathcal{D} \times \mathcal{A} \rightarrow \mathbb{Z}$ such that

$$\langle \alpha, x_\beta \rangle = \delta_{\alpha\beta}$$

we define Ξ_u^n to be the unique basis of $\mathcal{D}_n$ such that

$$\langle \Xi_u^n, \mathfrak{S}_v^n \rangle = \delta_{uv}$$

We characterize it with an explicit formula.

Definition 3.1.1. A permutation μ is said to be a *strict dominant permutation* if $\mathbf{c}(\mu)$ is a strict partition (that is, a strictly decreasing sequence of positive integers, padded at the end with 0s in our convention).

Theorem 3.1. For each permutation u and integer n with $\mathbf{m}(u) \leq n$ we have the equation

$$\Xi_u^n = \sum_{\ell(\alpha)=n} E_{u\mu^{-1}}^{\mathbf{c}(\mu)-\alpha; n} \alpha$$

where μ is any strict dominant permutation such that $0 \neq \mathbf{c}_n(\mu) \geq \ell(u)$ and $\mathbf{c}_{n+1}(\mu) = 0$.

Proof. By definition, the coefficient of α in Ξ_u^n is the coefficient of $\mathfrak{S}_u^n$ in x_α . The result can be derived from the Cauchy formula for double Schubert polynomials. Recall that for any permutation w , we have the formula

$$\mathfrak{S}_w(x; -y) = \sum_{u: \ell(u) + \ell(wu^{-1}) = \ell(w)} \mathfrak{S}_u(x) \mathfrak{S}_{wu^{-1}}(y)$$

Note that for any permutation μ as laid out in the statement of the theorem, $\ell(u\mu^{-1}) = \ell(\mu) - \ell(u)$. This is due to the lattice property of dominant permutations in weak order [10].

Thus,

$$\delta^{u\mu^{-1}} \mathfrak{S}_\mu(x; -y) = \mathfrak{S}_u(x; -y)$$

We have that

$$\mathfrak{S}_\mu(x; -y) = \sum_u \mathfrak{S}_u(x) \mathfrak{S}_{u\mu^{-1}}(y)$$

Expressing the second factor in the $e_{\alpha, \lambda}(y)$ basis, we have

$$\mathfrak{S}_\mu(x; -y) = \sum_{u, \alpha} \mathfrak{S}_u(x) E_{u\mu^{-1}}^{\mathbf{c}(\mu)-\alpha; n} e_{\mathbf{c}(\mu)-\alpha, \mathbf{c}(\mu)}(y)$$

An alternative expression for $\mathfrak{S}_\mu(x; -y)$ is

$$\mathfrak{S}_\mu(x; -y) = \sum_{\alpha} x_{\alpha} e_{\mathbf{c}(\mu)-\alpha, \mathbf{c}(\mu)}(y)$$

from which we see that the coefficient is correct. □

This is not stable for fixed u as n increases, and this is expected.

Example 3.1.2. Suppose $w = [1, 4, 5, 2, 3]$. Then

$$\Xi_w^3 = [022] - [013]$$

If we chose $[1, 3, 5, 7, 2, 4, 6]$ instead, we have

$$\Xi_{1357246} = [0015] - [0033] - [0114] + [0123]$$

For $w = [4, 2, 7, 1, 3, 5, 6]$,

$$\Xi_{4271356}^3 = -[026] + [035] + [125] - [134] + [206] - [215] - [305] + [314]$$

Thus the product structure of $\mathcal{D}$ encodes splitting of the Schubert polynomial into two sets of variables. In particular,

Lemma 3.1.3. *Let $a \geq 0$ be an integer and $w \in S_\infty$. Then we have*

$$[a] \cdot \Xi_w^n = \sum_{\substack{w' \xrightarrow{1} w \\ \ell(w, w')=a}} \Xi_{w'}^{n+1}$$

Proof. This follows from Proposition 2.1.5 with $i = 1$. $\square$

3.2. The coproduct. Consider the coproduct dual to the product on $\mathcal{A}$. We will denote it by

$$\Delta^* : \mathcal{D} \rightarrow \mathcal{D} \otimes \mathcal{D},$$

and we recall that it is characterized by the pairing identity

$$\langle \Delta^*(\alpha), x \otimes y \rangle = \langle \alpha, xy \rangle$$

for all $\alpha \in \mathcal{D}$ and $x, y \in \mathcal{A}$. Since the product on $\mathcal{A}$ preserves the length grading (multiplication of polynomials within a fixed $\mathcal{A}_n$, zero across $\mathcal{A}_p \otimes \mathcal{A}_q$ for $p \neq q$ with $p, q > 0$), Δ^* sends $\mathcal{D}_n$ into $\mathcal{D}_n \otimes \mathcal{D}_n$ for each n .

The Schubert structure constants now reappear as the coefficients of Δ^* in the dual Schubert basis.

Theorem 3.2 (Schubert LR coefficients as dual Schubert coproduct). *Let $w \in S_\infty$ with $\mathbf{m}(w) \leq n$. Then in $\mathcal{D}_n \otimes \mathcal{D}_n$,*

$$\Delta^*(\Xi_w^n) = \sum_{u,v} c_{u,v}^w \Xi_u^n \otimes \Xi_v^n,$$

where $c_{u,v}^w$ are the Schubert structure constants $\Xi_u^n \Xi_v^n = \sum_w c_{u,v}^w \Xi_w^n$, and the sum runs over pairs $u, v \in S_\infty$ with $\mathbf{m}(u), \mathbf{m}(v) \leq n$.

Proof. Expand $\Delta^*(\Xi_w^n) = \sum_{u,v} \lambda_{u,v} \Xi_u^n \otimes \Xi_v^n$ in the basis $\{\Xi_u^n \otimes \Xi_v^n\}$ of $\mathcal{D}_n \otimes \mathcal{D}_n$. Pairing with $\Xi_u^n \otimes \Xi_v^n$ and using the defining identity of Δ^* ,

$$\lambda_{u,v} = \langle \Delta^*(\Xi_w^n), \Xi_u^n \otimes \Xi_v^n \rangle = \langle \Xi_w^n, \Xi_u^n \Xi_v^n \rangle = \langle \Xi_w^n, \sum_z c_{u,v}^z \Xi_z^n \rangle = c_{u,v}^w.$$

$\square$

Remark. In particular, the Schubert Littlewood–Richardson coefficients are the structure constants of the coproduct Δ^* on $\mathcal{D}$ in the dual Schubert basis. Thus the otherwise mysterious nonnegativity of $c_{u,v}^w$ has the bialgebra-theoretic content that Δ^* has nonnegative integer coefficients in the basis $\{\Xi_u^n\}$. We will see in Corollary 8.4.7 that the same phenomenon recurs for the dual forest basis, with structure constants β_{ab}^c of Theorem 1.1.

Example 3.2.1. Take $w = 24135 \in S_5$ (so $\mathbf{m}(w) = 2$) and $n = 2$. Then in $\mathcal{D}_2 \otimes \mathcal{D}_2$,

$$\begin{aligned} \Delta^*(\Xi_{24135}^2) &= 1 \otimes \Xi_{2413}^2 + \Xi_{2413}^2 \otimes 1 \\ &\quad + \Xi_{132}^2 \otimes \Xi_{1423}^2 + \Xi_{1423}^2 \otimes \Xi_{132}^2 \\ &\quad + \Xi_{132}^2 \otimes \Xi_{231}^2 + \Xi_{231}^2 \otimes \Xi_{132}^2 \\ &\quad + \Xi_{21}^2 \otimes \Xi_{1423}^2 + \Xi_{1423}^2 \otimes \Xi_{21}^2. \end{aligned}$$

By Theorem 3.2, the coefficient of $\Xi_u^2 \otimes \Xi_v^2$ is the Schubert structure constant $c_{u,v}^{24135}$; for instance, $c_{132, 1423}^{24135} = c_{21, 1423}^{24135} = 1$, while $c_{u,v}^{24135} = 0$ for every pair (u, v) not appearing above.

4. THE RING OF BOUNDED RC GRAPHS

4.1. The module of bounded RC graphs.

Definition 4.1.1. Let $R \subseteq \mathbb{P} \times \mathbb{P}$ be a finite set. To each such set we associate an element w_R of S_∞ as follows. Given a pair (i, j) where $i, j > 0$ are integers, define $s(i, j) = s_{i+j-1}$. Totally order the grid such that $(i, j) < (a, b)$ if and only if $i < a$ or $i = a$ and $j > b$ (in other words, lexicographical order, except that the order on the second coordinate is reversed). By this ordering, index R as $r_1, r_2, \dots, r_m$ in increasing order. Then

$$w_R = s_{r_1} \cdots s_{r_m}$$

If $\ell(w_R) = m$, then we say that R is an *RC-graph*.

A *bounded RC-graph* is an RC-graph R together with a specified number of rows $n \geq \max\{i : (i, j) \in R \text{ for some } j\}$, with the added condition that $\mathbf{m}(w_R) \leq n$. The definition is as an ordered pair (R, n) , but it will be far more convenient to abuse notation and consider the number of rows a property of R itself. To denote the number of rows, we define the *height* of R to be $\mathbf{ht}(R) = n$. A bounded RC graph has an associated weight vector $\mathbf{wt}(R)$ where $\mathbf{wt}_i(R)$ is the number of elements in row i .

To a bounded RC graph R with $\mathbf{ht}(R) \geq 1$, we define the *trim* operation $\mathbf{trim}(R)$ to be the bounded RC graph with height $\mathbf{ht}(R) - 1$ and underlying set $\{(i - 1, j - 1) : (i, j) \in R, i \geq 2\}$. We also define

$$\uparrow^m R = \{(i + m, j) | (i, j) \in R\}$$

Let $\mathcal{BRC}$ be the free abelian group spanned by all bounded RC graphs. If $\mathcal{BRC}_n$ is the subgroup spanned by all bounded RC graphs R with $\mathbf{ht}(R) = n$, then we have a grading

$$\mathcal{BRC} = \bigoplus_{n=0}^{\infty} \mathcal{BRC}_n$$

There is an evident evaluation map $\text{ev} : \mathcal{BRC} \rightarrow \mathcal{A}$ defined on basis elements as

$$\text{ev}(R) = x_{\mathbf{wt}(R)}$$

which is a surjective homomorphism of graded modules. There is also a map $\alpha : \mathcal{BRC} \rightarrow \mathcal{D}$ defined by

$$\alpha(R) = \Xi_{w_R}^{\mathbf{ht}(R)}$$

which is also a surjective homomorphism of graded modules. In addition, there is $\omega : \mathcal{BRC} \rightarrow \mathcal{D}$ defined by

$$\omega(R) = [\mathbf{wt}(R)]$$

which is similarly surjective.

We can define elements $\mathcal{S}_w(n)$ as

$$\mathcal{S}_w(n) = \sum_{\substack{w_R = w \\ \mathbf{ht}(R) = n}} R$$

For a generating element $[a]$ of $\mathcal{D}$ and a bounded RC graph R , we define

$$[a] \cdot R = \sum_{\substack{\mathbf{trim}(R') = R \\ \mathbf{wt}_1(R') = a}} R'$$

which is an element of $\mathcal{BRC}$. By virtue of the fact that $\mathcal{D}$ is a free algebra generated by these elements, it is nearly a trivial observation that this is a left module action on $\mathcal{BRC}$. The consequences of this, however, are not at all trivial.

Lemma 4.1.2 ([4, Corollary 3.11]). *We have that the set $\{w' \mid w \searrow^1 w'\}$ is equal to the set of permutations w' such that there exists a permutation $v = s_{a_1} \cdots s_{a_m}$ for some integers $a_1 > a_2 > \cdots > a_m \geq 1$ such that $w = v \uparrow w'$.*

Lemma 4.1.3. *Let v be a permutation. If $v \searrow^1 v'$, let $a_1, \dots, a_k$ be the elements of $Q_1(v', v)$ in decreasing order. Then*

$$v = s_{a_1} \cdots s_{a_k} \uparrow v'$$

Proof. Suppose $v \in S_n$. We have by definition that there is a sequence of integers $b_1, \dots, b_p$, all distinct and greater than 1, such that

$$vt_{1,b_1} \cdots t_{1,b_p}(1) = n + 1$$

and

$$vt_{1,b_1} \cdots t_{1,b_p}(i + 1) = v'(i)$$

for all $i < n - 1$. This means that

$$vt_{1,b_1} \cdots t_{1,b_p} = s_n s_{n-1} \cdots s_1 \uparrow v'$$

This is because if $v'' = \uparrow v'$, then

$$v''(1) = 1$$

and

$$v''(i) = v(i-1) + 1$$

The cycle $s_n \cdots s_1$ sends $1 \mapsto n+1$ and $i \mapsto i-1$ if $1 < i \leq n+1$, hence

$$vt_{1,b_1} \cdots t_{1,b_p} = s_n s_{n-1} \cdots s_1 \uparrow v'$$

In particular,

$$v = s_n s_{n-1} \cdots s_1 \uparrow v' t_{1,b_p} \cdots t_{1,b_1}$$

This results in the factorization

$$v = s_n \cdots s_1 t_{1,v'(b_p-1)+1} \cdots t_{1,v'(b_1-1)+1} \uparrow v'$$

The value $v'(b_j-1)$ necessarily decreases as j decreases, since applying the corresponding reflection in the reverse order strictly increases the length with each application. Consequently, multiplying $s_n \cdots s_1$ on the right by these reflections removes the simple reflections $s_{v'(b_p-1)}, \dots, s_{v'(b_1-1)}$. The indices removed are precisely the complement of the elements of $Q_1(v', v)$, hence the elements of $Q_1(v', v)$ in decreasing order are what remain, as desired. $\square$

Definition 4.1.4. For a set of positive integers $A = \{a_1, \dots, a_m\}$ with $a_1 > a_2 > \cdots > a_m \geq 1$ and a positive integer i , define

$$\text{row}_i(A) = \{(i, a_j) \mid a_j \in A\}$$

Theorem 4.1. Let $a \geq 0$ be an integer and let $R \in \mathcal{BRC}$. Then

$$[a] \cdot R = \sum_{\substack{w' \searrow^1 w_R \\ \ell(w_R, w')=a}} (\text{row}_1(Q_1(w_R, w')) \cup \uparrow R)$$

where the right side denotes bounded RC graphs with $\text{ht}(R) + 1$ rows.

Proof. If $\text{trim}(R') = R$, then by definition $w_{R'} = s_{a_1} \cdots s_{a_m} \uparrow w_R$. It follows by Lemma 4.1.2 then $w_{R'} \searrow^1 w_R$. By Lemma 4.1.3, the integers $a_1, \dots, a_m$ are precisely the elements of $Q_1(w_R, w_{R'})$ in decreasing order. This establishes the result. $\square$

Lemma 4.1.5. Let $a \geq 0$ be an integer and let $w \in S_\infty$. For any valid n , we have

$$[a] \cdot \mathcal{S}_w(n) = \sum_{\substack{w \searrow^1 w' \\ \ell(w, w')=a}} \mathcal{S}_{w'}(n+1)$$

Proof. Theorem 4.1 establishes an explicit bijection between $[a] \cdot R$ and the set of bounded RC graphs R' such that $w_{R'} \searrow^1 w_R$ and $\ell(w_R, w_{R'}) = a$. The result follows by summing over all R such that $w_R = w$ and $\text{ht}(R) = n$. $\square$

Let t be an indeterminate commuting with all elements of $\mathcal{D}$. Write

$$\mathfrak{S}_{\mathcal{D}}(t) = \sum_{a=0}^{\infty} [a] t^a$$

Define

$$\mathfrak{S}_{\mathcal{D}}(x_1, x_2, \dots, x_n) = \mathfrak{S}_{\mathcal{D}}(x_1) \mathfrak{S}_{\mathcal{D}}(x_2) \cdots \mathfrak{S}_{\mathcal{D}}(x_n)$$

Theorem 4.2. Let $\emptyset \in \mathcal{BRC}$ denote the empty bounded RC graph with $\text{ht}(\emptyset) = 0$. Then

$$\mathfrak{S}_{\mathcal{D}}(x_1, \dots, x_n) \cdot \emptyset = \sum_{\substack{w \in S_\infty \\ \text{mw} \leq n}} \mathfrak{S}_w(x_1, \dots, x_n) \mathcal{S}_w(n)$$

Proof. The result for $n = 1$ is Lemma 4.1.5 together with Theorem 4.1. The general result follows by induction on n . Consider the inductive hypothesis

$$\mathfrak{S}_{\mathcal{D}}(x_2, \dots, x_n) \cdot \emptyset = \sum_{w \in S_{\infty}} \mathfrak{S}_w(x_2, \dots, x_n) \mathcal{S}_w(n-1)$$

Then applying $\mathfrak{S}_{\mathcal{D}}(x_1)$ to both sides, we have

$$\mathfrak{S}_{\mathcal{D}}(x_1) \mathfrak{S}_{\mathcal{D}}(x_2, \dots, x_n) \cdot \emptyset = \sum_{w \in S_{\infty}} \mathfrak{S}_w(x_2, \dots, x_n) \mathfrak{S}_{\mathcal{D}}(x_1) \mathcal{S}_w(n-1)$$

which is equal to

$$\sum_{a=0}^{\infty} \sum_{w \in S_{\infty}} x_1^a \mathfrak{S}_w(x_2, \dots, x_n) [a] \mathcal{S}_w(n-1)$$

Applying Lemma 3.1.3 to each term, we have

$$\sum_{a=0}^{\infty} \sum_{w \in S_{\infty}} x_1^a \mathfrak{S}_w(x_2, \dots, x_n) \sum_{\substack{w' \searrow_x w \\ \ell(w, w')=a}} \mathcal{S}_{w'}(n)$$

Bringing the polynomial inside the inner sum, this is

$$\sum_{w' \in S_{\infty}} \left(\sum_{w: w' \searrow_x w} x_1^{\ell(w, w')} \mathfrak{S}_w(x_2, \dots, x_n) \right) \mathcal{S}_{w'}(n)$$

which simplifies to

$$\sum_{w' \in S_{\infty}} \mathfrak{S}_{w'}(x_1, \dots, x_n) \mathcal{S}_{w'}(n)$$

as desired. $\square$

Corollary 4.1.6. *For each $w \in S_{\infty}$ and valid n , we have*

$$\text{ev}(\mathcal{S}_w(n)) = \mathfrak{S}_w^n(x_1, \dots, x_n)$$

4.2. The pipe dream visualization and roots. Given a pair of positive integers i, j and an RC graph R , define

$$R[i, j] = \{(a, b) \in R \mid (a, b) > (i, j)\}$$

Given an RC graph R and a pair of positive integers i, j , we define an ordered pair

$$\mathbf{rt}_R(i, j) = (w_{R[i, j]}^{-1}(i + j - 1), w_{R[i, j]}^{-1}(i + j))$$

There is a common visualization of RC graphs as pipe dreams. In this visualization, we draw an infinite grid in the first quadrant, and in each position (i, j) we draw either a crossing (if $(i, j) \in R$) or an elbow (if $(i, j) \notin R$). Then we draw pipes entering from the left edge of the grid, with the pipe entering at row i labeled i . The pipes travel through the grid, turning at elbows and crossing at crossings, and exit at the top of the grid, which is labeled with the same number.

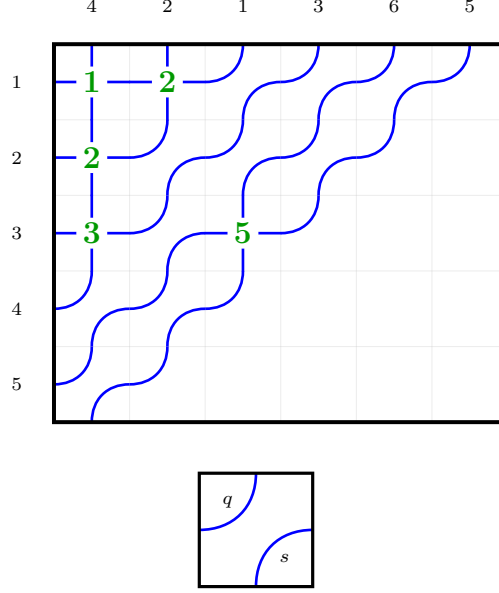
A modification to this common visualization that we use has the following additional features:

- We write the index of the simple reflections corresponding to the crossings in the grid area itself. Thus, at position (i, j) we write the number $i + j - 1$ if there is a crossing at that position.
- For a bounded RC graph R with $\text{ht}(R) = n$, we only draw the first n pipes entering from the left side of the grid and clip features outside of the first n rows. The pipes exiting at the top are still labeled since the width is not limited.
- *Note:* The labels at the top are not of the permutation w_R , but rather of the permutation w_R^{-1} .

See Figure 1 for an example of this visualization.

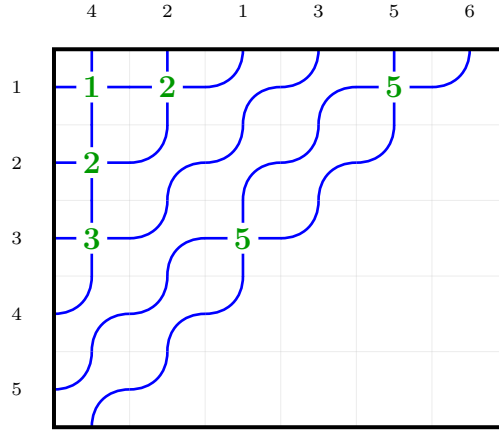
The main benefit of this is that visualizing $\mathbf{rt}_R(i, j)$ is easy. The pipes that pass through position (i, j) are labeled s and q for some $s, q > 0$. For a positive root in an unoccupied square, we will have the following labeling, where $q < s$:

FIGURE 1. The pipe dream visualization of the bounded RC graph R with $\text{ht}(R) = 5$ where $R = \{(1, 1), (1, 2), (2, 1), (3, 1), (3, 3)\}$



We observe that in the above RC graph, the position $(1, 5)$ does not have this configuration. Placing a crossing there would create a negative root, and the pipes would cross twice. *Note that this results in a collection of ordered pairs that is not an RC graph, if such a crossing exists.* For a valid RC graphs, only positive roots occur as crossings, and this happens if and only if pipes cross at most once.

FIGURE 2. An invalid set of crossings causing pipes to cross more than once, caused by inserting a crossing at a negative root



4.3. Zeroing out the last row. We proceed now to define a product on $\mathcal{BRC}$ turning it into a ring. To do this, we need to be able to define a function $\mathcal{Z} : \mathcal{BRC} \rightarrow \mathcal{BRC}$ trimming empty rows from the bottom instead of from the top. This is far more complicated.

See Figure 3 for an example of the zeroing operation (Algorithm 3).

Note that Algorithm 2 is *not* a realization of Sottile's Pieri formula for complete symmetric polynomials, despite preserving the relevant Bruhat relation. This is because the map

$$\left(\binom{k}{p} \right) \times \mathcal{RC}(w) \rightarrow \mathcal{RC}(n)$$

Algorithm 1 Basic monk insertion $i \xrightarrow{k} R$

-
- 1: **Input:** An RC graph R , parameter k , and an integer i with $1 \leq i \leq k$.
 - 2: **Output:** A modified RC graph R' with $\mathbf{wt}_j(R') = \mathbf{wt}_j(R)$ for $j \neq i$ and $\mathbf{wt}_i(R') = \mathbf{wt}_i(R) + 1$ such that $w_R \xrightarrow{k} w_{R'}$.
 - 3: Find leftmost position $(i, j) \notin R$ in row i such that if $\mathbf{rt}_R(i, j) = (a, b)$, then $a \leq k < b$.
 - 4: $R \leftarrow R \cup \{(i, j)\}$
 - 5: **if** there exists $(i', j') \in R$ with $\mathbf{rt}_R(i', j') \in \Phi^-$ **then**
 - 6: $R \leftarrow R \setminus \{(i', j')\}$
 - 7: Return to step 3 substituting $i = i'$.
 - 8: **end if**
 - 9: **return** R
-

Algorithm 2 Pseudo-Pieri insertion $I \xrightarrow{k} R$

-
- 1: **Input:** An RC graph R , parameter k , and sequence $I = \{i_1, \dots, i_m\}$ with $k \geq i_1 \geq i_2 \geq \dots \geq i_m \geq 1$.
 - 2: **Output:** A modified RC graph R' with $\mathbf{wt}_i(R') = \mathbf{wt}_i(R)$ for $i \notin I$ and $\mathbf{wt}_i(R') = \mathbf{wt}_i(R) + \#\{j \mid i_j = i\}$ for $i \in I$ such that $w_R \xrightarrow{k} w_{R'}$.
 - 3: **Initialize:** Let $L \leftarrow []$ (empty list of pairs (a, b) where $a \leq k < b$), and $U \leftarrow []$ (empty list of positions).
 - 4: **for** each $i \in (i_1, \dots, i_m)$ **do**
 - 5: Find leftmost position $(i, j) \notin R$ in row i satisfying $j > j'$ for all $(i, j') \in U$ such that either:
 - 6: a) $\mathbf{rt}_R(i, j) = (s, q)$ where $s \leq k < q$ and $q \notin \{b \mid (a, b) \in L\}$.
 - 7: b) $\mathbf{rt}_R(i, j) = (b_r, q)$ where $q > k$, $b_r < q$, and $q \notin \{b \mid (a, b) \in L\}$.
 - 8: $R \leftarrow R \cup \{(i, j)\}$ and $U \leftarrow U \cup \{(i, j)\}$
 - 9: Update L by adding (s, q) or replacing (a_r, b_r) with (a_r, q) and (a_r, b_r) .
 - 10: **if** there exists $(i', j') \in R$ with $\mathbf{rt}_R(i', j') \in \Phi^-$ **then**
 - 11: $R \leftarrow R \setminus \{(i', j')\}$
 - 12: Let $\mathbf{rt}_R(i', j') = (q, s)$
 - 13: **if** $(s, q) \in L$ **then**
 - 14: Remove (s, q) from L
 - 15: **else**
 - 16: Remove (a_r, q) from L , where $(a_r, q), (a_r, s) \in L$
 - 17: **end if**
 - 18: Return to step 5 to insert i' .
 - 19: **end if**
 - 20: **end for**
 - 21: **return** R
-

given by

$$(I, R) \mapsto I \xrightarrow{k} R$$

need not be injective.

Definition 4.3.1. Let R be an RC graph with $\mathbf{ht}(R) = n$ such that row n is empty. We define $\mathcal{Z}(R)$ to be the output of Algorithm 3, an RC graph with $n - 1$ rows, on input R .

Lemma 4.3.2. *Given an RC graph R , an integer k , and $k \geq i_1 \geq \dots \geq i_m \geq 1$, Algorithm 2 produces a valid RC graph R' satisfying*

$$\mathbf{wt}_i(R') = \mathbf{wt}_i(R) + \#\{j \mid i_j = i\}$$

and

$$w_R \xrightarrow{k} w_{R'}$$

Proof. Set $R_0 = R$ to be the initial RC graph. We claim that given R and L at the start of iteration p of the main loop, we have that R is a valid RC graph satisfying

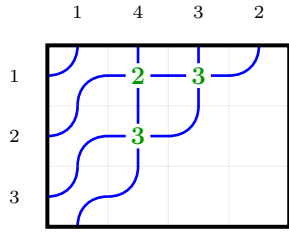
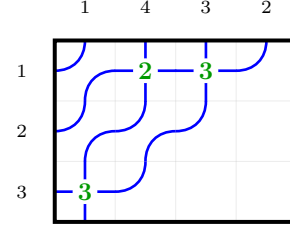
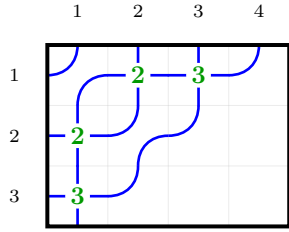
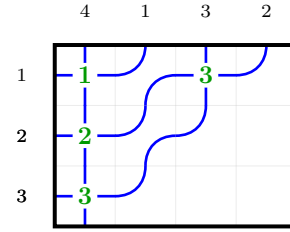
$$\mathbf{wt}_i(R) = \mathbf{wt}_i(R_0) + \#\{j \leq p - 1 \mid i_j = i\}$$

Algorithm 3 Map $\mathcal{Z}(R)$ zeroing out last row

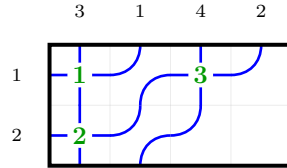
```

1: Input: A bounded RC graph  $R$  with  $\text{ht}(R) = n$  and row  $n$  empty.
2: Output: A bounded RC graph  $R'$  with  $\text{ht}(R') = n - 1$ ,  $|R'| = |R|$  (same number of crossings), and
    $w_R \xrightarrow{n} w_{R'}$ .
3: if  $R$  with last row removed is a valid bounded RC graph then
4:   return  $R$  with last row removed
5: else
6:   Let  $R_0 \leftarrow R$ .
7:   Find the maximal  $p$  and sequence  $\{(i_m, j_m)\}_{m=1}^p$  such that:
8:      $\text{rt}_{R_{m-1}}(i_m, j_m) = (n + m - 1, n + m)$  for each  $m$ , where  $R_m = R_{m-1} \setminus \{(i_m, j_m)\}$  for each  $m \geq 1$ .
9:      $R^- \leftarrow R \setminus \{(i_1, j_1), \dots, (i_p, j_p)\}$ .
10:    Let  $I \leftarrow (i_1, i_2, \dots, i_p)$ .
11:     $D \leftarrow R^- \cup \{(n, 1), (n, 2), \dots, (n, p)\}$ .
12:     $D' \leftarrow I \xrightarrow{n-1} D$ 
13:     $R' \leftarrow \{(i, j) \in D' \mid i < n\}$  as a bounded RC graph with  $n - 1$  rows.
14:   return  $R'$ .
15: end if

```

(a) Initial RC graph R (b) After deleting $(2, 2)$ with $\text{rt}_R(2, 2) = \alpha_3$ and moving to row 3(c) After insertion with $i_1 = 2$ 

(d) After rectification

(e) Final result $\mathcal{Z}(R) = (R', 2)$ after removing row 3FIGURE 3. Step-by-step computation of $\mathcal{Z}(R)$ for $R = \{(1, 2), (1, 3), (2, 2)\}$.

and

$$w_R = w_{R_0} t_{a_1 b_1} \cdots t_{a_{p-1} b_{p-1}}$$

where $L = [(a_1, b_1), \dots, (a_p, b_p)]$. This is clear for $p = 1$. For larger p , suppose we are inserting at row i_p . If we are in case (a) of Step 5, then adding (i_p, j) to R adds the root $t_s - t_q$ to the inversion set of w_R , and multiplying by t_{sq} gives the desired result. If we are in case (b) of Step 5, then adding (i_p, j) to R results in multiplication by $t_{b_r q}$. This commutes past the other reflections to meet $t_{a_r b_r}$. We thus have the situation

$$t_{a_r b_r} t_{b_r q} = t_{a_r q} t_{a_r b_r}$$

This ensures that the product remains as desired if the RC graph is valid. However, the condition that the RC graph R be valid is not guaranteed after this step, so we must rectify. Suppose we must rectify at row $i - 1$. We find a negative root $(i - 1, j')$ which must have been created by the insertion, and hence by the strong exchange property equal to $\mathbf{rt}_R(i, j)$ that was inserted. Removing this root (from both the graph and from L) returns us to the original permutation at the beginning of the iteration, and performing the insert at row $i - 1$ adds a new positive root, giving the desired result on the permutation. The weight is unchanged since we removed and added one crossing in row $i - 1$. If the algorithm ends here, we have instead multiplied by the new reflection obtained in the ultimate insert step. Otherwise, repeating this process for all necessary rectifications completes the proof of the claim. The claim iterated to $p = m + 1$ yields the desired result, and the method of updating L ensures that the b_j are all distinct, so that $w_R \xrightarrow{k} w_{R'}$. $\square$

Lemma 4.3.3. *Given a bounded RC graph R with $\mathbf{ht}(R) = n$ such that row n is empty, Algorithm 3 produces a valid bounded RC graph R' with $\mathbf{ht}(R') = n - 1$ satisfying*

$$\mathbf{wt}(R') = \mathbf{wt}(R)$$

Proof. Stripping out the crossings at roots $(n, n + 1), (n + 1, n + 2), \dots, (n + p - 1, n + p)$ removes exactly one crossing from each of rows $i_1, i_2, \dots, i_p$, and moving them to the last row to obtain R_1 ensures that $w_R = w_{R_1}$. By construction, the portion of the RC graph in rows with index less than n is valid (meaning its max descent is at most $n - 1$). Applying Algorithm 2 to insert at rows $i_1, i_2, \dots, i_p$ adds different crossings to preserve the number of elements in each row without adding descents past index n in the permutation, by Lemma 4.3.2. Removing row n then yields a valid bounded RC graph R' with $|R'| = n - 1$ and the desired properties. $\square$

Theorem 4.3. *Let w be a permutation with last descent at most n . Let $\mathcal{RC}(w, n)^0$ be the set of bounded RC graphs R with $\mathbf{ht}(R) = n$ such that $w_R = w$ and row n is empty. Then*

$$\mathcal{Z} : \mathcal{RC}(w, n)^0 \rightarrow \bigcup_{w \searrow^n w'} \mathcal{RC}(w', n - 1)$$

is a weight-preserving bijection.

To prove Theorem 4.3 for $n = 2$, we may characterize the bounded RC graphs R with $|R| = 2$ such that row 2 is empty and $\mathbf{m}(w_R) = 2$ as those for permutations $w = s_k s_{k-1} \cdots s_2$ for some $k > 1$. Algorithm 3 in this case moves the entirety of the crossings in row 1 to row 2. The procedure then inserts $k - 1$ crossings into row 1 in order from left to right, which must have roots $t_q - t_s$ such that $q = 1$. Thus R' is precisely $\{(1, 1), (1, 2), (1, 3), \dots, (1, k - 1)\}$, which is the unique bounded RC graph for the permutation $w' = s_{k-1} s_{k-2} \cdots s_1$ with one row. This establishes the base case.

For the induction step, we require the following lemmas.

Lemma 4.3.4. *Let R be a bounded RC graph with $\mathbf{ht}(R) = n \geq 2$ such that row n is empty. Then if $\mathcal{Z}(R) = R'$, we have*

$$w_R \searrow^n w_{R'}$$

Proof. Let $w = w_R$ and suppose $\mathbf{c}_n(w) = m$. By construction, the second to last step of Algorithm 3 yields a bounded RC graph $\tilde{R}$ with $|\tilde{R}| = n$ such that if $\tilde{w} = w_{\tilde{R}}$, then

$$\tilde{w}(n) > \tilde{w}(n + 1) < \tilde{w}(n + 2) < \cdots < \tilde{w}(n + m)$$

and

$$w \xrightarrow{n-1} \tilde{w}$$

via reflections t_{ab} such that $n \leq b \leq n + m$. Let $w' = w_{R'}$. We also have

$$\tilde{w} \xrightarrow{n} \varphi_{n,N}(w')$$

via only reflections t_{nb} such that $b > n + m$. We claim that

$$w \xrightarrow{n} \varphi_{n,N}(w')$$

This can only fail if $\tilde{w}(n) \neq w(n)$ by the observations above. However, the pipe labeled n , by virtue of the fact that we have $(n, 1), \dots, (n, m)$ populated, will always lie to the right of pipes $n + 1, n + 2, \dots, n + m$ in the wiring diagram for $\tilde{w}$ or any intermediate stage. Inserting into any row that contained a crossing $(n, n + p)$, there must be a valid empty space to the left of the pipe labeled n since we have deleted these crossings. By virtue of the fact that the leftmost is always chosen, no reflection (i, n) will ever be inserted. This ensures that $\tilde{w}(n) = w(n)$, as desired. $\square$

Lemma 4.3.5. *Let (R, n) be a bounded RC graph with $n \geq 2$ such that row n is empty. Then*

$$\mathcal{Z}(\text{trim}(R)) = \text{trim}(\mathcal{Z}(R))$$

Proof. This is almost a trivial observation. In terms of the word of R , $\text{trim}(R, n)$ preserves a suffix. Removing all of the initial roots before trimming is therefore the same as removing the initial roots that still remain after trimming, by the exchange property. Afterwards, in performing the insertion algorithm, there is no dependency on rows with a lower index, the modification only proceeds downward in row number. Therefore, trimming the first row at any point only stops the process earlier, and does not change rows with higher index than 1 in the outcome. $\square$

Lemma 4.3.6. *Let w be a permutation with last descent at most n . Let $\mathcal{RC}(w, n)^0$ be the set of bounded RC graphs R with $\text{ht}(R) = n$ such that $w_R = w$ and row n is empty. Then*

$$\mathcal{Z} : \mathcal{RC}(w, n)^0 \rightarrow \mathcal{RC}(n - 1)$$

is injective.

Proof. If $n = 2$ this is clear. Suppose now that $n > 2$ and that the result holds for $n - 1$. Let $(R, n) \in \mathcal{RC}(w, n)^0$. If $(R', n) \in \mathcal{RC}(w, n)^0$ is another bounded RC graph such that

$$\mathcal{Z}(R) = \mathcal{Z}(R')$$

then by Lemma 4.3.5, we have

$$\mathcal{Z}(\text{trim}(R)) = \mathcal{Z}(\text{trim}(R'))$$

hence R and R' agree on all rows except possibly row 1 by the inductive hypothesis. Since $w_R = w_{R'}$, it follows that $R = R'$ from Theorem 4.1, establishing injectivity. $\square$

Proof of Theorem 4.3. By the previous lemma, it suffices to show that $\mathcal{Z}$ is surjective. We have by the transition formula that if $\mathbf{m}(w) \leq n$, then

$$\mathfrak{S}_w(x_1, \dots, x_{n-1}, 0) = \sum_{\substack{w \searrow w' \\ \ell(w) = \ell(w')}} \mathfrak{S}_{w'}(x_1, \dots, x_{n-1})$$

Note that this sum is without multiplicity. The right hand side is equal to

$$\sum_{\substack{w \searrow w' \\ \ell(w) = \ell(w')}} \sum_{R' \in \mathcal{RC}_{n-1}(w')} x^{\mathbf{wt}(R')}$$

and the left hand side is equal to

$$\sum_{R \in \mathcal{RC}(w, n)^0} x^{\mathbf{wt}(R)}$$

since the only RC graphs that contribute to the left hand side are those with row n empty. This implies that

$$|\mathcal{RC}_n^0(w)| = \sum_{\substack{w \searrow w' \\ \ell(w) = \ell(w')}} |\mathcal{RC}_{n-1}(w')|$$

Now, the set

$$\{\mathcal{Z}(R) \mid R \in \mathcal{RC}_n(w)^0\}$$

has the same cardinality as $\mathcal{RC}_n(w)^0$ by injectivity (Lemma 4.3.6), and is a subset of $\bigcup_{w \searrow w'}^n \mathcal{RC}_{n-1}(w')$. Since the right hand side has the same cardinality as $\mathcal{RC}_n(w)^0$, we conclude that $\mathcal{Z}$ is surjective, as desired. $\square$

We may extend $\mathcal{Z}$ to an endomorphism $\mathcal{BRC} \rightarrow \mathcal{BRC}$ by defining $\mathcal{Z}(R) = 0$ if the last row of R is not empty. Then we have the following result.

Corollary 4.3.7 (Combinatorial Lascoux–Scützenberger transition formula). *Let w be a permutation with last descent at most n . Then*

$$\mathcal{Z}(\mathcal{S}_w(n)) = \sum_{\substack{w \searrow w' \\ \ell(w)=\ell(w')}}^n \mathcal{S}_{w'}(n-1)$$

4.4. Definition of the ring product.

Definition 4.4.1. Suppose we have two bounded RC graphs R_1 with $\text{ht}(R_1) = m$ and R_2 with $\text{ht}(R_2) = n$. The product of these is defined by

$$R_1 \sqcup R_2 = \sum_{\substack{\text{clip}^m(R')=R_1 \\ \text{trim}^n(R')=R_2}} R'$$

where R' ranges over bounded RC graphs with $\text{ht}(R') = m + n$.

Associativity of $\sqcup$ rests on two structural identities relating clip^p and trim^p : a functoriality of clip under iteration, and a commutativity of clip with trim .

Lemma 4.4.2. *Let R be a bounded RC graph with $\text{ht}(R) = n$, and let $0 \leq p \leq p+q \leq n$. Then*

$$\text{clip}^p(\text{clip}^{p+q}(R)) = \text{clip}^p(R).$$

Proof. By definition, $\text{clip}^k(R')$ is computed by iterating the row-zeroing map $\mathcal{Z}$ on the bottom row, one row at a time, until the height drops to k . Set $r = n - (p+q)$, so that $\text{clip}^{p+q}(R)$ is obtained from R by r successive applications of $\mathcal{Z}$ to the bottom row, and $\text{clip}^p(R)$ is obtained from R by $r+q$ such applications. The first r applications of $\mathcal{Z}$ in the computation of $\text{clip}^p(R)$ produce exactly $\text{clip}^{p+q}(R)$; the remaining q applications then produce $\text{clip}^p(\text{clip}^{p+q}(R))$. Hence the two compositions agree. $\square$

Lemma 4.4.3 (Commutativity of clip and trim). *Let R be a bounded RC graph with $\text{ht}(R) = p+q+r$. Then*

$$\text{trim}^p(\text{clip}^{p+q}(R)) = \text{clip}^q(\text{trim}^p(R)).$$

Proof. Both sides have height q . Lemma 4.3.5 asserts $\mathcal{Z}(\text{trim}(R')) = \text{trim}(\mathcal{Z}(R'))$ for any R' whose last row is empty. Since clip^{p+q} on a graph of height $p+q+r$ and clip^q on a graph of height $q+r$ are both equal to the r -fold iterate of $\mathcal{Z}$ on the bottom row (cf. Lemma 4.4.2), iterating Lemma 4.3.5 r times gives

$$\text{trim}^p(\text{clip}^{p+q}(R)) = \text{trim}^p(\mathcal{Z}^r(R)) = \mathcal{Z}^r(\text{trim}^p(R)) = \text{clip}^q(\text{trim}^p(R)),$$

as required. $\square$

Theorem 4.4. *The product $\sqcup$ is associative and $\mathcal{BRC}$ is a ring when equipped with $\sqcup$.*

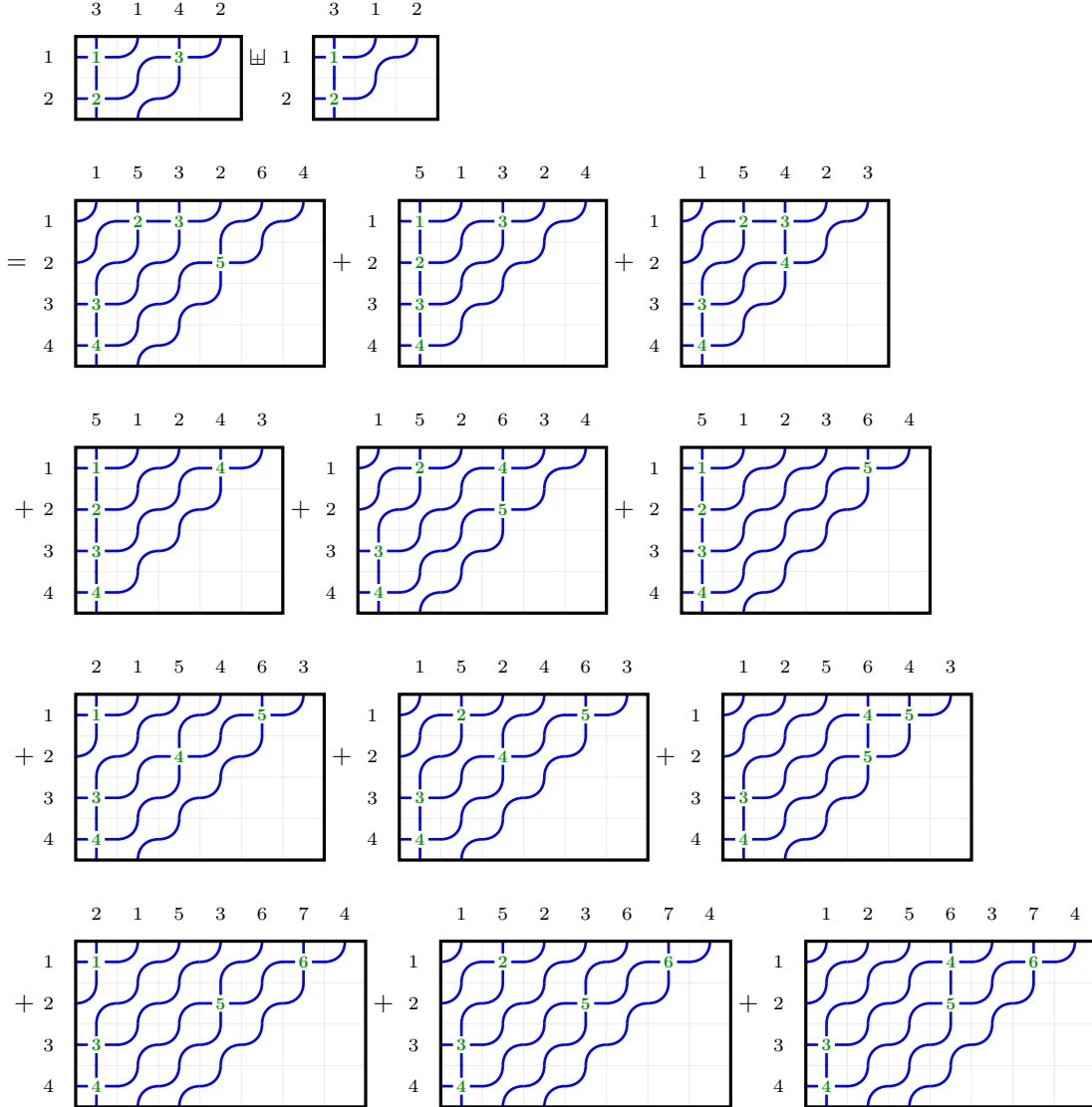
Proof. Let R_1, R_2, R_3 be bounded RC graphs of heights p, q, r respectively, and let R be a bounded RC graph with $\text{ht}(R) = p+q+r$. We claim that R appears as a term in $(R_1 \sqcup R_2) \sqcup R_3$ if and only if it appears as a term in $R_1 \sqcup (R_2 \sqcup R_3)$, and in both cases with multiplicity 1. Unwinding the definition of $\sqcup$ in each bracketing yields the following conditions on R :

	Left bracketing $(R_1 \sqcup R_2) \sqcup R_3$	Right bracketing $R_1 \sqcup (R_2 \sqcup R_3)$
(i)	$\text{clip}^p(\text{clip}^{p+q}(R)) = R_1$	$\text{clip}^p(R) = R_1$
(ii)	$\text{trim}^p(\text{clip}^{p+q}(R)) = R_2$	$\text{clip}^q(\text{trim}^p(R)) = R_2$
(iii)	$\text{trim}^{p+q}(R) = R_3$	$\text{trim}^q(\text{trim}^p(R)) = R_3$

Row (i) is equivalent on both sides by Lemma 4.4.2. Row (ii) is equivalent on both sides by Lemma 4.4.3. Row (iii) is equivalent on both sides because $\text{trim}^{p+q} = \text{trim}^q \circ \text{trim}^p$ by definition of trim . Thus the term R contributes to the same bracketed product on both sides. Since the membership conditions are equalities (not weighted by any combinatorial multiplicity) and R is determined to lie in either product by these three conditions, the multiplicity is 1 in each case. We conclude that $(R_1 \sqcup R_2) \sqcup R_3 = R_1 \sqcup (R_2 \sqcup R_3)$.

To tie up the claim that $\mathcal{BRC}$ is a ring, distributivity is directly imposed by the definition of the product as extending from the definition bilinearly, and the identity element is the empty RC graph of height 0. $\square$

Example 4.4.4. We compute the product of two bounded RC graphs $(R_1, 2) \in \mathcal{BRC}(2413, 2)$ and $(R_2, 2) \in \mathcal{BRC}(231, 2)$:



4.5. Proof of the Littlewood-Richardson rule for the dual Schubert basis.

Definition 4.5.1. Define a map $\Xi : \mathcal{BRC} \rightarrow \mathcal{D}$ by

$$\Xi(R) = \Xi_{w_R}^{\text{ht}(R)}$$

and $\omega : \mathcal{BRC} \rightarrow \mathcal{D}$ by

$$\omega(R) = [\text{wt}(R)]$$

Let $\mathcal{R}$ be the subring of $\mathcal{BRC}$ generated by the single-row RC graphs $\text{row}(i)$ for $i \geq 0$. Let a be a sequence of elements of a ring indexed by the nonnegative integers. For a permutation w and an integer $n \geq \mathfrak{m}(w)$,

define a function $\Xi_w^n(a)$ by

$$\Xi_w^n(a) = \sum_{\alpha} d_{\alpha,w}^n a_{\alpha_1} \cdots a_{\alpha_n}$$

where the sum is over all weak compositions α of length n , and $d_{\alpha,w}^n$ is the coefficient of α in $\Xi_w^n \in \mathcal{D}$.

Lemma 4.5.2. *We have that the elements*

$$\Xi_w^n(\mathbf{row})$$

for integers $n \geq 0$ and w that ranges over all permutations in S_{∞} with $n \geq \mathbf{m}(w)$ form a $\mathbb{Z}$ -basis of $\mathcal{R}$, and $\omega : \mathcal{R} \rightarrow \mathcal{D}$ is an isomorphism of graded rings.

Proof. First, $\omega : \mathcal{BRC} \rightarrow \mathcal{D}$ is a ring homomorphism because

$$\omega(R_1 \sqcup R_2) = \omega(R_1) \sqcup \omega(R_2),$$

which follows from the weight concatenation property of $\sqcup$ (equivalently, from weight preservation under `clip` and `trim` in Definition 4.4.1). Restricting to $\mathcal{R}$ gives a graded ring map $\omega|_{\mathcal{R}} : \mathcal{R} \rightarrow \mathcal{D}$.

Since $\omega(\mathbf{row}(i)) = [i]$ for all $i \geq 0$, the image of $\omega|_{\mathcal{R}}$ contains the free generators $[i]$ of $\mathcal{D}$, hence $\omega|_{\mathcal{R}}$ is surjective. Conversely, by freeness of $\mathcal{D}$, there is a unique graded ring homomorphism

$$\iota : \mathcal{D} \rightarrow \mathcal{R}, \quad \iota([i]) = \mathbf{row}(i).$$

Then $(\omega|_{\mathcal{R}}) \circ \iota = \text{id}_{\mathcal{D}}$, so $\omega|_{\mathcal{R}}$ is also injective. Therefore $\omega : \mathcal{R} \xrightarrow{\sim} \mathcal{D}$ is an isomorphism of graded rings.

Finally, the family $\{\Xi_w^n\}_{n \geq 0, n \geq \mathbf{m}(w)}$ is a $\mathbb{Z}$ -basis of $\mathcal{D}$ by construction, and

$$\omega(\Xi_w^n(\mathbf{row})) = \Xi_w^n.$$

Applying the inverse isomorphism ι shows that $\{\Xi_w^n(\mathbf{row})\}$ is a $\mathbb{Z}$ -basis of $\mathcal{R}$. □

Definition 4.5.3. Let Ω be the $\mathbb{Z}$ -submodule of $\mathcal{BRC}$ generated by the RC graphs $R_1 - R_2$ such that $w_{R_1} = w_{R_2}$ and $\mathbf{ht}(R_1) = \mathbf{ht}(R_2)$.

Lemma 4.5.4. Ω is a two-sided ideal of $\mathcal{BRC}$.

Proof. The fact that Ω is a left ideal follows from the fact that a product $R \sqcup R_1$ is uniquely determined by $R \sqcup \text{empty}(\mathbf{ht}(R_1))$ and R_1 , and the contribution of R_1 is precisely that it replaces the last $\mathbf{ht}(R_1)$ rows of each term. In particular, precisely the same $\mathbf{ht}(R)$ occur in precisely the same number of terms, to which $\uparrow^{\mathbf{ht}(R)} R_1$ is added as the last $\mathbf{ht}(R_1)$ rows. It follows that $R \sqcup R_1 - R \sqcup R_2$ is a linear combination of terms of the form $R'_1 - R'_2$ such that $\mathbf{ht}(R'_1) = \mathbf{ht}(R'_2)$ and $w_{R'_1} = w_{R'_2}$, and Ω is a left ideal.

The fact that Ω is a right ideal follows from the fact that a product $R_1 \sqcup R$ is uniquely determined by $R_1 \sqcup \text{empty}(\mathbf{ht}(R))$ and R , and the contribution of R is precisely that it replaces the last $\mathbf{ht}(R)$ rows of each term. In particular, precisely the same $\mathbf{ht}(R)$ occur in precisely the same number of terms, to which $\uparrow^{\mathbf{ht}(R)} R$ is added as the last $\mathbf{ht}(R)$ rows. It follows that $R_1 \sqcup R - R_2 \sqcup R$ is a linear combination of terms of the form $R'_1 - R'_2$ such that $\mathbf{ht}(R'_1) = \mathbf{ht}(R'_2)$ and $w_{R'_1} = w_{R'_2}$, and Ω is a right ideal. □

Theorem 4.5. *We have that $\alpha : \mathcal{BRC} \rightarrow \mathcal{D}$ is a surjective homomorphism of rings with kernel Ω , and the homomorphism $\iota : \mathcal{D} \rightarrow \mathcal{BRC}$ sending $[i] \mapsto \mathbf{row}(i)$ is a right inverse to α . In particular, $\mathcal{BRC} \cong \mathcal{D} \ltimes \Omega$ as rings.*

Proof. We have an exact sequence

$$0 \rightarrow \Omega \rightarrow \mathcal{BRC} \rightarrow \mathcal{D} \rightarrow 0$$

where the map $\mathcal{BRC} \rightarrow \mathcal{D}$ is α . We have a homomorphism $\iota : \mathcal{D} \rightarrow \mathcal{BRC}$ sending $[i] \mapsto \mathbf{row}(i)$ inducing an isomorphism on $\mathcal{R}$ by Lemma 4.5.2, and the composition $\mathcal{D} \rightarrow \mathcal{BRC} \rightarrow \mathcal{D}$ is the identity since $\alpha(\mathbf{row}(i)) = [i]$. Since Ω is a two-sided ideal, we can conclude that α is in fact a homomorphism of rings. Since $\mathcal{BRC} = \mathcal{R} \oplus \Omega$ by virtue of the fact that the sequence is split (which is due to the evident section ι), and $\mathcal{R} \cong \mathcal{D}$, we have $\mathcal{BRC} \cong \mathcal{D} \ltimes \Omega$ as rings. □

Definition 4.5.5. Given RC graphs $U \in \mathcal{RC}_p(u)$ and $V \in \mathcal{RC}_q(v)$ and a permutation w , define

$$d_{U,V}^w = \#\{R \in \mathcal{RC}_{p+q}(w) : \text{clip}^p(R) = U, \text{trim}^p(R) = V\}$$

Proof of Theorem 1.2. Fix $u, v \in S_\infty$ with last descents at most p, q respectively. The element

$$\Xi_u^p \sqcup \Xi_v^q \in \mathcal{D}_{p+q}$$

depends only on u, v, p, q , so its expansion coefficients $d_{u,v}^w(p, q)$ in the dual Schubert basis are well defined and *a priori* independent of any auxiliary RC-graph choices.

Choose any $U \in \mathcal{RC}_p(u)$ and $V \in \mathcal{RC}_q(v)$; such graphs exist because u and v have last descents at most p and q . By definition of α ,

$$\alpha(U) = \Xi_u^p, \quad \alpha(V) = \Xi_v^q.$$

By Theorem 4.5, $\alpha : \mathcal{BRC} \rightarrow \mathcal{D}$ is a ring homomorphism, hence

$$\Xi_u^p \sqcup \Xi_v^q = \alpha(U) \sqcup \alpha(V) = \alpha(U \sqcup V).$$

By the definition of $\sqcup$ on $\mathcal{BRC}$,

$$U \sqcup V = \sum_{\substack{R \in \mathcal{RC}_{p+q} \\ \text{clip}^p(R)=U, \text{trim}^p(R)=V}} R,$$

and applying α termwise,

$$(4.5.1) \quad \Xi_u^p \sqcup \Xi_v^q = \sum_{\substack{R \in \mathcal{RC}_{p+q} \\ \text{clip}^p(R)=U, \text{trim}^p(R)=V}} \Xi_{w_R}^{p+q}.$$

The dual Schubert polynomials $\{\Xi_w^{p+q}\}_{\mathfrak{m}(w) \leq p+q}$ are linearly independent in $\mathcal{D}_{p+q}$. Comparing coefficients of Ξ_w^{p+q} in (4.5.1) with the defining expansion

$$\Xi_u^p \sqcup \Xi_v^q = \sum_w d_{u,v}^w(p, q) \Xi_w^{p+q}$$

yields

$$d_{u,v}^w(p, q) = d_{U,V}^w = \#\{R \in \mathcal{RC}_{p+q}(w) : \text{clip}^p(R) = U, \text{trim}^p(R) = V\}.$$

Since the LHS is independent of the choices of U, V , so is the RHS; this gives the theorem and shows in particular that $d_{u,v}^w(p, q) \in \mathbb{Z}_{\geq 0}$. $\square$

5. THE RING OF RC GRAPH CRYSTALS

The key polynomials κ_a (also called *Demazure characters* in type A) form a $\mathbb{Z}$ -basis of $\mathbb{Z}[x_1, x_2, \dots]$ indexed by weak compositions a . Their origin lies in the work of Demazure [12, 13], who introduced isobaric divided difference operators π_i in order to compute the characters of the B -modules $H^0(X_w, \mathcal{L}_\lambda)$ associated to Schubert varieties X_w in G/B . In type A , the resulting characters become polynomials in the x_i , and one obtains κ_a for any weak composition a by setting $\kappa_\lambda = x^\lambda$ when λ is a partition (a dominant weight) and applying the appropriate operator π_i at each descent of a to interpolate between rearrangements of λ .

A purely combinatorial development of key polynomials was carried out by Lascoux and Schützenberger [23], who introduced the *right key* $K_+(T)$ and *left key* $K_-(T)$ of a semistandard Young tableau T and proved that

$$\kappa_a = \sum_T x^{\text{wt}(T)},$$

where the sum runs over semistandard Young tableaux T of shape $\text{sort}(a)$ whose right key $K_+(T)$ is bounded above (entrywise) by the key tableau associated to a . They additionally proved that Schubert polynomials decompose as a positive sum of key polynomials, given a combinatorial proof in terms of compatible sequences and the right-key map by Reiner and Shimozono [35]. The intermediate basis of *Demazure atoms* (the “standard bases” of [23]) was later given an explicit nonnegative monomial expansion by Mason [27] via semi-skyline augmented fillings.

From a representation-theoretic perspective, Kashiwara [20] introduced *Demazure crystals*: for a dominant integral weight λ with crystal $B(\lambda)$ and a Weyl group element w with reduced expression $w = s_{i_1} \cdots s_{i_\ell}$, the Demazure crystal $B_w(\lambda) \subseteq B(\lambda)$ is defined inductively from an extremal weight element by closure under the operators $\{e_{i_k}^n : n \geq 0\}$, and its character coincides with the Demazure character. Littelmann [24] obtained parallel results in the path model. Assaf and Schilling [1], building on the crystal structure on reduced factorizations of Morse and Schilling [29], exhibited an explicit Demazure crystal structure on the set of

RC graphs of a permutation w , thereby giving a representation-theoretic proof of Lascoux–Schützenberger key-positivity. It is this Demazure crystal structure on RC graphs that we will repeatedly invoke below.

The nonnegativity of the dual key structure constants $k_{a,b}^c$ of Theorem 5.4 is, in retrospect, a consequence of the theory of *excellent filtrations* for B -modules introduced by Polo [33] and van der Kallen [41], with existence of such filtrations on tensor products of Demazure modules established in general by Mathieu [28]; see also Joseph [19]. Restricting a Demazure module $V_c(\lambda)$ for GL_{p+q} to the Borel of the Levi $GL_p \times GL_q$ yields a B -module with an excellent filtration whose Demazure subquotients account for the $k_{a,b}^c$. This was pointed out by the anonymous user “Henry V” on MathOverflow [18]. While this abstract nonnegativity is classical, no explicit positive combinatorial rule for these coefficients appears in the literature to our knowledge.

5.1. Little bumps and the crystal action on RC graphs.

Definition 5.1.1 (Crystal action on RC graphs, highest weights, extremal weights). Following Morse and Schilling [29], who defined an $A_{\ell-1}$ -crystal structure on reduced factorizations of a permutation w , and the adaptation to RC graphs by Assaf and Schilling [1], we equip $\mathcal{RC}(w)$ with Kashiwara raising and lowering operators e_i, f_i for $i \geq 1$. The operator pair (e_i, f_i) acts only on rows i and $i+1$ of an RC graph R , viewed as adjacent factors in the corresponding reduced factorization.

The action is governed by a bracketing procedure on the column indices appearing in rows i and $i+1$. Process the entries of row i in decreasing order of column index: for each column b in row i , pair it with the smallest column $a > b$ appearing in row $i+1$ that has not yet been paired in an earlier step; if no such a exists, b is left unpaired. Let $R_i(R)$ denote the set of unpaired column indices in row i , and $L_i(R)$ the set of unpaired column indices in row $i+1$.

If $R_i(R)$ is nonempty, $f_i(R)$ is obtained by removing the smallest unpaired entry $b \in R_i(R)$ from row i and inserting an entry into row $i+1$ at the largest column $b' < b$ such that b' does not already appear in row $i+1$; if $R_i(R) = \emptyset$ or no such b' exists, $f_i(R) = \emptyset$. Dually, if $L_i(R)$ is nonempty, $e_i(R)$ is obtained by removing the largest unpaired entry $a \in L_i(R)$ from row $i+1$ and inserting an entry into row i at the smallest column $a' > a$ such that a' does not already appear in row i ; if $L_i(R) = \emptyset$ or no such a' exists, $e_i(R) = \emptyset$.

The *weight* of R as an element of the crystal is simply $\mathbf{wt}(R)$, the row weight itself. A *highest weight* element is an RC graph R with $e_i(R) = \emptyset$ for all $i \geq 1$. Every $R \in \mathcal{RC}(w)$ corresponds to a unique highest weight element $\mathbb{Y}(R)$, which can always be obtained by greedily applying available e_i until none apply. The *Demazure crystal* $\text{Dem}(R)$ generated by a highest weight $\mathbb{Y}(R)$ is the closure of $\{\mathbb{Y}(R)\}$ under the operators $\{f_i^k : i \geq 1, k \geq 0\}$ subject to the Demazure truncation rule of [1]. The *extremal weight* $\text{extwt}(R)$ of R is the weak composition that is the lexicographically smallest weight of any element of $\text{Dem}(R)$.

The significance of this elaborate structure is the following.

Theorem 5.1. *If $A \in \mathcal{BRC}$ is such that $\text{extwt}(A) = a$, then*

$$\kappa_a = \sum_{R \in \text{Dem}(A)} x^{\mathbf{wt}(R)}$$

It will be advantageous to have an alternative, crystal-friendly definition of the map $\mathcal{Z}$ for the purposes of proving results about key polynomials.

Definition 5.1.2 (Little bumps). Given a reduced word $a_1 \cdots a_m$ for a permutation w and an index $1 \leq p \leq m$, the *Little bump* $\text{little}_p(a_1 \cdots a_m)$ is defined recursively as follows. If $a_p > 1$, replace a_p with $a_p - 1$, and if $a_p = 1$ then instead replace all a_q with $q \neq p$ with $a_q + 1$. Denote the new word by $a'_1 \cdots a'_m$. If this resulting word is reduced, we are done. Otherwise, there is a unique index $j \neq p$ such that the word obtained by deleting a'_j from the new word is reduced by the deletion property of Coxeter groups. We then define

$$\text{little}_p(a_1 \cdots a_m) = \text{little}_j(a'_1 \cdots a'_m)$$

Definition 5.1.3 (Crystal-friendly transition map). Define a function $\mathcal{L} : \mathcal{BRC}^0(n) \rightarrow \mathcal{BRC}(n+1)$ as follows. Assume $n = \mathbf{m}(w_R)$, and suppose $\mathbf{rt}_R(i, j) = (n, n+1)$, where $i < n$. Set

$$R' = (R \setminus \{(i, j)\}) \cup \{(n, 1)\}$$

Then define

$$\mathcal{L}(R) = (i \overset{n-1}{\rightsquigarrow} R') \setminus \{(n, 1)\}$$

which we assign height $n + 1$. Then define $\mathcal{Z} : \mathcal{BRC}^0(n) \rightarrow \mathcal{BRC}(n - 1)$ by iterating $\mathcal{L}$ and stopping as soon as the result has $\mathbf{m}(w_R) \leq n - 1$, then reducing the number of rows to $n - 1$.

Proposition 5.1.4. *Let $(R, n) \in \mathcal{BRC}^0(n)$ be a bounded RC graph such that $\mathbf{m}(w_R) = n$. Suppose*

$$\text{word}(R) = a_1 a_2 \cdots a_k$$

and

$$\text{seq}(R) = r_1 r_2 \cdots r_k$$

suppose $\mathbf{rt}_R(i, j) = (n, n + 1)$, and let p be the index such that $r_p = i$ and $r_p + j - 1 = a_p$. Then

$$\text{word}(\mathcal{L}(R)) = \text{little}_p(\text{word}(R))$$

Proof. Consider the biword

$$\begin{array}{cccccccc} r_1 & r_2 & \cdots & r_{p-1} & r_p & r_{p+1} & \cdots & r_k \\ a_1 & a_2 & \cdots & a_{p-1} & a_p & a_{p+1} & \cdots & a_k \end{array}$$

The first deletion yields

$$\begin{array}{cccccccc} r_1 & r_2 & \cdots & r_{p-1} & r_{p+1} & \cdots & r_k & n \\ a_1 & a_2 & \cdots & a_{p-1} & a_{p+1} & \cdots & a_k & n \end{array}$$

Necessarily, we must have $a_p > 1$. To see this, note that if we have $a_p = 1$, then we would have $r_p = 1$ by the compatible sequence condition. The only way for position $(1, 1)$ in an RC graph to be the final right descent is if it is the unique right descent. By assumption, that descent would have to be n , so that $n = 1$. In that case, however, it would not be possible for the last row to be empty, which is included in the hypotheses of the proposition.

We must also have that $r_p < a_p$, because there is a sequence of chute moves involving this crossing that moves it to position $(n, 1)$ by [4, Theorem 3.7]. Assume first that $p = k$. Then the insertion algorithm will insert $a_p - 1 = n - 1$ into row r_k . If this word is reduced, then we are done, and this clearly coincides with the Little bump. If it is not reduced, the descent $n - 1$ occurs somewhere to the left, say at position j . The insertion algorithm would delete this letter then try to insert another one. Recursively, this is the same as applying $\mathcal{L}$ to the biword obtained by removing all letters after position j and then reinserting them afterwards. By the inductive hypothesis, this coincides with applying the Little bump at position j , which would indeed be the next step in the recursion of applying the Little bump at position k . Thus, the two coincide by induction. $\square$

We invoke the following theorem of Hamaker and Young.

Theorem 5.2 ([17, Theorem 4.4]). *Let w be a reduced word for a permutation $w \in S_\infty$. Then*

$$Q(w) = Q(\text{little}_p(w))$$

for any valid p , where Q denotes the Edelman-Green recording tableau [14].

Proposition 5.1.5. *We have that*

$$\mathcal{Z} : \mathcal{RC}_n(w)^0 \rightarrow \bigcup_{w \overset{n}{\searrow} w'} \mathcal{RC}_{n-1}(w')$$

is an isomorphism of crystal graphs.

Proof. The fact that this is a weight-preserving bijection follows directly from repeated application of [25, Lemma 7], which states that the Little bump on the last descent realizes the Lascoux-Schützenberger transition formula. The crystal isomorphism follows from Theorem 5.2. $\square$

Proposition 5.1.6. *We have the equality $\mathcal{Z} = \mathcal{Z}$.*

Proof. Let R be a bounded RC graph, say with permutation w , with r rows and last descent r such that row r is empty. We prove this by induction on $\ell(w)$ as well as on $s - r$, where $s > r$ is the maximal index such that $w(r) > w(s)$. If $s = r + 1$, then the definitions of the two maps are identical.

By the exchange property of Coxeter groups, there is a unique pair (i, j) such that $\mathbf{rt}_R(i, j) = (r, r + 1)$. Set

$$R' = R \setminus \{(i, j)\}$$

and assume that $\mathbf{ht}(R') = r + 1$. By the inductive hypothesis, we have $\mathcal{Z}(R') = \mathcal{Z}(R')$. We have thus established that $\mathcal{Z}$ and $\mathcal{Z}$ induce the same bijection between $\mathcal{RC}(ws_r)^0$ and

$$\bigcup_{ws_r \xrightarrow{r+1} w'} \mathcal{RC}(w')$$

Inspection of both algorithms reveals that the removal of this crossing and the subsequent reinsertion in the first step of both algorithms is precisely the same, and holding this fixed then applying the inductive hypothesis yields the result. $\square$

Corollary 5.1.7 ($\mathcal{Z}$ intertwines with crystal operators). *For any bounded RC graph R and any positive integer i satisfying $1 \leq i \leq \mathbf{ht}(R) - 1$, we have*

$$\mathcal{Z}(e_i(R)) = e_i(\mathcal{Z}(R))$$

and

$$\mathcal{Z}(f_i(R)) = f_i(\mathcal{Z}(R))$$

whenever the left-hand sides are defined.

Proof. This follows from Proposition 5.1.6 and the fact that $\mathcal{Z}$ is a crystal isomorphism by Proposition 5.1.5. $\square$

5.2. The dual key basis and the ring of RC graph crystals.

Definition 5.2.1 (The dual key basis). For a weak composition a of length n , define $\kappa_a^* \in \mathcal{D}_n$ to be the unique element such that $\langle \kappa_a^*, \kappa_b \rangle = \delta_{a,b}$ for all weak compositions b .

Theorem 5.3. *Let c be a weak composition. Then the dual key polynomial κ_c^* is equal to*

$$\kappa_c^* = \sum_{\substack{R = \mathbb{Y}(R) \\ \text{extwt}(R) = c}} \Xi_{w_R}^{\mathbf{ht}(R)}$$

Proof. The statement of the theorem is equivalent to Lascoux and Schützenberger’s formula for Schubert polynomials in terms of key polynomials through Theorem 5.1. $\square$

Definition 5.2.2 (The submodule $\Theta \subseteq \mathcal{BRC}$). Let Θ be the $\mathbb{Z}$ -submodule of $\mathcal{BRC}$ generated by the elements $R_1 - R_2$ such that $\mathbb{Y}(R_1) = \mathbb{Y}(R_2)$.

Lemma 5.2.3. Θ is a two-sided ideal of $\mathcal{BRC}$.

Proof. The fact that Θ is a left ideal follows from the fact that the products $R \boxplus R_1$ and $R \boxplus R_2$ have the same first $\mathbf{ht}(R)$ rows, and the rows below are simply shifts of R_1 and R_2 . Since the crystal raising/lowering operators act directly on adjacent rows, the same crystal operators will apply to the rows below, so that $R \boxplus R_1$ and $R \boxplus R_2$ have the same crystal structure in the trailing rows. In passing to highest weights $\mathbb{Y}(R \boxplus R_1)$ and $\mathbb{Y}(R \boxplus R_2)$, we may pass through the partial highest weights $R \boxplus Y(R_1) = R \boxplus Y(R_2)$ by independence of path choice to the highest weight, so that in particular the highest weights of the bijectively corresponding terms of $R \boxplus R_1$ and $R \boxplus R_2$ are the same.

To prove that Θ is a right ideal, the same argument applies, but we must invoke Corollary 5.1.7 to guarantee that the crystals remain isomorphic as needed. $\square$

Lemma 5.2.4. *The homomorphism $\alpha : \mathcal{BRC} \rightarrow \mathcal{D}$ factors through the quotient $\mathcal{BRC}/\Theta$.*

Proof. This is because the generators of Θ are all in Ω . $\square$

Definition 5.2.5. For a composition a , define an element $K_a \in \mathcal{BRC}/\Theta$ by

$$K_a = \sum_{\substack{R \in \text{Dem}_a \\ R = \mathbb{Y}(R)}} R$$

Proposition 5.2.6. The elements $K_a \in \mathcal{BRC}/\Theta$ form an additive basis of a subring isomorphic to $\mathcal{D}$ under the homomorphism $\alpha : \mathcal{BRC}/\Theta \rightarrow \mathcal{D}$, and

$$\alpha(K_a) = \kappa_a^*$$

Proof. The last statement is simply Theorem 5.3. The K_a actually consist of pairwise disjoint sets of RC graphs, the set of all such is clearly independent. To show that they span a subring, consider the homomorphism $\alpha : \mathcal{BRC}/\Theta \rightarrow \mathcal{D}$. We have that $\alpha(K_a) = \kappa_a^*$, and since these are linearly independent, it follows that the kernel of α intersects trivially with the span of the K_a . Since α is a homomorphism, it follows that α is injective on the span of the K_a , and since

$$\alpha(K_a) \sqcup \alpha(K_b) = \kappa_a^* \sqcup \kappa_b^* = \sum_c k_{ab}^c \kappa_c^* = \sum_c k_{a,b}^c \alpha(K_c),$$

it follows that $K_a \sqcup K_b$ is a linear combination of the K_c , so the span of the K_a is closed under multiplication, so that α is an isomorphism and we are done. $\square$

Theorem 5.4 (LR rule for dual keys). Let a, b be weak compositions of lengths p and q , and let c be a weak composition of length $p + q$. Define integers $k_{a,b}^c$ by

$$\kappa_a^* \sqcup \kappa_b^* = \sum_c k_{a,b}^c \kappa_c^*.$$

For any highest-weight RC graph C with $\text{extwt}(C) = c$, let $N(C)$ be the number of RC graphs R with $\mathbb{Y}(R) = C$ such that $\text{clip}^p(R)$ and $\text{trim}^p(R)$ are both highest weights and $\text{extwt}(\text{clip}^p(R)) = a$, $\text{extwt}(\text{trim}^p(R)) = b$. Then $N(C)$ depends only on (a, b, c) , and

$$k_{a,b}^c = N(C);$$

in particular $k_{a,b}^c \in \mathbb{Z}_{\geq 0}$.

Proof of Theorem 5.4. The coefficient of K_c in $K_a \sqcup K_b$ is the same as the coefficient of κ_c^* in $\kappa_a^* \sqcup \kappa_b^*$, which is $k_{a,b}^c$ by definition. Since each of the summands of K_c that are Demazure crystals of RC graphs are distinct from the summands of any other key element and each other, we need only consider one of the summands to read off the coefficient. The result follows. $\square$

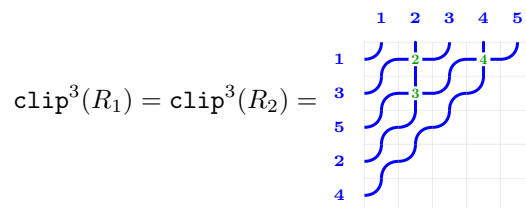
Example 5.2.7 (Computing $\kappa_a^* \sqcup \kappa_b^*$ for $a = (0, 1, 2)$, $b = (0, 1, 0, 2)$). Let $a = (0, 1, 2)$ (a 3-row composition) and $b = (0, 1, 0, 2)$ (a 4-row composition), so $p = 3$, $q = 4$, and $p + q = 7$. We illustrate the coefficient $k_{a,b}^c$ for $c = (0, 0, 0, 1, 2, 0, 3)$.

The permutation w_c whose principal RC graph has Demazure crystal highest weight c is

$$w_c = (1, 2, 3, 5, 7, 4, 10, 6, 8, 9).$$

The key element K_c is the sum of all 7-row RC graphs whose Demazure crystal highest weight equals c . Among these, exactly two satisfy $\text{clip}^3(R) \in \text{Dem}_a$ and $\text{trim}^3(R) \in \text{Dem}_b$:

Both graphs have permutation $w_c = (1, 2, 3, 5, 7, 4, 10, 6, 8, 9)$. They differ only in their first two rows: R_1 has row data $[(6, 4), (9,), ()]$ in the top three rows, while R_2 has $[(9, 4), (6,), ()]$. Applying clip^3 (retaining only the top 3 rows and re-indexing) to each yields the same 3-row graph with extremal weight $a = (0, 1, 2)$:



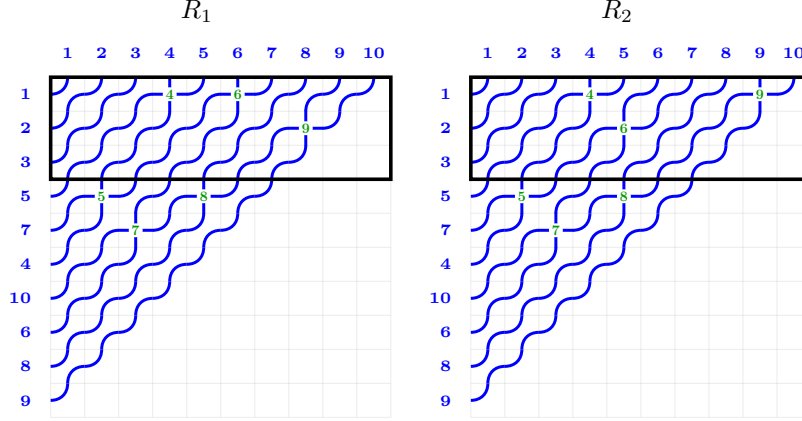
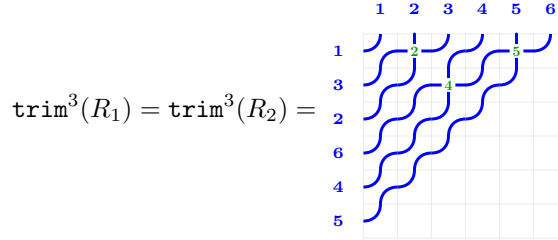


FIGURE 4. The two 7-row RC graphs R_1 (left) and R_2 (right) in Dem_c satisfying $\text{clip}^3(R_i) \in \text{Dem}_a$ and $\text{trim}^3(R_i) \in \text{Dem}_b$. The outlined box marks the top 3 rows (the clip region).

Applying trim^3 (keeping the bottom 4 rows) gives the same 4-row graph with extremal weight $b = (0, 1, 0, 2)$:



Since there are exactly 2 such RC graphs, Theorem 5.4 gives $k_{a,b}^c = 2$.

6. THE RING OF FOREST CLASSES OF RC GRAPHS

The *forest polynomials* $\mathfrak{P}_a$ were introduced by Nadeau and Tewari in [31] and are closely related to Schubert polynomials. They can be defined by reduced words and compatible sequences, but the invariant preserved by these words is distinct from the permutation. Given a reduced word/compatible sequence pair $(\mathbf{r}, \mathbf{s})$, there is an insertion algorithm that produces a corresponding invariant $P_{\mathfrak{P}}(\mathbf{r}, \mathbf{s})$ that represents an equivalence relation on the words, where the compatible sequence can vary arbitrarily. We therefore shorten the notation to $P_{\mathfrak{P}}(\mathbf{r})$. The equivalence classes consist of words that are all for the same permutation, so that if we define

$$R \equiv_{\mathfrak{P}} R'$$

to mean that $P_{\mathfrak{P}}(\text{word}(R)) = P_{\mathfrak{P}}(\text{word}(R'))$, then the forest polynomials partition the set of RC graphs for the given permutation. The forest polynomials are indexed in the original work by “indexed forests,” hence the name. These correspond bijectively to weak compositions, so we can define a weak composition $\mathbf{c}_{\mathfrak{P}}(R)$ by associating an indexed forest to this equivalence class as in the article. Define $\mathcal{C}_{\mathfrak{P}}(a)$ for a composition a to be the set of equivalence classes of RC graphs under this relation such that $\mathbf{c}_{\mathfrak{P}}(R) = a$.

6.1. Indexed forests, forest invariant.

Definition 6.1.1 (Indexed forests and codes). An *indexed forest* F is a pair $(S, T_{\bullet})$ where S is a set of positive integers, called the *support* of F . Writing

$$S = I_1 \cup I_2 \cup \dots \cup I_k$$

where $I_1, I_2, \dots, I_k$ are the maximal intervals contained in S , we have $T_{\bullet}$ as a sequence of binary trees $T_1, T_2, \dots, T_k$ where T_j has $|I_j|$ internal nodes for each j . The internal nodes of F are denoted by $\text{IN}(F)$. For $v \in \text{IN}(F)$, we denote by $\text{left}(v)$ and $\text{right}(v)$ the left and right children of v respectively, which may not be in $\text{IN}(F)$ if they are leaves, or may be empty.

There is a natural labeling $\lambda_F : \text{IN}(F) \rightarrow S$ sending each internal node, say $v \in T_j$, to the unique element of I_j that is at the same index as $v \in T_j$ in the inorder traversal of T_j .

The function $\rho_F : \text{IN}(F) \rightarrow S$ is defined for an internal node $v \in T_j$ by

$$\rho_F(v) = \begin{cases} \lambda_F(v) & \text{if } \text{left}(v) \notin \text{IN}(F) \\ \lambda_F(\text{left}(v)) & \text{if } \text{left}(v) \in \text{IN}(F). \end{cases}$$

For each indexed forest F , we assign a weak composition $\mathbf{c}(F)$ by

$$\mathbf{c}_j(F) = \#\{v \in \text{IN}(F) : \rho_F(v) = j\}$$

Then the *forest polynomial* $\mathfrak{P}_F(x)$ is defined by

$$\mathfrak{P}_F(x) = \sum_{\tau} \prod_{v \in \text{IN}(F)} x_{\tau(v)}$$

where the sum is over all functions $\tau : \text{IN}(F) \rightarrow \mathbb{Z}_{>0}$ such that $\tau(v) \leq \rho_F(v)$ for all $v \in \text{IN}(F)$, $\tau(v) \leq \tau(\text{left}(v))$ for all $v \in \text{IN}(F)$ such that $\text{left}(v) \in \text{IN}(F)$, and $\tau(v) < \tau(\text{right}(v))$ for all $v \in \text{IN}(F)$ such that $\text{right}(v) \in \text{IN}(F)$.

It is easiest for our purposes to index forest polynomials by weak compositions rather than indexed forests, so we define $\mathfrak{P}_a(x) = \mathfrak{P}_F(x)$ where F is the unique indexed forest such that $\mathbf{c}(F) = a$.

Definition 6.1.2 (Injective words, forest invariant). In [32], an alphabet $\overline{\mathbb{Z}}$ is defined as the set of pairs of integers (i, j) in lexicographical order, written with the notation $i^{[j]}$. They define $\text{val}(i^{[j]}) = i$. An *injective word* is a word $w = w_1 w_2 \cdots w_k$ such that $w_i \neq w_j$ for all $i \neq j$. An *LBS (local binary search) labeling* of an indexed forest F is an injective function $\rho : \text{IN}(F) \rightarrow \overline{\mathbb{Z}}$ such that

- (1) $\rho(\text{left}(v)) < \rho(v)$ for all $v \in \text{IN}(F)$ such that $\text{left}(v) \in \text{IN}(F)$.
- (2) $\rho(v) < \rho(\text{right}(v))$ for all $v \in \text{IN}(F)$ such that $\text{right}(v) \in \text{IN}(F)$.
- (3) For all $v \in \text{IN}(F)$, $\text{val}(\rho(v)) = \lambda_F(u)$ for some $u \in \text{IN}(F)$.

Following [31], given any injective word $w = w_1 \cdots w_m$ in $\overline{\mathbb{Z}}$, we associate to it inductively an indexed forest $F(w)$, an LBS labeling $P(w)$ of $F(w)$, and an injective “time-stamp” labeling $Q(w) : \text{IN}(F(w)) \rightarrow \{1, \dots, m\}$ as follows. If $m = 0$, then $F(w)$ is the empty forest. Otherwise write $w = w'a$ with $w' = w_1 \cdots w_{m-1}$, and assume $F' := F(w')$, $P' := P(w')$, and $Q' := Q(w')$ have already been constructed; let S' be the support of F' , and let $S' = I_1 \cup \cdots \cup I_k$ be its decomposition into maximal intervals (listed left to right). For each j , denote by u_j the root of the tree of F' supported on I_j , and set $a_j := P'(u_j)$. We obtain $F(w)$ by adjoining a new internal node u to F' , with attachments determined by $i := \text{val}(a)$:

- If $i \notin S'$, then u is the root of a new tree, with left child u_j when $i - 1 \in I_j$ (otherwise no left child), and right child u_j when $i + 1 \in I_j$ (otherwise no right child).
- If $i \in I_j$, then we compare a with a_j in the order on $\overline{\mathbb{Z}}$:
 - if $a > a_j$, then u_j becomes the left child of u ; additionally, if $\max(I_j) + 2 \in S'$ (necessarily $\max(I_j) + 2 = \min(I_{j+1})$), then u_{j+1} becomes the right child of u ;
 - if $a < a_j$, then u_j becomes the right child of u ; additionally, if $\min(I_j) - 2 \in S'$ (necessarily $\min(I_j) - 2 = \max(I_{j-1})$), then u_{j-1} becomes the left child of u .

The remaining roots of F' become roots of $F(w)$ unchanged. We then extend P' to $P(w)$ by setting $P(w)(u) := a$, and extend Q' to $Q(w)$ by setting $Q(w)(u) := m$. The pair $(P(w), Q(w))$ is the Ω -insertion of w .

The importance of this algorithm is

Theorem 6.1 ([31, Theorem 5.1]). *The correspondence $w \mapsto (P(w), Q(w))$ is a bijection between the following two sets:*

- (1) *Injective words w with letters in $\overline{\mathbb{Z}}$, and*
- (2) *Pairs (P, Q) such that $P \in \text{LBS}(F)$ and $Q \in \text{Dec}(F)$ for a common indexed forest F .*

Definition 6.1.3 (Injectification, $\equiv_{\mathfrak{P}}$). For a word $\mathbf{r}$ in the alphabet $\mathbb{Z}$, we may canonically turn $\mathbf{r}$ into an injective word in the alphabet $\overline{\mathbb{Z}}$ by assigning superscripts consecutively starting at 1 to break ties between equal letters.

Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be reduced words. Then we say that $\mathbf{r}_1 \equiv_{\mathfrak{P}} \mathbf{r}_2$ if $P(\overline{\mathbf{r}_1}) = P(\overline{\mathbf{r}_2})$. Naturally, we may apply this to RC graphs as well: for RC graphs R_1 and R_2 , we say that $R_1 \equiv_{\mathfrak{P}} R_2$ if $\overline{\text{word}(R_1)} \equiv_{\mathfrak{P}} \overline{\text{word}(R_2)}$. For a permutation w , we denote by $\mathcal{C}_{\mathfrak{P}}(w)$ the set of equivalence classes of reduced words for w under $\equiv_{\mathfrak{P}}$.

Theorem 6.2 ([31, Theorem 1.4]). *Let $w \in S_{\infty}$. Then we have*

$$\mathfrak{S}_w(x) = \sum_{C \in \mathcal{C}_{\mathfrak{P}}(w)} \mathfrak{P}_{\mathfrak{c}_{\mathfrak{P}}(C)}(x)$$

6.2. The dual forest basis and the ring of forest classes of RC graphs.

Definition 6.2.1 (The dual forest basis). For a weak composition a of length n , define $\mathfrak{P}_a^* \in \mathcal{D}_n$ to be the unique element such that $\langle \mathfrak{P}_a^*, \mathfrak{P}_b \rangle = \delta_{a,b}$ for all weak compositions b .

We record the result that the coefficients of $\mathfrak{P}_c^*$ in the monomial-dual basis are exactly the forest expansion coefficients of monomials computed by Nadeau, Spink, and Tewari, and that under the Borel-type isomorphism $H^\bullet(\text{QFl}_n) \cong \mathbb{Z}[x_1, \dots, x_n]/\text{QSym}_n^+$ of [6] the same integers compute the cap product against the geometric homology basis $\{[X(F)]\}$ of $H_\bullet(\text{QFl}_n)$.

Proposition 6.2.2. *Let a be a weak composition of length n , and let F_a denote the corresponding indexed forest. Let $\{e_{\mathbf{b}}\}_{\mathbf{b} \in \mathbb{Z}_{\geq 0}^n}$ denote the basis of $\mathcal{D}_n$ dual to the monomial basis $\{x^{\mathbf{b}}\}$ of $\mathcal{A}_n$. Then*

$$\mathfrak{P}_a^* = \sum_{\mathbf{b} \in \mathbb{Z}_{\geq 0}^n} \epsilon_{F_a}(\mathbf{b}) e_{\mathbf{b}} \in \mathcal{D}_n,$$

where $\epsilon_{F_a}(\mathbf{b}) \in \{-1, 0, +1\}$ is the path sign of [30, Proposition 10.15].

The same integers admit two further interpretations:

- (1) (Volume polynomial / Postnikov–Stanley-style dual.) The volume polynomial $V_{F_a}(\boldsymbol{\lambda}) \in \mathbb{Q}[\lambda_1, \dots, \lambda_n]$ of [30, Section 10.5], which is dual to $\mathfrak{P}_{F_a}$ under the divided-power D -pairing of loc. cit., expands as

$$V_{F_a}(\boldsymbol{\lambda}) = \sum_{\mathbf{b} \in \mathbb{Z}_{\geq 0}^n} \epsilon_{F_a}(\mathbf{b}) \frac{\boldsymbol{\lambda}^{\mathbf{b}}}{\mathbf{b}!} \quad ([30, \text{Proposition 10.14}]).$$

Up to the divided-power normalization, $V_{F_a}(\boldsymbol{\lambda})$ and $\mathfrak{P}_a^*$ are the same linear functional on $\mathcal{A}_n$, in exact analogy with the relationship between the Postnikov–Stanley dual Schubert polynomials [34] and our Ξ_u^n noted in the introduction.

- (2) (Geometric Kronecker dual on QFl_n .) Under the Borel isomorphism $\pi_n : \mathcal{A}_n \twoheadrightarrow \mathcal{A}_n/\text{QSym}_n^+ \cdot \mathcal{A}_n \cong H^\bullet(\text{QFl}_n)$ of [6, Theorem A and Corollary 12.5], the dual functional $\mathfrak{P}_a^*$ kills the kernel ideal and descends to the Kronecker dual of $[\mathfrak{P}_{F_a}] \in H^\bullet(\text{QFl}_n)$ paired against the homology basis $\{[X(F)] : F \in \text{Forest}_n\}$ of [6, Corollary 11.6]. Concretely, for $F_a \in \text{Forest}_n$,

$$\mathfrak{P}_a^* = \pi_n^*([X(F_a)]) \quad \text{in } \mathcal{D}_n,$$

and equivalently

$$\int_{X(F_a)} \overline{x^{\mathbf{b}}} = \epsilon_{F_a}(\mathbf{b}) \quad \text{in } H^\bullet(\text{QFl}_n)$$

for all $\mathbf{b} \in \mathbb{Z}_{\geq 0}^n$, where $\overline{x^{\mathbf{b}}} = \pi_n(x^{\mathbf{b}})$. For a with $F_a \notin \text{Forest}_n$ both sides vanish: $\mathfrak{P}_{F_a} \in \text{QSym}_n^+ \cdot \mathcal{A}_n$ by [30, Theorem 9.7], so $\mathfrak{P}_a^*$ vanishes on this submodule.

Proof. The first identity is immediate: applying $\mathfrak{P}_a^*$ to the forest expansion of monomials $x^{\mathbf{b}} = \sum_G \epsilon_G(\mathbf{b}) \mathfrak{P}_G$ [30, Proposition 10.15] gives $\langle \mathfrak{P}_a^*, x^{\mathbf{b}} \rangle = \sum_G \epsilon_G(\mathbf{b}) \langle \mathfrak{P}_a^*, \mathfrak{P}_G \rangle = \epsilon_{F_a}(\mathbf{b})$ by the defining duality $\langle \mathfrak{P}_a^*, \mathfrak{P}_b \rangle = \delta_{a,b}$.

Item (1) is then a direct comparison of [30, Proposition 10.14] with the displayed expansion.

For item (2), the splitting $\mathcal{A}_n = \bigoplus_{F_a \in \text{Forest}_n} \mathbb{Z}\mathfrak{P}_a \oplus \text{QSym}_n^+ \cdot \mathcal{A}_n$ [30, Theorem 9.7] forces $\mathfrak{P}_a^*$ to annihilate the kernel of π_n when $F_a \in \text{Forest}_n$, hence to factor through π_n . The induced functional pairs with $[\mathfrak{P}_b]$ to $\delta_{a,b}$, which is precisely the defining property of $[X(F_a)]$ [6, Theorem 12.12], so $\mathfrak{P}_a^* = \pi_n^*([X(F_a)])$. The second equality of item (2) is then the previous display read modulo $\text{QSym}_n^+ \cdot \mathcal{A}_n$. $\square$

In particular, the algebraic dual basis $\{\mathfrak{P}_a^*\} \subset \mathcal{D}_n$ and the geometric homology basis $\{[X(F)]\} \subset H_\bullet(\mathrm{QFl}_n)$ are not merely Kronecker-dual to the same basis: they are the same linear functional on $\mathcal{A}_n$, viewed once before and once after passing to the Borel quotient. The bialgebra structure on $\mathcal{D}$ and the geometric coproduct on $\bigoplus_n H_\bullet(\mathrm{QFl}_n)$ induced by the variable-splitting comorphism $\iota_{p,q}^*$ of Section 8.5 are correspondingly related: $\pi^* = \bigoplus_n \pi_n^*$ identifies the geometric coproduct with the restriction of the concatenation coproduct on $\mathcal{D}$ to $\bigoplus_n \pi_n^*(H_\bullet(\mathrm{QFl}_n))$.

Theorem 6.3. *We have the following.*

I Let $R \in \mathcal{BRC}$ be a bounded RC graph. Then the forest polynomial $\mathfrak{P}_{\mathfrak{c}_{\mathfrak{P}}(R)}$ is equal to

$$\mathfrak{P}_{\mathfrak{c}_{\mathfrak{P}}(R)}(x) = \sum_{R' \equiv_{\mathfrak{P}} R} x^{\mathfrak{wt}(R')}$$

II Let c be a weak composition. Then the dual forest polynomial $\mathfrak{P}_c^$ is equal to*

$$\mathfrak{P}_c^* = \sum_{[R] \in \mathcal{C}_{\mathfrak{P}}(c)} \Xi_{w_R}^{\mathfrak{ht}(R)}$$

Proof of Theorem 6.3. Part I of this theorem follows from the fact that for any equivalence class of words W that have the same P -LBS, $\mathfrak{P}_{\mathfrak{c}_{\mathfrak{P}}(W)}(x)$ is the sum of the compatible sequences on the words in W ([31, Proposition 5.10]), which are simply the RC graphs in the corresponding equivalence class of RC graphs. Part II follows from Theorem 6.2 and the definition of the dual Schubert basis. $\square$

Guo and Woodruff [16] also published an alternative RC graph formula for forest polynomials.

Proposition 6.2.3. *Let c be a weak composition of length p . Let $\mathcal{RC}_{\mathfrak{P}}(c)$ be the set of RC graphs with $\mathfrak{wt}(R) = c$ such that $\mathrm{word}(\mathbb{Y}_{\mathfrak{P}}(R))$ is minimal in lexicographical order among all RC graphs R' such that $\mathfrak{c}_{\mathfrak{P}}(R) = \mathfrak{c}_{\mathfrak{P}}(R')$.*

$$[c] = \sum_{R \in \mathcal{RC}_{\mathfrak{P}}(c)} \mathfrak{P}_{\mathfrak{c}_{\mathfrak{P}}(R)}^*$$

Remark. The coefficients of the expansion of a monomial into forest polynomials, or equivalently the coefficients of the expansion of a dual forest polynomial into weak composition monomials, is given in [31, Proposition 10.15].

Definition 6.2.4 (The submodule $\Gamma \subseteq \mathcal{BRC}$). Let Γ be the $\mathbb{Z}$ -submodule of $\mathcal{BRC}$ generated by the elements $R_1 - R_2$ such that $R_1 \equiv_{\mathfrak{P}} R_2$.

Lemma 6.2.5. Γ is a two-sided ideal of $\mathcal{BRC}$.

Proof. The fact that Γ is a left ideal follows from the fact that the products $R \sqcup R_1$ and $R \sqcup R_2$ have the same first $\mathfrak{ht}(R)$ rows, and the rows below are simply shifts of R_1 and R_2 . In terms of the resulting words, the shifted $\mathrm{word}(R_1)$ and $\mathrm{word}(R_2)$ are suffixes of $\mathrm{word}(R \sqcup R_1)$ and $\mathrm{word}(R \sqcup R_2)$. The $\equiv_{\mathfrak{P}}$ equivalence is determined by local swaps of pairs of letters that are related in a shift-invariant [31, Proposition 5.8] and prefix-invariant [31, Lemma 5.9] way, from which the result follows, since the prefixes are identical and the suffixes are related in the same way as $\mathrm{word}(R_1)$ and $\mathrm{word}(R_2)$.

To prove that Γ is a right ideal, assume $\mathrm{word}(R_1)$ and $\mathrm{word}(R_2)$ differ by a single local swap of adjacent letters a and b in the same equivalence class, with a to the left of b in $\mathrm{word}(R_1)$ and to the right of b in $\mathrm{word}(R_2)$. It suffices to consider only the case where R is empty to show that

$$R_1 \sqcup R - R_2 \sqcup R \in \Gamma$$

for any R . According to [31], the local swap of a and b is a commutation move, and we may reduce to the case where a and b are the last two letters of $\mathrm{word}(R_1)$, and that b is the last descent of w_{R_1} . For any empty R , the position a will remain the same. For any term not identical to R_1 , the suffix of the word will be ab' where $b' > b$. For each b' , there will be precisely one matching term with suffix $b'a$ and identical prefix. The local swap of a and b' in the suffix is a valid local swap, so the resulting terms are all in Γ , and we are done. $\square$

Lemma 6.2.6. *The homomorphism $\alpha : \mathcal{BRC} \rightarrow \mathcal{D}$ factors through the quotient $\mathcal{BRC}/\Gamma$.*

Proof. This is because the generators of Γ are all in Ω . $\square$

Definition 6.2.7. For a composition a , define an element $F_a \in \mathcal{BRC}/\Gamma$ by

$$F_a = \sum_{[R] \in \mathcal{C}_{\mathfrak{P}}(a)} [R]$$

Proposition 6.2.8. *The elements $F_a \in \mathcal{BRC}/\Gamma$ form an additive basis of a subring isomorphic to $\mathcal{D}$ under the homomorphism $\alpha : \mathcal{BRC}/\Gamma \rightarrow \mathcal{D}$, and*

$$\alpha(F_a) = \mathfrak{P}_a^*$$

Proof. As before, the last statement is simply Theorem 6.3. The F_a actually consist of pairwise disjoint sets of RC graphs, the set of all such is clearly independent. To show that they span a subring, consider the homomorphism $\alpha : \mathcal{BRC}/\Gamma \rightarrow \mathcal{D}$. We have that $\alpha(F_a) = \mathfrak{P}_a^*$, and since these are linearly independent, it follows that the kernel of α intersects trivially with the span of the F_a . Since α is a homomorphism, it follows that α is injective on the span of the F_a , and since

$$\alpha(F_a) \sqcup \alpha(F_b) = \mathfrak{P}_a^* \sqcup \mathfrak{P}_b^* = \sum_c f_{ab}^c \mathfrak{P}_c^* = \sum_c f_{a,b}^c \alpha(F_c),$$

it follows that $F_a \sqcup F_b$ is a linear combination of the F_c , so the span of the F_a is closed under multiplication, so that α is an isomorphism and we are done. $\square$

Theorem 6.4 (LR rule for dual forests). *Let a, b be weak compositions of lengths p and q , and let c be a weak composition of length $p + q$. Define integers $f_{a,b}^c$ by*

$$\mathfrak{P}_a^* \sqcup \mathfrak{P}_b^* = \sum_c f_{a,b}^c \mathfrak{P}_c^*.$$

For any $C \in \mathcal{C}_{\mathfrak{P}}(c)$, let $N(C)$ be the number of $R \in [C]$ such that $\text{clip}^p(R) \in \mathcal{C}_{\mathfrak{P}}(a)$, $\text{trim}^p(R) \in \mathcal{C}_{\mathfrak{P}}(b)$, $\text{wt}(\text{clip}^p(R)) = a$, and $\text{wt}(\text{trim}^p(R)) = b$. Then $N(C)$ depends only on (a, b, c) , and

$$f_{a,b}^c = N(C);$$

in particular $f_{a,b}^c \in \mathbb{Z}_{\geq 0}$.

Proof of Theorem 6.4. The coefficient of F_c in $F_a \sqcup F_b$ is the same as the coefficient of $\mathfrak{P}_c^*$ in $\mathfrak{P}_a^* \sqcup \mathfrak{P}_b^*$, which is $f_{a,b}^c$ by definition. Since each of the summands of F_c that are equivalence classes of RC graphs are distinct from the summands of any other forest element and each other, we need only consider one of the summands to read off the coefficient. The result follows. $\square$

Example 6.2.9 (Computing $\mathfrak{P}_a^* \sqcup \mathfrak{P}_b^*$ for $a = (0, 0, 2, 0, 2)$, $b = (0, 1, 0, 0, 2)$). Let

$$a = (0, 0, 2, 0, 2), \quad b = (0, 1, 0, 0, 2), \quad c = (0, 0, 1, 0, 1, 0, 2, 0, 0, 3).$$

Here $p = q = 5$, so by Theorem 6.4 the coefficient $f_{a,b}^c$ counts the R in a fixed forest class $\mathcal{C}_{\mathfrak{P}}(c)$ of 10-row RC graphs satisfying

$$\text{clip}^5(R) \in \mathcal{C}_{\mathfrak{P}}(a), \quad \text{trim}^5(R) \in \mathcal{C}_{\mathfrak{P}}(b), \quad \text{wt}(\text{clip}^5(R)) = a, \quad \text{wt}(\text{trim}^5(R)) = b.$$

Writing an RC graph as the set of ordered pairs $(i, j) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ recording the (row, column) of each crossing, there are exactly two such R , namely

$$\begin{aligned} R_1 &= \{(3, 1), (3, 2), (5, 3), (5, 8), (7, 1), (10, 1), (10, 2)\}, \\ R_2 &= \{(3, 1), (3, 6), (5, 1), (5, 8), (7, 1), (10, 1), (10, 2)\}. \end{aligned}$$

Both have permutation

$$w_{R_1} = w_{R_2} = (1, 2, 4, 3, 6, 5, 9, 7, 8, 13, 10, 11, 12),$$

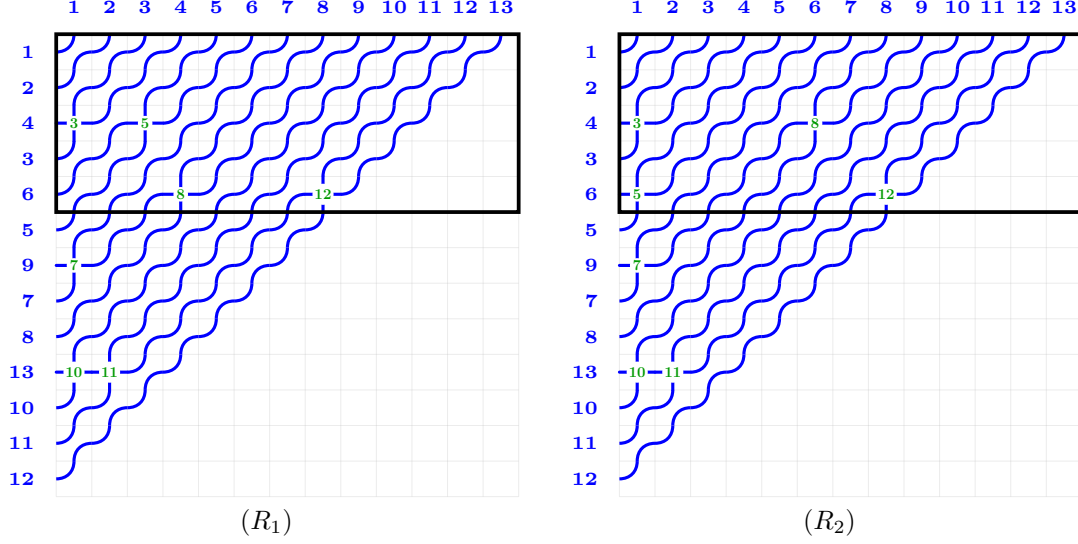
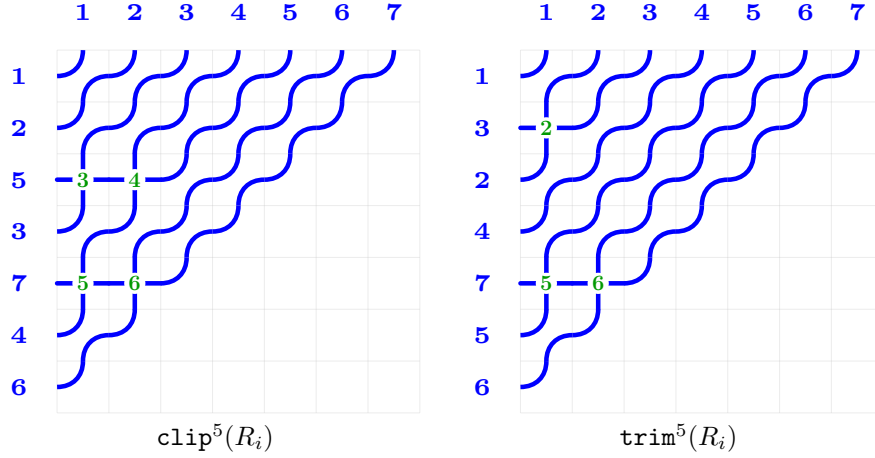
and they share a common clip/trim pair

$$\text{clip}^5(R_i) = \{(3, 1), (3, 2), (5, 1), (5, 2)\}, \quad \text{trim}^5(R_i) = \{(2, 1), (5, 1), (5, 2)\},$$

with $\text{wt}(\text{clip}^5(R_i)) = a$ and $\text{wt}(\text{trim}^5(R_i)) = b$.

Therefore Theorem 6.4 gives

$$f_{a,b}^c = 2.$$

FIGURE 5. The two 10-row witness pipe dreams in the forest class for c .FIGURE 6. Common clip/trim pair for both witnesses R_1, R_2 .

7. PRODUCT RULE FOR DUAL SLIDE POLYNOMIALS

7.1. Slide polynomials.

Definition 7.1.1. For a word $\mathbf{r} = \mathbf{r}_1\mathbf{r}_2\cdots\mathbf{r}_k$, a sequence of positive integers $a_1a_2\cdots a_k$ is *compatible* with $\mathbf{r}$ if $a_i \leq \mathbf{r}_i$ for all i , $a_i \leq a_{i+1}$ for all i , and $a_i < a_{i+1}$ if $\mathbf{r}_i < \mathbf{r}_{i+1}$.

Slide polynomials $\mathfrak{F}(\mathbf{r})$ for a word $\mathbf{r}$ are defined by summing the monomials x^a over all compatible sequences a of $\mathbf{r}$.

Assaf and Searles [2] derived an expansion of Schubert polynomials in terms of slide polynomials as follows. A *quasi-Yamanouchi* RC graph is an RC graph R such that if R' is another RC graph with $\text{word}(R) = \text{word}(R')$ and $\text{ht}(R) = \text{ht}(R')$, then $\text{wt}(R) \leq \text{wt}(R')$ in dominance order and $\text{flat}(\text{wt}(R'))$ refines $\text{flat}(\text{wt}(R))$. The set of RC graphs A such that $\text{word}(A) = \text{word}(R)$ and $\text{ht}(A) = \text{ht}(R)$ is a partially ordered set with R as its smallest element. We then define

$$\mathbb{QY}(A) = R$$

Theorem 7.1 ([2, Theorem 3.13]). *For a permutation $w \in S_\infty$, we have the formula*

$$\mathfrak{S}_w(x) = \sum_R \mathfrak{F}_{\mathbf{wt}(R)}(x)$$

where the sum is over all quasi-Yamanouchi RC graphs R for w .

Corollary 7.1.2. *For a weak composition c of length p , we have the formula*

$$\mathfrak{F}_c^* = \sum_{R=\mathbb{QY}(R)} \Xi_{w_R}^p$$

where the sum is over all quasi-Yamanouchi RC graphs R such that $\mathbf{wt}(R) = c$.

Remark. The expansion of dual slides in the weak composition basis is given by the expansion of monomials into slides in [32, Theorem 5.2].

Definition 7.1.3 (The submodule $\Sigma \subseteq \mathcal{BRC}$). Let Σ be the $\mathbb{Z}$ -submodule of $\mathcal{BRC}$ generated by the elements $R_1 - R_2$ such that $\mathbb{QY}(R_1) = \mathbb{QY}(R_2)$.

Lemma 7.1.4. Σ is a two-sided ideal of $\mathcal{BRC}$.

Proof. That Σ is a left ideal follows from the same mechanism used for Θ : in $R \sqcup R_1$ and $R \sqcup R_2$, the first $\mathbf{ht}(R)$ rows agree, while the trailing rows are shifts of R_1 and R_2 . Since the crystal operators act on adjacent rows, they descend in the same way to the trailing block. The quasi-Yamanouchi map $\mathbb{QY}$ is defined by applying a sequence of crystal raising operators determined by the weight ([2]); hence applying $\mathbb{QY}$ to $R \sqcup R_1$ and to $R \sqcup R_2$ produces graphs whose trailing rows have a common partial quasi-Yamanouchi reduction $R \sqcup \mathbb{QY}(R_1) = R \sqcup \mathbb{QY}(R_2)$, and so $\mathbb{QY}(R \sqcup R_1) = \mathbb{QY}(R \sqcup R_2)$.

That Σ is a right ideal follows from Corollary 5.1.7: multiplication on the right by a fixed R induces an isomorphism of crystal graphs on the relevant subblock, and the $\mathbb{QY}$ images of corresponding terms therefore agree. $\square$

Lemma 7.1.5. *The homomorphism $\alpha : \mathcal{BRC} \rightarrow \mathcal{D}$ factors through the quotient $\mathcal{BRC}/\Sigma$.*

Proof. This is because the generators of Σ are all in Ω : if $\mathbb{QY}(R_1) = \mathbb{QY}(R_2)$ then $w_{R_1} = w_{\mathbb{QY}(R_1)} = w_{\mathbb{QY}(R_2)} = w_{R_2}$ and $\mathbf{ht}(R_1) = \mathbf{ht}(R_2)$, so $\alpha(R_1) = \alpha(R_2)$. $\square$

Definition 7.1.6. For a weak composition a , define an element $\mathcal{F}_a \in \mathcal{BRC}/\Sigma$ by

$$\mathcal{F}_a = \sum_{R:\mathbb{QY}(R)=R, \mathbf{wt}(R)=a} R.$$

Proposition 7.1.7. *The elements $\mathcal{F}_a \in \mathcal{BRC}/\Sigma$ form an additive basis of a subring isomorphic to $\mathcal{D}$ under the homomorphism $\alpha : \mathcal{BRC}/\Sigma \rightarrow \mathcal{D}$, and*

$$\alpha(\mathcal{F}_a) = \mathfrak{F}_a^*.$$

Proof. The last identity is the Corollary above. Distinct $\mathcal{F}_a$ are supported on disjoint sets of quasi-Yamanouchi RC graphs (indexed by their common weight a), so the $\mathcal{F}_a$ are linearly independent in $\mathcal{BRC}/\Sigma$. Considering the homomorphism $\alpha : \mathcal{BRC}/\Sigma \rightarrow \mathcal{D}$, since the $\mathfrak{F}_a^*$ are linearly independent, the kernel of α intersects the span of the $\mathcal{F}_a$ trivially; hence α is injective on $\sum_a \mathbb{Z}\mathcal{F}_a$. Since

$$\alpha(\mathcal{F}_a) \sqcup \alpha(\mathcal{F}_b) = \mathfrak{F}_a^* \sqcup \mathfrak{F}_b^* = \sum_c s_{a,b}^c \mathfrak{F}_c^* = \sum_c s_{a,b}^c \alpha(\mathcal{F}_c),$$

the product $\mathcal{F}_a \sqcup \mathcal{F}_b$ lies in the span of the $\mathcal{F}_c$, so this span is closed under multiplication and the restriction of α is a ring isomorphism. $\square$

Theorem 7.2. *Let a, b be weak compositions of lengths p and q . Write*

$$\mathfrak{F}_a^* \sqcup \mathfrak{F}_b^* = \sum_c s_{a,b}^c \mathfrak{F}_c^*$$

For each c , fix an RC graph C with $\mathbf{wt}(\mathbb{QY}(C)) = c$. Then $s_{a,b}^c$ is the number of distinct RC graphs R with $\mathbf{word}(R) = \mathbf{word}(C)$ with $\mathbf{wt}(\mathbf{clip}^p(R)) = a, \mathbf{wt}(\mathbf{trim}^q(R)) = b$ such that $\mathbf{clip}^p(R), \mathbf{trim}^q(R)$ are quasi-Yamanouchi, and is in particular nonnegative.

Proof. The coefficient of $\mathcal{F}_c$ in $\mathcal{F}_a \sqcup \mathcal{F}_b$ equals the coefficient of $\mathfrak{F}_c^*$ in $\mathfrak{F}_a^* \sqcup \mathfrak{F}_b^*$, namely $s_{a,b}^c$, by Proposition 7.1.7. Expanding the product $\mathcal{F}_a \sqcup \mathcal{F}_b$ via the $(\text{clip}^p, \text{trim}^p)$ -decomposition of $\sqcup$ (Theorem 4.1) and reading off the coefficient of $\mathcal{F}_c$ from a fixed representative C with $\text{wt}(\mathbb{Q}\mathbb{Y}(C)) = c$, the contributing terms are precisely the RC graphs R with $\text{word}(R) = \text{word}(C)$ whose clip and trim have quasi-Yamanouchi weights a and b respectively. $\square$

7.2. A general ideal-quotient Littlewood–Richardson principle. The dual Schubert, dual key, and dual forest LR rules of the previous sections all fit a common template, which we record here.

Theorem 7.3 (General LR principle via ideals of $\mathcal{BRC}$). *Let $\sim$ be an equivalence relation on $\bigsqcup_n \mathcal{RC}_n$ satisfying:*

- (H1) *If $R_1 \sim R_2$ then $\text{ht}(R_1) = \text{ht}(R_2)$ and $w_{R_1} = w_{R_2}$ (in particular $\alpha(R_1) = \alpha(R_2)$).*
- (H2) *$\sim$ is a congruence for $\sqcup$: $R_1 \sim R_2$ implies $R \sqcup R_1 \sim R \sqcup R_2$ and $R_1 \sqcup R \sim R_2 \sqcup R$ for every R .*
- (H3) *clip^p and trim^p descend to $\sim$: if $R_1 \sim R_2$ and $1 \leq p \leq \text{ht}(R_1)$, then $\text{clip}^p(R_1) \sim \text{clip}^p(R_2)$ and $\text{trim}^p(R_1) \sim \text{trim}^p(R_2)$.*
- (H4) *There is an index set J and a function $\lambda : (\mathcal{RC}/\sim) \rightarrow J$ such that, writing $I_\sim = \text{span}_{\mathbb{Z}}\{R_1 - R_2 : R_1 \sim R_2\} \subseteq \mathcal{BRC}$ and*

$$E_\lambda := \sum_{[R]: \lambda([R])=\lambda} [R] \in \mathcal{BRC}/I_\sim,$$

the family $\{\alpha(E_\lambda)\}_{\lambda \in J}$ is a $\mathbb{Z}$ -basis of $\mathcal{D}$.

Then $I_\sim$ is a two-sided ideal of $\mathcal{BRC}$ contained in $\Omega = \ker \alpha$, the elements $\{E_\lambda\}$ span a subring of $\mathcal{BRC}/I_\sim$ isomorphic to $\mathcal{D}$ via α , and the structure constants $b_{\mu,\nu}^\lambda$ defined by

$$\mathfrak{B}_\mu^* \sqcup \mathfrak{B}_\nu^* = \sum_\lambda b_{\mu,\nu}^\lambda \mathfrak{B}_\lambda^*, \quad \mathfrak{B}_\lambda^* := \alpha(E_\lambda),$$

are nonnegative integers. More precisely, for any function $\text{rep}: \mathcal{RC} \rightarrow \mathcal{RC}$ that sends every element of a $\sim$ -class to a single common representative of that class (that is, $\text{rep}(R) \sim R$ for all R , and $\text{rep}(R) = \text{rep}(S)$ whenever $R \sim S$), we have the positive combinatorial formula

$$b_{\mu,\nu}^\lambda = \#\{R \sim C : \text{clip}^p(R) = \text{rep}(\text{clip}^p(R)), \text{trim}^p(R) = \text{rep}(\text{trim}^p(R)), \\ \lambda([\text{clip}^p(R)]) = \mu, \lambda([\text{trim}^p(R)]) = \nu\}$$

for any fixed C with $\lambda([C]) = \lambda$. The count is independent of the choice of rep .

Proof. By (H1) every generator $R_1 - R_2$ of $I_\sim$ lies in $\Omega = \ker \alpha$, so $I_\sim \subseteq \Omega$ and α descends to a surjective ring homomorphism $\bar{\alpha}: \mathcal{BRC}/I_\sim \rightarrow \mathcal{D}$. By (H2), $I_\sim$ is a two-sided ideal of $\mathcal{BRC}$, so $\mathcal{BRC}/I_\sim$ inherits a ring structure.

By construction of E_λ in (H4), each $\sim$ -class $[R]$ appears in E_λ with multiplicity exactly 1 (when $\lambda([R]) = \lambda$) or 0 (otherwise), and the supports of E_λ and $E_{\lambda'}$ in $\mathcal{BRC}/I_\sim$ are disjoint whenever $\lambda \neq \lambda'$; consequently the coefficient $b_{\mu,\nu}^\lambda$ may be read off from the contribution to a single chosen class $[C]$ with $\lambda([C]) = \lambda$. In particular the $\{E_\lambda\}$ are linearly independent. Under $\bar{\alpha}$ they map to the $\mathbb{Z}$ -basis $\{\mathfrak{B}_\lambda^*\}$ of $\mathcal{D}$ by (H4), so $\bar{\alpha}$ restricts to a $\mathbb{Z}$ -module isomorphism $\bigoplus_\lambda \mathbb{Z}E_\lambda \xrightarrow{\cong} \mathcal{D}$. Since $\bar{\alpha}$ is a ring homomorphism, the span of the E_λ is closed under multiplication and the isomorphism is one of rings. Consequently

$$E_\mu \sqcup E_\nu = \sum_\lambda b_{\mu,\nu}^\lambda E_\lambda \quad \text{in } \mathcal{BRC}/I_\sim.$$

To read off $b_{\mu,\nu}^\lambda$, fix any C with $\lambda([C]) = \lambda$ and let $S_1 = \text{rep}(R_1)$, $S_2 = \text{rep}(R_2)$ be the chosen representatives of the μ - and ν -classes (for any R_1, R_2 in those classes). By Theorem 4.1 the product $\sqcup$ on $\mathcal{BRC}$ admits a $(\text{clip}^p, \text{trim}^p)$ -decomposition: for any S_1, S_2 with $\text{ht}(S_1) = p$, $\text{ht}(S_2) = q$, the set $\{R \in \mathcal{RC}_{p+q} : \text{clip}^p(R) = S_1, \text{trim}^p(R) = S_2\}$ enumerates the summands of $S_1 \sqcup S_2$ with multiplicity one. Projecting to $\mathcal{BRC}/I_\sim$, (H3) ensures that the class of R depends only on the classes of $\text{clip}^p(R)$ and $\text{trim}^p(R)$, so the contribution of $S_1 \sqcup S_2$ to the coefficient of $[C]$ in E_λ is exactly the count of $R \sim C$ for which $\text{clip}^p(R) = S_1$ and $\text{trim}^p(R) = S_2$, i.e. for which $\text{clip}^p(R)$ and $\text{trim}^p(R)$ are themselves fixed points of rep and represent the μ - and ν -classes. This is the stated formula. Being a cardinality, it is nonnegative.

Finally, the count is independent of the choice of rep . The element $E_\lambda \in \mathcal{BRC}/I_\sim$ is defined directly as $\sum_{[R]: \lambda([R])=\lambda} [R]$, with no reference to rep , so the structure constants in $E_\mu E_\nu = \sum_\lambda b_{\mu,\nu}^\lambda E_\lambda$ are intrinsic. To

see that the formula computes the correct value for every valid rep, fix classes $[T_1], [T_2]$ with $\lambda([T_1]) = \mu$, $\lambda([T_2]) = \nu$, and consider, for any $S_1 \in [T_1]$ and $S_2 \in [T_2]$, the count

$$N(S_1, S_2) := \#\{R \sim C : \text{clip}^p(R) = S_1, \text{trim}^p(R) = S_2\}.$$

By the $(\text{clip}^p, \text{trim}^p)$ -decomposition of Theorem 4.1, $N(S_1, S_2)$ equals the coefficient of $[C]$ in the product $[S_1] \boxplus [S_2] \in \mathcal{BRC}/I_\sim$ (using (H2) to make the product well-defined and (H3) to ensure each summand of $S_1 \boxplus S_2$ has $\sim$ -class depending only on $[S_1]$ and $[S_2]$). In particular $N(S_1, S_2)$ depends only on the classes $[T_1], [T_2]$ and on $[C]$, not on the choice of representatives S_1, S_2 . The rep-formula selects, for each such class-pair, the single representative pair $(\text{rep}(T_1), \text{rep}(T_2))$ and counts $N(\text{rep}(T_1), \text{rep}(T_2))$; another valid rep' counts $N(\text{rep}'(T_1), \text{rep}'(T_2))$, and these are equal. Summing over class-pairs yields the same total $b_{\mu, \nu}^\lambda$. $\square$

Remark. In the present paper, Theorem 7.3 is instantiated by

- $\sim$ = equality on $\mathcal{RC}$, with $I_\sim = 0$ and $\lambda([R]) = (w_R, \text{ht}(R))$, recovering Theorem 1.2 (dual Schubert basis);
- $\sim$ = Demazure-crystal equivalence, with $I_\sim = \Theta$ and λ the extremal weight, recovering Theorem 5.4 (dual key basis);
- $\sim \equiv \mathfrak{p}$, with $I_\sim = \Gamma$ and $\lambda = \mathfrak{c}\mathfrak{p}$, recovering Theorem 6.4 (dual forest basis);
- $\sim$ = “same reduced word and same ht ” on $\mathcal{RC}$, with $\lambda([R]) = \text{wt}(\mathbb{Q}\mathbb{Y}(R))$, recovering Theorem 7.2 (dual slide basis).

The verifications of (H1)–(H4) in each case are precisely the lemmas and propositions of §2, §5, §6, and §7.

8. THE LITTLEWOOD-RICHARDSON RULE FOR FOREST POLYNOMIALS

In this section we build up the necessary theory to state and prove the Littlewood-Richardson rule for forest polynomials, Theorem 1.1, which expresses the coefficient of an individual forest polynomial in the expansion of a product of two forest polynomials as the cardinality of an explicitly defined set. Explicitly, it is the characterization of the coefficient β_{ab}^c in the equation

$$\mathfrak{P}_a \mathfrak{P}_b = \sum_c \beta_{ab}^c \mathfrak{P}_c$$

enumeratively.

The statement of Theorem 1.1 concerns forest polynomials directly rather than their duals, and although its proof draws on machinery developed in preparation for the dual LR rules, a reader interested only in this result need not work through the earlier LR rules first.

That the coefficients β_{ab}^c are nonnegative integers is not new. Bergeron, Gagnon, Nadeau, Spink, and Tewari [5] prove constructively that the forest polynomials multiply with nonnegative integer (in fact, in the equivariant setting, Graham-positive) structure constants, by extracting them from a positive straightening algorithm: an iterative rewriting procedure in an augmented version of the positive Thompson monoid that produces, given a and b , a positive expansion of $\mathfrak{P}_a \mathfrak{P}_b$. The authors of [5] are explicit, however, that this is not an enumerative rule in the spirit of Schubert calculus: the output of their algorithm depends on the rewriting choices made along the way, and a given β_{ab}^c is recovered only as the cumulative sum of contributions accumulated through the procedure, rather than as the cardinality of any prescribed combinatorial set.

Outline of the proof. The proof proceeds in four steps.

In §8.1 we define an associative product $\boxtimes$ on $\mathcal{BRC}_n$, the *squash product*, built directly out of the row-decomposition operations of §4 (a horizontal concatenation of two RC graphs of common height, followed by clip to recover an n -row representative). The product behaves quite poorly as a global product on $\mathcal{BRC}$. However, on full Grassmannian RC graphs of height n , $\boxtimes$ turns out fortuitously to coincide with the plactic monoid product on semistandard Young tableaux of shape with parts at most n , modulo the customary transposition between RC-graph and tableau conventions (Theorem 8.1). Furthermore, it models the standard theory of the polynomial ring in n variables as a module over the ring of symmetric polynomials with basis the Schubert polynomials indexed by permutations in S_n . This leverage (a fairly well-behaved right module action of the ring of Grassmannian RC graphs of a fixed height) is the key to expressing Schubert multiplication in terms of $\boxtimes$.

The squash product induces a canonical projection $\mathcal{RC}_n : \mathcal{M}_n \rightarrow \mathcal{BRC}_n$, where $\mathcal{M}_n$ is the *multiplactic algebra* obtained by tensoring the height-stratified plactic algebras together.

The multiplactic algebra carries enough structure to express Schubert multiplication. With a specific choice of sum of elementary tensors corresponding to elementary symmetric factorizations, $\mathcal{M}_n$ supports a signed expansion whose image under $\mathcal{RC}_n$ recovers the Schubert structure constants $c_{u,v}^w$ correctly (Theorem 8.3). That is, for permutations u and v , there are elements of $\mathcal{M}_n$ whose images under $\mathcal{RC}_n$ are precisely the sums of RC graphs in $\mathcal{RC}_n(u)$ and $\mathcal{RC}_n(v)$ respectively, with apparently extraneous signed terms in $\mathcal{M}_n$ that cancel out under $\mathcal{RC}_n$, whose product yields the correct RC graphs with the correct multiplicities for the Schubert structure constants $c_{u,v}^w$ upon projection of the product in $\mathcal{M}_n$ to $\mathcal{BRC}_n$. Those familiar with quantum Schubert polynomials and the Fomin-Kirillov algebra may be less surprised by this apparent miracle than those who are not.

This signed model is genuinely *cancellative*: distinct terms in the signed expansion in $\mathcal{M}_n$ that project under $\mathcal{RC}_n$ to the same RC graph with opposite signs, completely canceling, unfortunately can contribute nontrivial terms to the product. That is to say, simply removing these terms causes the correctness to fail, and it is not simply an artifact of presentation.

For forest polynomials, however, the obstruction is weaker. Removing the extraneous terms when we restrict to forest RC graphs in order to model multiplication of forest polynomials squeaks by well enough to provide an LR rule, despite not being consistent enough to descend to a ring structure on labeled indexed forests.

8.1. The multiplactic algebra.

Definition 8.1.1. Let R_1 and R_2 be RC graphs with $\text{ht}(R_1) = \text{ht}(R_2)$. Let N be the position of the last unfixed point of w_{R_1} . We define

$$R_1 \sqcup R_2 = R_1 \cup \{(i, j + N) \mid (i, j) \in R_2\}$$

This is an RC graph with $|R_1 \sqcup R_2| = |R_1| + |R_2|$ and $\text{ht}(R_1 \sqcup R_2) = \text{ht}(R_1) + N$, though at most only the first $\text{ht}(R_1)$ rows are occupied. It is not too difficult to see that

$$w_{R_1 \sqcup R_2} = w_{R_1} \uparrow^N w_{R_2}$$

We then define the *squash product* of R_1 and R_2 to be

$$R_1 \boxtimes R_2 = \text{clip}^{\text{ht}(R_1)}(R_1 \sqcup R_2)$$

Proposition 8.1.2. Let $u \in S_\infty$ be a permutation such that $\mathbf{m}(u) \leq n$ and let v be an n -Grassmannian permutation. Then we have that

$$\boxtimes : \mathcal{RC}_n(u) \otimes \mathcal{RC}_n(v) \rightarrow \bigoplus_{w \in S_\infty} \mathcal{RC}_n(w)^{\oplus c_{uv}^w}$$

is an isomorphism of crystal graphs.

Proof. Clearly the map is weight preserving. We defer the assertion that the image is precisely correct by allowing us to expand the codomain to the set $\mathcal{RC}_n$. Let us show that the function commutes with the crystal operators. Let $R_1 \in \mathcal{RC}_n(u)$ and $R_2 \in \mathcal{RC}_n(v)$, and let $R = R_1 \sqcup R_2$. Consider $f_i(R_1 \boxtimes R_2)$. If $\varphi_i(R_1) > \varepsilon_i(R_2)$, then the unpaired elements in row i in R_1 exhaust the unpaired elements in row $i + 1$ in R_2 . Therefore, only elements of R_1 are affected by f_i , as in the definition of the tensor product. If instead $\varphi_i(R_1) \leq \varepsilon_i(R_2)$, then all unpaired elements of R_1 in row i are paired with elements of R_2 in row $i + 1$, and so only elements of R_2 are affected by f_i . The same reasoning applies to e_i . The fact that the isomorphism induces the specific decomposition follows from [21]. $\square$

Definition 8.1.3. We define an RC graph R of $\text{ht}(R) = k$ to be *full Grassmannian* if w_R is a k -Grassmannian permutation (that is, it has at most one descent, which would be at position k). We define $\mathcal{G}_n$ to be the $\mathbb{Z}$ -submodule of $\mathcal{BRC}_n$ generated by all full Grassmannian RC graphs with $\text{ht}(R) = n$.

We define Plac_n to be the plactic algebra on $[n]$, the free abelian group on the set of words in $[n]$ modulo the Knuth relations. We take as the basis of Plac_n the set of row reading words of semistandard Young tableaux in *French notation* (upside-down).

Recall that the Young/Ferrers diagram of a partition (in French notation) is a set of “boxes” on a grid, with λ_i boxes in row i for the first $\ell(\lambda)$ rows, where the rows are indexed from the bottom up. The notation SSYT_n denotes the set of semistandard Young tableaux, which are labelings of the boxes of such diagrams with elements of $[n]$ such that the numbers strictly increase along the columns and weakly increase along the

rows. The shape of a semistandard Young tableau is the partition corresponding to the underlying Young diagram, and the weight is the weak composition whose i -th part counts the number of boxes labeled i .

Consider the set $I(v)$ of right inversions of an n -Grassmannian permutation v , that is to say the pairs (i, j) with $i < j$ and $v(i) > v(j)$. An RC graph $R \in \mathcal{RC}_n(v)$ corresponds bijectively to a labeling $T_R : I(v) \rightarrow [n]$ by defining

$$T_R(\mathbf{rt}(i, j)) = i$$

for an element $(i, j) \in R$ (see, for example, [3]). If R_0 is the bottom RC graph of v , then there is a bijection between the elements of $I(v)$ and the boxes of the Young diagram of v given by

$$(i, j) \mapsto (i + \ell(\text{shape}(v)) - n, j)$$

Hence there is a linear isomorphism

$$P_n : \mathbb{Z}^{\oplus \text{SSYT}_n} \rightarrow \mathcal{G}_n$$

sending T to the RC graph R such that $T_R = T$.

Lemma 8.1.4. *Suppose R is full n -Grassmannian such that all rows after row r for some $r \leq n$ are empty. Then $\text{clip}^r(R)$ is full r -Grassmannian, and if $T \in \text{SSYT}(n)$ satisfies $P_n(T) = R$, then $P_r(T) = \text{clip}^r(R)$.*

Proof. From Theorem 4.3 and the fact that Schur polynomials are stable, we have that $\text{clip}^r(R)$ is full r -Grassmannian of the same shape and weight. Invoking the more specific Proposition 5.1.6 and theory of Kashiwara crystals, we have that $P_r(T) = \text{clip}^r(R)$. $\square$

Theorem 8.1. *We have that the bilinear map $\boxtimes : \mathcal{G}_n \times \mathcal{G}_n \rightarrow \mathcal{G}_n$ is an associative product such that $P_n : \text{Plac}_n \rightarrow \mathcal{G}_n$ is an isomorphism of rings.*

Proof. Consider the function

$$\tau_n : \text{Plac}_n \otimes \text{Plac}_n \rightarrow \bigoplus_{N=n}^{\infty} \mathcal{BRC}_N$$

defined on basis elements by

$$\tau_n(T_1 \otimes T_2) = P_n(T_1) \sqcup P_n(T_2)$$

Proposition 8.1.2 shows that

$$\text{clip}^n(\tau_n(T_1 \otimes T_2)) = P_n(T_1) \boxtimes P_n(T_2) = P_n(T_1 T_2)$$

due to the rigidity of crystal graph isomorphisms. Since P_n is therefore a homomorphism of (potentially nonassociative) algebras as well as bijective, it is an isomorphism, from which it follows that $\boxtimes$ itself is associative. $\square$

Theorem 8.2. *The module $\mathcal{BRC}_n$ is a right $\mathcal{G}_n$ -module under the product $\boxtimes$. Considering $\mathcal{BRC}_n$ as a right module over Λ_n via the injective homomorphism $\Lambda_n \hookrightarrow \text{Plac}_n \simeq \mathcal{G}_n$, the Λ_n -submodule generated by $\mathcal{S}_w(n)$ for $w \in S_n$ is isomorphic to the polynomial ring $\mathbb{Z}[x_1, \dots, x_n]$ as a Λ_n -module via the homomorphism $\mathcal{S}_w(n) \mapsto \mathfrak{S}_w(x)$, and in particular is a free right Λ_n -module with basis $\{\mathcal{S}_w(n) \mid w \in S_n\}$.*

Proof. The fact that $\mathcal{BRC}_n$ is a right $\mathcal{G}_n$ module follows from Proposition 8.1.2 and the statement that the tensor product of crystal graphs is associative. The injective homomorphism of rings $\Lambda_n \hookrightarrow \text{Plac}_n \simeq \mathcal{G}_n$ is given by sending a Schur polynomial s_λ to the sum of all words of semistandard tableaux of shape λ is realized through P_n . The map $\mathcal{BRC}_n \rightarrow \mathbb{Z}[x_1, \dots, x_n]$ sending R to $x^{\mathbf{rt}(R)}$ is clearly a surjective homomorphism of abelian groups. Restricting to the submodule Schub_n , since $\mathcal{S}_w(n) \mapsto \mathfrak{S}_w^n(x)$ for all w with $\mathbf{m}(w) \leq n$, we obtain an isomorphism of $\mathbb{Z}$ -modules at the very least. Proposition 8.1.2 ensures that the restriction to this submodule is in fact a Λ_n -isomorphism. Thus, the Λ_n -submodule generated by the $\mathcal{S}_w(n)$ is free with basis $\{\mathcal{S}_w(n) \mid w \in S_n\}$. $\square$

Definition 8.1.5 (The multiplactic algebra). Consider the ring $\mathcal{M}_n$ defined as

$$\mathcal{M}_n = \bigotimes_{k=1}^n \mathcal{G}_k$$

We then define a linear homomorphism

$$\mathcal{M}_n \rightarrow \mathcal{BRC}_n$$

by

$$\mathcal{RC}(A_1 \otimes A_2 \otimes \cdots \otimes A_k) = A_1 \boxtimes A_2 \boxtimes \cdots \boxtimes A_k$$

where we increase the size of the left factor to match the right factor as needed, finally resizing the end result to height n .

We may avoid the clunky tensor product notation by noting that we may identify which tensor factor an element of $\mathcal{M}_n$ belongs to by its height, and so we may write elements of $\mathcal{M}_n$ as formal products of full-Grassmannian RC graphs, with the understanding that the k -th factor of the product is the submodule $\mathcal{G}_k$.

For each $1 \leq k \leq n$ and binary vector $\beta \in \{0, 1\}^k$, define $\mathbf{E}^{k;\beta}$ to be the unique k -Grassmannian RC graph for a partition with a single column satisfying

$$\mathbf{wt}_j(\mathbf{E}^{k;\beta}) = \beta_j$$

for $1 \leq j \leq k$. Equivalently, the rows of $\mathbf{E}^{k;\beta}$ that carry a crossing are exactly those indexed by the support of β , namely $\{j \in \{1, \dots, k\} : \beta_j = 1\}$, and the degree (column length) of $\mathbf{E}^{k;\beta}$ is $|\beta| := \beta_1 + \cdots + \beta_k$.

Define $\mathbf{E}_p^k$ by

$$\mathbf{E}_p^k = \sum_{\substack{\beta \in \{0,1\}^k \\ |\beta|=p}} \mathbf{E}^{k;\beta}$$

where the sum is over all binary vectors of length k with exactly p ones, in bijection with the $\binom{k}{p}$ size- p subsets of $\{1, \dots, k\}$.

Proposition 8.1.6. *For any $p_1, p_2, \dots, p_m$ and $k_1 \leq k_2 \leq \dots \leq k_m \leq n$ with $p_i \leq k_i$ for all i , we have that*

$$\mathcal{RC}_n(\mathbf{E}_{p_1}^{k_1} \mathbf{E}_{p_2}^{k_2} \cdots \mathbf{E}_{p_m}^{k_m}) = \sum_w c_w \mathcal{S}_w(n)$$

where $c_w \in \mathbb{Z}_{\geq 0}$ are the coefficients such that

$$e_{p_1, k_1}^n \cdots e_{p_m, k_m}^n = \sum_w c_w \mathfrak{S}_w^n(x) \in \mathcal{A}_n$$

Proof. We prove this by induction on m . For an elementary tensor $A_1 \otimes \cdots \otimes A_m$, the result follows directly from the definition of $\mathcal{RC}_n$. Assuming the inductive hypothesis that this formula holds for tensors of length less than m , then for a tensor of length m we have that

$$\mathcal{RC}_n(\mathbf{E}_{p_1}^{k_1} \mathbf{E}_{p_2}^{k_2} \cdots \mathbf{E}_{p_m}^{k_m}) = \mathcal{RC}_n(\mathbf{E}_{p_1}^{k_1} \mathbf{E}_{p_2}^{k_2} \cdots \mathbf{E}_{p_{m-1}}^{k_{m-1}}) \boxtimes \mathcal{RC}_n(\mathbf{E}_{p_m}^{k_m}) = \sum_{w'} c_{w'}' \mathcal{S}_{w'}(n) \boxtimes \mathbf{E}_{p_m}^{k_m}$$

by Theorem 8.2, since $\mathbf{E}_{p_m}^{k_m} \in \mathcal{G}_{k_m}$, and we are ensured by the increasing order of k_i that the left factor can be clipped to height k_m without the permutation changing. We then apply Proposition 8.1.2 to obtain the result for a single term of this type by induction. Invoking the fact that the image lies in Schub_n , the result follows by applying Theorem 8.2. $\square$

Definition 8.1.7. Suppose $\mathbf{m}(w) \leq n$. We define an element $\mathbf{S}_w(n) \in \mathcal{M}_n$ by

$$\mathbf{S}_w(n) = \sum_{\alpha} E_w^{\alpha; n} \mathbf{E}_{\alpha_1, 1}^1 \mathbf{E}_{\alpha_2, 2}^2 \cdots \mathbf{E}_{\alpha_n, n}^n \mathbf{E}_{\alpha_{n+1}, n}^n \cdots \mathbf{E}_{\alpha_N, n}^n$$

where we recall that $E_w^{\alpha; n}$ represents the coefficient of the elementary basis element of $\mathcal{A}_n$ indexed by the weak composition α in the expansion of $\mathfrak{S}_w^n$. We observe by Proposition 8.1.6 that $\mathcal{RC}_n(\mathbf{S}_w(n)) = \mathcal{S}_w(n)$, so that $\mathbf{S}_w(n)$ is a lift of $\mathcal{S}_w(n)$ to $\mathcal{M}_n$, which usually has signs (*but not always*; see, for example, [43] for a classification of permutations for which $\mathbf{S}_w(n)$ has a single SEM term).

Theorem 8.3. *Let $u, v \in S_{\infty}$ satisfy $\mathbf{m}(u), \mathbf{m}(v) \leq n$. Then*

$$\mathcal{RC}_n(\mathbf{S}_u(n) \mathbf{S}_v(n)) = \sum_w c_{uv}^w \mathcal{S}_w(n)$$

where c_{uv}^w are the Schubert structure constants.

Proof. Explicitly, write

$$\mathbf{S}_u(n)\mathbf{S}_v(n) = \sum_{\alpha, \beta} E_u^{\alpha; n} E_v^{\beta; n} \mathbf{E}_{\alpha_1, 1}^1 \cdots \mathbf{E}_{\alpha_N, n}^n \mathbf{E}_{\beta_1, 1}^1 \cdots \mathbf{E}_{\beta_N, n}^n$$

The relations of the ring allow us to slide smaller terms in $\mathbf{S}_v(n)$ past strictly larger terms in $\mathbf{S}_u(n)$, so we may rearrange the product in increasing order of height. The result then follows from Proposition 8.1.6 and the fact that the image of the element under $\mathcal{RC}_n$ lies in Schub_n , so that we may invoke Theorem 8.2. $\square$

Remark. It may be unsurprising to the reader at this point that the kernel of $\mathcal{RC}_n : \mathcal{M}_n \rightarrow \mathcal{BRC}_n$ is not an ideal. If it were, then the product on $\mathcal{M}_n$ would descend to a term by term product such that

$$\mathcal{S}_u(n) \cdot \mathcal{S}_v(n) = \sum_w c_{uv}^w \mathcal{S}_w(n)$$

and the problem of finding a positive formula for the Schubert structure constants would be solved, due to the fact that $\mathcal{S}_u(n)$ and $\mathcal{S}_v(n)$ have completely nonnegative coefficients. Unfortunately, this is simply not true. The negative terms in $\mathbf{S}_u(n)$ are essential in making the product work.

8.2. The lift product. We now define a canonical representative of an RC graph in $\mathcal{M}_n$, an elementary tensor without multiplicity, that allows us to project back to define an associative product, but as necessitated by the remark above this does not correctly model Schubert polynomial multiplication. However, we will see that it is correct enough to give a Littlewood-Richardson rule for forest polynomials.

Proposition 8.2.1. *Let $u \in S_\infty$ be a permutation and suppose $R \in \mathcal{RC}_n(u)$. Let λ be a partition and let w be the corresponding dominant permutation with $\mathfrak{c}^*(w) = \lambda$. Assume that $\ell(uw^{-1}) = \ell(w) - \ell(u)$. Let c be a weak composition such that $c_i \leq \lambda_{N+1-i}$ for all i , where $N = \ell(\lambda)$. Then the number of sequences of binary vectors $\beta_1, \dots, \beta_N$ such that $\beta_i \in \{0, 1\}^{\lambda_{N+1-i}}$ with $|\beta_i| = c_i$ satisfying*

$$\mathbf{E}^{\lambda_N; \beta_1} \boxtimes \cdots \boxtimes \mathbf{E}^{\lambda_1; \beta_N} = R$$

is equal to the coefficient of x^c in $\mathfrak{S}_{uw^{-1}}(x)$.

Proof. Recall the Cauchy formula for $\mathfrak{S}_w(x; -y)$:

$$\mathfrak{S}_w(x; -y) = \sum_u \mathfrak{S}_u(x) \mathfrak{S}_{uw^{-1}}(y)$$

Also,

$$\mathfrak{S}_w(x; -y) = E_{\lambda_1, \lambda_1}(x; -y_1) \cdots E_{\lambda_N, \lambda_N}(x; -y_N)$$

Replace the factorial elementary symmetric polynomial $E_{p,p}(x; -y_i)$ for each p with

$$\mathbf{E}_p^p(-y_i) \in \mathcal{M}_n[y_1, \dots, y_N],$$

defined by

$$\mathbf{E}_p^p(-y_i) = \sum_{j=0}^p y_i^j \mathbf{E}_{p-j}^p.$$

The result follows from Proposition 8.1.6. $\square$

Definition 8.2.2. Define a function $\text{lift}_n : \mathcal{RC}_n \rightarrow \mathcal{M}_n$ as follows. For an RC graph $R \in \mathcal{RC}_n$, let λ be a partition of the form

$$(\overbrace{n, \dots, n}^p, n-1, n-2, \dots, 1)$$

where p is sufficiently large such that $\ell(w_R \mathfrak{d}_\lambda^{-1}) = \ell(\mathfrak{d}_\lambda) - \ell(w_R)$. Let c be the code of the permutation $w_R \mathfrak{d}_\lambda^{-1}$. Then we define

$$\text{lift}_n(R) = E_1 \cdots E_{\ell(c)}$$

where $|E_i| = \lambda_{N+1-i} - c_{N+1-i}$ and $\text{ht}(E_i) = \lambda_{N+1-i}$ for all i , with E_i all elementary symmetric full Grassmannian RC graphs, which is guaranteed to exist and be unique by Proposition 8.2.1.

Some justification that lift_n is well-defined is in order. For each choice of λ , the existence and uniqueness of the factorization follows from Proposition 8.2.1. However, λ depends on a choice of p “sufficiently large.” We must show that the factorization is independent of this choice. If we increase p by one, then λ is replaced by $\lambda' = (\overbrace{n, \dots, n}^{p+1}, n-1, n-2, \dots, 1)$. The code of $w_R(\mathfrak{d}_{\lambda'})^{-1}$ is then obtained from the code of $w_R(\mathfrak{d}_{\lambda})^{-1}$ by prepending n , so the same factorization works for both choices of p .

Definition 8.2.3. We define a product $*$ on $\mathcal{BRC}_n$ by

$$R_1 * R_2 = \mathcal{RC}_n(\text{lift}_n(R_1)\text{lift}_n(R_2))$$

Lemma 8.2.4 (Weight-additivity of the lift product). *For any $R_1, R_2 \in \mathcal{BRC}_n$,*

$$\mathbf{wt}(R_1 * R_2) = \mathbf{wt}(R_1) + \mathbf{wt}(R_2),$$

where addition of weight vectors is componentwise. Equivalently, the weight-evaluation map

$$\text{ev} : \mathcal{BRC}_n \rightarrow \mathbb{Z}[x_1, \dots, x_n], \quad \text{ev}(R) = x^{\mathbf{wt}(R)},$$

*is a homomorphism of rings from $(\mathcal{BRC}_n, *)$ to the polynomial ring.*

Proof. We first observe that the squash product preserves row weights: by Definition of $\sqcup$, the columns of R_2 are shifted strictly past those of R_1 but their row indices are unchanged, so $\mathbf{wt}(R_1 \sqcup R_2)_i = \mathbf{wt}(R_1)_i + \mathbf{wt}(R_2)_i$ for $i \leq \text{ht}(R_1)$ and 0 otherwise. Since the rows beyond $\text{ht}(R_1)$ of $R_1 \sqcup R_2$ are empty, each iterate of $\mathcal{Z}$ used to form $\text{clip}^{\text{ht}(R_1)}(R_1 \sqcup R_2)$ acts on an empty last row and merely truncates it; in particular the row weights for rows $1, \dots, \text{ht}(R_1)$ are preserved. Hence

$$\mathbf{wt}(R_1 \boxtimes R_2)_i = \mathbf{wt}(R_1)_i + \mathbf{wt}(R_2)_i \quad (1 \leq i \leq \text{ht}(R_1))$$

whenever R_1 and R_2 have the same height. By induction, the same identity holds for any $\boxtimes$ -product of RC graphs of common height: row weights add componentwise.

Now apply this to elementary multiplactic factors. For $\beta \in \{0, 1\}^k$, we have $\mathbf{wt}_j(\mathbf{E}^{k;\beta}) = \beta_j$ by definition. If $\text{lift}_n(R) = E_1 \cdots E_m$ with $E_i = \mathbf{E}^{k_i;\beta_i}$ a full-Grassmannian elementary RC graph, then $R = \mathcal{RC}_n(E_1 \otimes \cdots \otimes E_m)$ is the iterated $\boxtimes$ -product of the E_i .

Padding each E_i by empty rows up to common height n , the row-weight-additivity of $\boxtimes$ established above gives

$$\mathbf{wt}_j(R) = \sum_{i: k_i \geq j} (\beta_i)_j \quad (1 \leq j \leq n).$$

Applying the same identity to $R_1 * R_2 = \mathcal{RC}_n(\text{lift}_n(R_1)\text{lift}_n(R_2))$, whose multiplactic factorization is the concatenation of those of R_1 and R_2 , yields

$$\mathbf{wt}(R_1 * R_2) = \mathbf{wt}(R_1) + \mathbf{wt}(R_2)$$

componentwise. The ring-homomorphism statement is immediate:

$$\text{ev}(R_1 * R_2) = x^{\mathbf{wt}(R_1) + \mathbf{wt}(R_2)} = x^{\mathbf{wt}(R_1)} x^{\mathbf{wt}(R_2)} = \text{ev}(R_1)\text{ev}(R_2).$$

□

Lemma 8.2.5. *Let R be an RC graph and suppose*

$$\text{lift}_n(R) = A_1 \cdots A_n$$

with $\text{ht}(A_i) = i$ for all i . Let $1 \leq j \leq n$ and let

$$R^{\text{left}} = \mathcal{RC}_n(A_1 \cdots A_j)$$

and

$$R^{\text{right}} = \mathcal{RC}_n(A_{j+1} \cdots A_n)$$

Then

$$\text{lift}_n(R^{\text{left}}) = A_1 \cdots A_j$$

and

$$\text{lift}_n(R^{\text{right}}) = A_{j+1} \cdots A_n$$

Proof. This follows from the fact that if a partition λ is split into an initial section λ_1 of length $n - j$ and a final section λ_2 of length j , if we define c_1 and c_2 to be the codes of $w_{R^{\text{left}}} \mathfrak{d}_{\lambda_1}^{-1}$ and $w_{R^{\text{right}}} \mathfrak{d}_{\lambda_2}^{-1}$, then the code of $w_R(\mathfrak{d}_{\lambda_1} \mathfrak{d}_{\lambda_2})^{-1}$ is obtained by concatenating c_1 and c_2 . The result then follows from Proposition 8.2.1. Note that $\mathfrak{d}_{\lambda_1} \mathfrak{d}_{\lambda_2}$ is the dominant permutation with code $\lambda_1 \cup \lambda_2$, where the union is the multiset union considering these as partitions. $\square$

Lemma 8.2.6 (Lift factors at disjoint height ranges commute). *Let $A \in \mathcal{M}_n$ be a tensor of elementary RC factors all of whose heights are at most k , and let $B \in \mathcal{M}_n$ be a tensor of elementary RC factors all of whose heights are strictly greater than k . Then*

$$A \cdot B = B \cdot A \quad \text{in } \mathcal{M}_n,$$

where the product is the lift product $*$ on $\mathcal{BRC}_n$ transferred to $\mathcal{M}_n$ via lift_n . Equivalently, given any RC graph R admitting a lift-factorization $R = R^{\text{left}} * R^{\text{right}}$ with $\text{ht}(R^{\text{left}}) \leq k < \text{ht}(R^{\text{right}})$, the factors R^{left} and R^{right} commute under $*$ with any analogous factors arising from other RC graphs.

Proof. Elementary symmetric RC graphs of different heights commute trivially, and the lift map turns this into commutativity of the corresponding $\boxtimes$ -factors in $\mathcal{M}_n$. $\square$

Theorem 8.4. *The product $*$: $\mathcal{BRC}_n \times \mathcal{BRC}_n \rightarrow \mathcal{BRC}_n$ is associative and is generated by the elementary symmetric full-Grassmannian RC graphs of height at most n .*

Proof. We want to prove that for RC graphs R_1, R_2 , and R_3 ,

$$R_1 * (R_2 * R_3) = (R_1 * R_2) * R_3$$

Assume first that R_3 is elementary symmetric of length k . We may uniquely factorize

$$\begin{aligned} R_1 &= R_1^{\text{left}} * R_1^{\text{right}} \\ R_2 &= R_2^{\text{left}} * R_2^{\text{right}} \end{aligned}$$

where $\text{ht}(R_1^{\text{left}}) \leq k$ and $\text{ht}(R_1^{\text{right}}) > k$, and similarly for R_2^{left} and R_2^{right} . By Lemma 8.2.6, the “left” factors commute with the “right” factors, so we have that

$$\begin{aligned} R_1 * (R_2 * R_3) &= (R_1^{\text{left}} * R_1^{\text{right}}) * ((R_2^{\text{left}} * R_2^{\text{right}}) * R_3) \\ &= (R_1^{\text{left}} * R_2^{\text{left}}) * R_3 * (R_1^{\text{right}} * R_2^{\text{right}}), \end{aligned}$$

and similarly

$$\begin{aligned} (R_1 * R_2) * R_3 &= ((R_1^{\text{left}} * R_1^{\text{right}}) * (R_2^{\text{left}} * R_2^{\text{right}})) * R_3 \\ &= (R_1^{\text{left}} * R_2^{\text{left}}) * R_3 * (R_1^{\text{right}} * R_2^{\text{right}}). \end{aligned}$$

Thus both bracketings are consistently written as

$$(R_1^{\text{left}} * R_2^{\text{left}}) * R_3 * (R_1^{\text{right}} * R_2^{\text{right}}),$$

so they are equal. The key point is that the corresponding lift factors commute, so this regrouping is canonical.

An arbitrary RC graph is itself a product of elementary symmetric RC graphs of weakly increasing heights according to Proposition 8.2.1, so an induction argument proves the result for arbitrary R_3 . $\square$

8.3. The slide quotient.

Proposition 8.3.1. *Let $A = A_1 \otimes \cdots \otimes A_m$ be an elementary tensor of full Grassmannian elementary symmetric polynomial RC graphs in weakly increasing order of height, considered as an element of the tensor product of Demazure crystals. Then*

$$A \mapsto \mathcal{RC}_n(A)$$

extends to an isomorphism of Demazure crystals from the connected component of A in the tensor product to the connected component of $\mathcal{RC}_n(A) \subseteq \mathcal{RC}_n(w_A)$.

Proof. This is seen to be true by invoking Proposition 8.1.2, associativity of the tensor product of crystals, and the fact that the Demazure crystal generated by A is isomorphic to the connected component of A in the case of complete Kashiwara crystals of left to right increasing lengths [21]. $\square$

Definition 8.3.2 (Quasi-crystal structure on $\mathcal{RC}_n(w)$). Define operators $\ddot{e}_i$ and $\ddot{f}_i$ on $\mathcal{RC}_n(w)$ by

$$\ddot{e}_i(R) = \begin{cases} e_i(R) & \text{if } \text{word}(R) = \text{word}(e_i(R)) \\ \perp & \text{otherwise} \end{cases}$$

$$\ddot{f}_i(R) = \begin{cases} f_i(R) & \text{if } \text{word}(R) = \text{word}(f_i(R)) \\ \perp & \text{otherwise} \end{cases}$$

where we interpret $\text{word}(\perp)$ as a formal symbol distinct from every reduced word, so that the word-equality condition fails whenever $e_i(R)$ (resp. $f_i(R)$) is undefined in the underlying crystal. Define the weight map $\mathbf{wt} : R \mapsto \mathbf{wt}(R) \in \mathbb{Z}^n$ (the weight lattice of type A_{n-1} , with simple roots $\alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1}$), and set

$$\ddot{\varepsilon}_i(R) = \begin{cases} \max\{k \geq 0 : \ddot{e}_i^k(R) \neq \perp\} & \text{if } \ddot{e}_i(R) \neq \perp \text{ or } \ddot{f}_i(R) \neq \perp, \\ +\infty & \text{if } \ddot{e}_i(R) = \ddot{f}_i(R) = \perp, \end{cases}$$

$$\ddot{\varphi}_i(R) = \ddot{\varepsilon}_i(R) + \mathbf{wt}_i(R) - \mathbf{wt}_{i+1}(R),$$

with the convention $m + (+\infty) = +\infty$ for $m \in \mathbb{Z}$.

These data equip $\mathcal{RC}_n(w)$ with a seminormal quasi-crystal structure (verified in Proposition 8.3.3). Following [11], we define the quasi-tensor product $A \tilde{\otimes} B$ of two quasi-crystals as the ordinary crystal tensor product, except that whenever i is an index such that $\ddot{\varphi}_i(A) > 0$ and $\ddot{\varepsilon}_i(B) > 0$, we set $\ddot{e}_i(A \tilde{\otimes} B) = \ddot{f}_i(A \tilde{\otimes} B) = \perp$ (undefined).

Proposition 8.3.3. *The structure $(\mathcal{RC}_n(w), \ddot{e}_i, \ddot{f}_i, \mathbf{wt}, \ddot{\varepsilon}_i, \ddot{\varphi}_i)_{i \geq 1}$ defined above is a seminormal quasi-crystal of type A_∞ in the sense of [11, Definitions 3.1, 3.9].*

Proof. We verify the six axioms of [11, Definition 3.1] in turn. Throughout, $R \in \mathcal{RC}_n(w)$ and $i \geq 1$.

Axiom (4) (inverse property). Suppose $\ddot{e}_i(R) = R' \neq \perp$. By definition $e_i(R) = R'$ in the underlying crystal and $\text{word}(R) = \text{word}(R')$. The crystal operators e_i and f_i are mutually inverse partial bijections on $\mathcal{RC}_n(w)$ ([1, 29]), so $f_i(R') = R$. Combined with $\text{word}(R') = \text{word}(R) = \text{word}(f_i(R'))$, this gives $\ddot{f}_i(R') = R$. The converse implication is symmetric.

Axiom (2) (weight law and string increments for $\ddot{e}_i$). If $\ddot{e}_i(R) \neq \perp$ then $\ddot{e}_i(R) = e_i(R)$, and the underlying crystal gives $\mathbf{wt}(\ddot{e}_i R) = \mathbf{wt}(R) + \alpha_i$.

Now consider $\ddot{\varepsilon}_i$. Suppose $\ddot{e}_i(R) = R' \neq \perp$. Then $\ddot{f}_i(R') = R \neq \perp$ by axiom (4), so $\ddot{\varepsilon}_i(R')$ is computed by the max-formula. Iterating axiom (4) shows that $\ddot{e}_i^{j+1}(R) = \ddot{e}_i^j(R')$ for all $j \geq 0$ whenever either side is defined, hence

$$\ddot{\varepsilon}_i(R') = \max\{k \geq 0 : \ddot{e}_i^k(R') \neq \perp\} = \max\{k \geq 0 : \ddot{e}_i^{k+1}(R) \neq \perp\} = \ddot{\varepsilon}_i(R) - 1.$$

For $\ddot{\varphi}_i$, the weight law gives $\mathbf{wt}_i(R') - \mathbf{wt}_{i+1}(R') = \mathbf{wt}_i(R) - \mathbf{wt}_{i+1}(R) + 2$, so by definition

$$\ddot{\varphi}_i(R') = \ddot{\varepsilon}_i(R') + \mathbf{wt}_i(R') - \mathbf{wt}_{i+1}(R') = (\ddot{\varepsilon}_i(R) - 1) + (\mathbf{wt}_i(R) - \mathbf{wt}_{i+1}(R) + 2) = \ddot{\varphi}_i(R) + 1,$$

as required.

Axiom (3) (string increments for $\ddot{f}_i$). By Proposition 3.3 of [11], axiom (3) follows from axioms (2) and (4), both of which we have verified.

Axiom (1) ($\ddot{\varphi}_i - \ddot{\varepsilon}_i = \langle \mathbf{wt}, \alpha_i^\vee \rangle$). For A_{n-1} , $\langle \mathbf{wt}(R), \alpha_i^\vee \rangle = \mathbf{wt}_i(R) - \mathbf{wt}_{i+1}(R)$. If $\ddot{\varepsilon}_i(R)$ is finite then $\ddot{\varphi}_i(R) = \ddot{\varepsilon}_i(R) + \mathbf{wt}_i(R) - \mathbf{wt}_{i+1}(R)$ by definition. If $\ddot{\varepsilon}_i(R) = +\infty$ then $\ddot{\varphi}_i(R) = +\infty + m = +\infty$ for any $m \in \mathbb{Z}$, and the identity holds with the convention $+\infty - +\infty$ interpreted as $\langle \mathbf{wt}(R), \alpha_i^\vee \rangle$ ([11, §2] adopts $m + (\pm\infty) = \pm\infty$, so the axiom in the form $\ddot{\varphi}_i = \ddot{\varepsilon}_i + \langle \mathbf{wt}, \alpha_i^\vee \rangle$ is verified directly).

We must also check $\ddot{\varphi}_i(R) \geq 0$ when finite (seminormality). Suppose $\ddot{\varepsilon}_i(R)$ is finite, so $\ddot{e}_i(R) \neq \perp$ or $\ddot{f}_i(R) \neq \perp$. The $\ddot{e}_i$ -string through R has the form $R_0, R_1, \dots, R_L$ with $R_j = \ddot{f}_i^j(R_0)$, $R = R_{\ddot{\varepsilon}_i(R)}$, $L = \ddot{\varepsilon}_i(R) + \beta$ where $\beta = \max\{k : \ddot{f}_i^k(R) \neq \perp\}$. By axiom (4) the string is well-defined, the word is constant along it, and the weight decreases by α_i at each step. Applying the weight law iteratively, $\mathbf{wt}_i(R_0) - \mathbf{wt}_{i+1}(R_0) = \mathbf{wt}_i(R) - \mathbf{wt}_{i+1}(R) + 2\ddot{\varepsilon}_i(R)$, and $\mathbf{wt}_i(R_L) - \mathbf{wt}_{i+1}(R_L) = \mathbf{wt}_i(R_0) - \mathbf{wt}_{i+1}(R_0) - 2L$. Since $\ddot{e}_i(R_0) = \perp$ and $\ddot{f}_i(R_L) = \perp$ by maximality, we have $\ddot{\varepsilon}_i(R_0) = 0$ and $\ddot{\varphi}_i(R_L)$ computed by definition; iterating axiom (2) backwards from R_L along $\ddot{e}_i$ gives $\ddot{\varphi}_i(R_L) = \ddot{\varphi}_i(R_0) - L$, and combined with the identity at R_0 this yields $\ddot{\varphi}_i(R) = \beta \geq 0$.

Axiom (5). If $\ddot{e}_i(R) = -\infty$ then both operators are $\perp$. By construction $\ddot{e}_i \in \mathbb{Z}_{\geq 0} \cup \{+\infty\}$, so this case does not occur; the axiom is vacuous.

Axiom (6). If $\ddot{e}_i(R) = +\infty$ then both operators are $\perp$. This is true by definition of $\ddot{e}_i$: we set $\ddot{e}_i(R) = +\infty$ precisely when $\ddot{e}_i(R) = \ddot{f}_i(R) = \perp$.

Seminormality. By [11, Definition 3.9], a quasi-crystal is seminormal if $\ddot{e}_i(R), \ddot{\varphi}_i(R) \geq 0$ and either equals $\max\{k : \ddot{e}_i^k(R) \neq \perp\}$ (resp. $\ddot{f}_i^k$) or $+\infty$. The first condition was checked above in the proof of axiom (1). For the second, when $\ddot{e}_i(R)$ is finite it equals the max-formula by definition; when $+\infty$, the convention is met. The same holds for $\ddot{\varphi}_i$ by the argument at the end of axiom (1). $\square$

Theorem 8.5. *Let $\mathcal{C} \subseteq \mathcal{RC}_n(w)$ be a connected component of the quasi-crystal $(\mathcal{RC}_n(w), \ddot{e}_i, \ddot{f}_i)$, and let $R_0 \in \mathcal{C}$ be its unique quasi-Yamanouchi element. Then*

$$\sum_{R \in \mathcal{C}} x^{\mathbf{wt}(R)} = \mathfrak{F}_{\mathbf{wt}(R_0)}(x);$$

that is, the character of $\mathcal{C}$ is the slide polynomial indexed by $\mathbf{wt}(R_0)$.

Proof. By construction $\ddot{e}_i$ and $\ddot{f}_i$ preserve $\mathbf{word}(R)$, so all elements of $\mathcal{C}$ share a common word $\mathbf{r} = \mathbf{word}(R_0)$. Conversely, an arbitrary $R \in \mathcal{RC}_n(w)$ with $\mathbf{word}(R) = \mathbf{r}$ is reached from R_0 by a sequence of crystal operators e_i, f_i that fix the word: each such step changes $\mathbf{wt}$ by moving a single crossing between rows i and $i+1$ without altering inversions, hence is realized by the corresponding $\ddot{e}_i$ or $\ddot{f}_i$. Thus $\mathcal{C} = \{R \in \mathcal{RC}_n(w) : \mathbf{word}(R) = \mathbf{r}\}$.

The map $R \mapsto \mathbf{wt}(R)$ identifies $\mathcal{C}$ with the set of weight vectors realized by such RC graphs. By the BJS compatible-sequence formulation [9], this set is exactly the set of sequences a compatible with $\mathbf{r}$, with $\mathbf{wt}(R_0)$ corresponding to the unique compatible sequence whose flat composition is coarsest. Summing $x^{\mathbf{wt}(R)}$ over $\mathcal{C}$ therefore gives $\sum_{a \text{ compatible with } \mathbf{r}} x^a = \mathfrak{F}(\mathbf{r}) = \mathfrak{F}_{\mathbf{wt}(R_0)}(x)$. $\square$

Lemma 8.3.4 (Quasi-tensor swap by character). *Let X, Y be seminormal quasi-crystals of type A_∞ whose connected components have characters that are slide polynomials (in the sense of Theorem 8.5). Then $X \ddot{\otimes} Y$ and $Y \ddot{\otimes} X$ are isomorphic as quasi-crystals; the isomorphism is in general not natural at the level of the underlying sets, but the multisets of connected components agree.*

Proof. By [11, Theorem 5.1] both $X \ddot{\otimes} Y$ and $Y \ddot{\otimes} X$ are seminormal quasi-crystals, and by [11, Proposition 4.6] each decomposes uniquely (as a quasi-crystal, hence on the level of underlying sets up to choice of isomorphism on each component) into a disjoint union of its connected components. The character of either side is the formal sum, over components, of the slide polynomial of each component (Theorem 8.5); since the weight map is additive under the quasi-tensor product, this character coincides for $X \ddot{\otimes} Y$ and $Y \ddot{\otimes} X$. Slide polynomials $\{\mathfrak{F}_a : a \text{ a weak composition}\}$ are linearly independent in $\mathbb{Z}[[x_1, x_2, \dots]]$ ([2]), so the multiset of slide polynomials appearing in either decomposition is the same, hence the multiset of connected components is the same. Each component is itself a quasi-crystal determined up to isomorphism by its slide polynomial (Theorem 8.5 together with the fact that a connected seminormal type- A_∞ quasi-crystal is determined by its character), so a (non-canonical) component-by-component choice of isomorphism furnishes the asserted quasi-crystal isomorphism $X \ddot{\otimes} Y \cong Y \ddot{\otimes} X$. $\square$

Lemma 8.3.5 (Plactic case of the lift product). *Let A, B be full n -Grassmannian RC graphs in $\mathcal{G}_n$. Then $A * B = A \boxtimes B$, and the map*

$$A \ddot{\otimes} B \longmapsto A * B$$

*extends to an isomorphism of quasi-crystals onto the connected component of $A * B$ in $\mathcal{RC}_n(w_{A*B})$.*

Proof. The identity $A * B = A \boxtimes B$ on $\mathcal{G}_n$ follows from Theorem 8.1: the lift map lift_n , restricted to $\mathcal{G}_n$, is the unique multiplactic factorization compatible with $\boxtimes$, so the product of two lifts under $*$ recovers the $\boxtimes$ -product. Equivalently, $\text{lift}_n(A)\text{lift}_n(B)$ in $\mathcal{M}_n$ has $\mathcal{RC}_n$ -image equal to $\text{clip}^n(A \sqcup B) = A \boxtimes B$.

It thus suffices to prove the quasi-crystal isomorphism for $\boxtimes$ on $\mathcal{G}_n$. Via the ring isomorphism $P_n : \text{Plac}_n \rightarrow \mathcal{G}_n$ of Theorem 8.1, the statement reduces to the plactic algebra: writing $T_A = P_n^{-1}(A)$, $T_B = P_n^{-1}(B) \in \text{SSYT}_n$, the map

$$T_A \otimes T_B \longmapsto T_A \cdot T_B$$

(plactic concatenation followed by Knuth equivalence) is the standard Kashiwara crystal isomorphism from the connected component of $T_A \otimes T_B$ in $B(\mu_A) \otimes B(\mu_B)$ onto $B(\nu) \subseteq B(\mu_A) \otimes B(\mu_B)$, where $\nu = \text{shape}(T_A T_B)$. This is the Littlewood–Richardson / RSK content of the plactic monoid [20, 24].

To upgrade this Kashiwara isomorphism to a quasi-crystal isomorphism, we identify the quasi-crystal structure on $\mathcal{G}_n \subseteq \mathcal{BRC}_n$ with the natural quasi-crystal on injective-word images of SSYT reading words. The reading-word map $T \mapsto \overline{\text{rw}(T)}$ is a quasi-crystal morphism from $(\text{SSYT}_n, \check{e}_i, \check{f}_i)$ (with quasi-crystal operators defined by reading-word preservation) to the quasi-crystal of injective words, and the quasi-tensor of [11, §5] is built precisely so that

$$\overline{\text{rw}(T_A)} \check{\otimes} \overline{\text{rw}(T_B)} \mapsto \overline{\text{rw}(T_A T_B)}$$

becomes a quasi-crystal isomorphism onto its connected component ([11, Theorem 5.11]). Pulling back along P_n and the reading-word map yields the desired quasi-crystal isomorphism $A \check{\otimes} B \rightarrow A * B = A \boxtimes B$. $\square$

Lemma 8.3.6. *The map*

$$\text{lift}_n : \mathcal{RC}_n \rightarrow \mathcal{M}_n$$

is an isomorphism of Demazure crystals onto its image, with elements of $\mathcal{M}_n$ given the crystal tensor product structure in increasing order of height.

Proof. Let $\text{lift}_n(A) = E_1 \otimes \cdots \otimes E_p$ in weakly increasing-height order, the unique factorization guaranteed by Proposition 8.2.1. Proposition 8.3.1 asserts precisely that the map

$$\Phi : E_1 \otimes \cdots \otimes E_p \mapsto \mathcal{RC}_n(E_1 \otimes \cdots \otimes E_p) = A$$

extends to an isomorphism of Demazure crystals. Since Φ is a bijective morphism of Demazure crystals, it admits an inverse Demazure crystal morphism on the target, and that inverse is lift_n : indeed, given A' in the connected component of A , the unique preimage $\Phi^{-1}(A')$ is a tensor of full-Grassmannian elementary symmetric RC graphs in increasing height order *with factors of the same size* whose $\mathcal{RC}_n$ -image is A' , so by the uniqueness clause of Proposition 8.2.1 based on the size of the factors it equals $\text{lift}_n(A')$. $\square$

Proposition 8.3.7 (The lift product is a quasi-crystal isomorphism). *Let A, B be RC graphs. Then the map*

$$A \check{\otimes} B \mapsto A * B$$

*extends to an isomorphism of quasi-crystals from the quasi-tensor product of the connected components of A and B in the seminormal quasi-crystal of Proposition 8.3.3 onto the connected component of $A * B$ in $\mathcal{RC}_n(w_{A*B})$.*

Proof. We proceed in three steps: a reduction to a single elementary symmetric right factor, the Kashiwara argument via disjoint union under a height restriction, and the removal of that restriction.

Step 1: Reduction to B a single elementary symmetric RC graph. Write $\text{lift}_n(B) = E_1 \otimes \cdots \otimes E_k$ in weakly increasing-height order (Proposition 8.2.1). The lift product $*$ on $\mathcal{BRC}$ is associative, since it is the projection under $\mathcal{RC}_n$ of the associative multiplication on $\mathcal{M}_n$ and Proposition 8.2.1 guarantees that $\mathcal{RC}_n$ respects this multiplication on lift factorizations. Likewise the quasi-tensor product $\check{\otimes}$ is associative on seminormal quasi-crystals by [11, Theorem 5.1]. Hence

$$A * B = ((A * E_1) * E_2) * \cdots * E_k, \quad A \check{\otimes} B = ((A \check{\otimes} E_1) \check{\otimes} E_2) \check{\otimes} \cdots \check{\otimes} E_k,$$

where on the right, the quasi-crystal of B is identified with $E_1 \check{\otimes} \cdots \check{\otimes} E_k$ via Lemma 8.3.6. By induction on k , it suffices to prove the proposition when $B = E$ is a single full Grassmannian elementary symmetric RC graph.

Step 2: The case $B = E$ with $\text{ht}(A) \leq \text{ht}(E)$. Work at the common height $n = \text{ht}(E)$, padding A with empty rows if needed. Under this height restriction, E is full n -Grassmannian, A is an arbitrary element of $\mathcal{RC}_n$, and the squash product reduces to a disjoint union followed by a clip: by the definition of $\boxtimes$ in §8.1,

$$A \boxtimes E = \text{clip}^n(A \sqcup E),$$

and on lifts, $\text{lift}_n(A) \cdot E$ in $\mathcal{M}_n$ has $\mathcal{RC}_n$ -image $A \boxtimes E$, so $A * E = A \boxtimes E$ in this case (Lemma 8.3.5 on $\mathcal{G}_n$, extended to general A via $\mathcal{BRC}_n$ being a right $\mathcal{G}_n$ -module under $\boxtimes$, Theorem 8.2).

The map $A \boxtimes E \mapsto A \sqcup E$ is a Kashiwara crystal isomorphism onto the connected component of $A \sqcup E$ in $\mathcal{RC}_n(w_{A \sqcup E})$. This is precisely the pairing argument given in the proof of Proposition 8.1.2: the unpaired row- i crossings of A pair against the unpaired row- $(i+1)$ crossings of E at the seam in exactly the order dictated by the standard Kashiwara tensor rule, so e_i, f_i on $A \sqcup E$ act as on $A \boxtimes E$. (Note that the underlying words *interleave* under $\sqcup - w_{A \sqcup E} = w_A \uparrow^N w_E$ where N is the position of the last unfixed point of w_A – but

this affects only how the seam is identified at the level of the word, not the crystal action.) Postcomposing with the Kashiwara crystal morphism clip^n (Proposition 5.1.6) gives the Kashiwara crystal isomorphism

$$A \otimes E \mapsto A * E$$

onto the connected component of $A * E$ in $\mathcal{RC}_n(w_{A * E})$.

To upgrade to a quasi-crystal isomorphism, restrict on both sides to operators that preserve the underlying word. On the right, this restriction yields exactly the seminormal quasi-crystal of Proposition 8.3.3. On the left, an inspection of the standard Kashiwara tensor rule shows that an operator e_i on $A \otimes E$ alters $w_{A \sqcup E} = w_A \uparrow^N w_E$ iff one of the following occurs:

- it acts on A and changes w_A , i.e. $\check{e}_i(A) = \perp$;
- it acts on E and changes w_E , i.e. $\check{e}_i(E) = \perp$;
- it acts on A while A has an unpaired row- i crossing that, under the shift $\uparrow^N$, would pair with an unpaired row- $(i+1)$ crossing of E – equivalently $\check{\varphi}_i(A) > 0$ and $\check{e}_i(E) > 0$ (a seam inversion).

The first two cases are exactly the gating by $\check{e}_i$ on the individual factors, and the third is the quasi-tensor seam rule of CGM [11, §5]. Hence the word-preserving restriction of $A \otimes E$ is $A \check{\otimes} E$, and the restricted isomorphism is a quasi-crystal isomorphism $A \check{\otimes} E \rightarrow A * E$.

Step 3: Removing the height constraint. Suppose $\text{ht}(A) > \text{ht}(E)$. Write $\text{lift}_n(A) = A_1 \otimes \cdots \otimes A_m$ in weakly increasing-height order (Proposition 8.2.1); then $A = A_1 * A_2 * \cdots * A_m$. Set

$$A^{\text{left}} = A_1 * \cdots * A_\ell, \quad A^{\text{right}} = A_{\ell+1} * \cdots * A_m,$$

where ℓ is the largest index with $\text{ht}(A_\ell) \leq \text{ht}(E)$ (so $\text{ht}(A^{\text{left}}) \leq \text{ht}(E) < \text{ht}(A^{\text{right}})$), with the obvious conventions if $\ell = 0$ or $\ell = m$). By associativity of $*$, $A = A^{\text{left}} * A^{\text{right}}$ and

$$(8.3.1) \quad A * E = A^{\text{left}} * A^{\text{right}} * E.$$

We now build the quasi-crystal isomorphism $A \check{\otimes} E \rightarrow A * E$ in two stages, viewing the abstract quasi-tensor in its given order $A^{\text{left}} \check{\otimes} A^{\text{right}} \check{\otimes} E$ throughout. We do *not* claim that this quasi-tensor is naturally isomorphic, in the order $A^{\text{left}} \otimes A^{\text{right}} \otimes E$, to one in any other order at the level of abstract Kashiwara crystals – it generally is not. What we claim is that $*$ itself is well-defined on the abstract quasi-tensor regardless of the order of factors (since $*$ on $\mathcal{BRC}$ is associative), and that the resulting map onto $A * E$ is a quasi-crystal isomorphism. The naturality the abstract tensor lacks is supplied by $*$, which ‘snaps’ the factors into the canonical lift order of $A * E$.

Stage 3a (handling A^{right}): For each individual factor A_j with $j > \ell$, we have $\text{ht}(A_j) > \text{ht}(E)$, so the roles of A and E in Step 2 reverse: the squash product $E \boxtimes A_j$ at the common height $\text{ht}(A_j)$ realizes $E * A_j$, and the same disjoint-union/clip pairing argument provides a quasi-crystal isomorphism $E \check{\otimes} A_j \rightarrow E * A_j$. Iterating across $j = \ell + 1, \dots, m$ and using that $\check{\otimes}$ is associative on seminormal quasi-crystals [11, Theorem 5.1], we obtain a quasi-crystal isomorphism $E \check{\otimes} A^{\text{right}} \rightarrow E * A^{\text{right}}$. By Lemma 8.3.4, the abstract quasi-tensors $A^{\text{right}} \check{\otimes} E$ and $E \check{\otimes} A^{\text{right}}$ are isomorphic as quasi-crystals (non-canonically); composing this swap isomorphism with $E \check{\otimes} A^{\text{right}} \rightarrow E * A^{\text{right}}$ yields a quasi-crystal isomorphism

$$\Theta_{\text{right}} : A^{\text{right}} \check{\otimes} E \longrightarrow E * A^{\text{right}}.$$

The non-naturality of the swap is harmless: what is preserved under Θ_{right} is exactly the quasi-crystal structure (weights, $\check{e}_i, \check{f}_i$), which is all we need.

Stage 3b (handling A^{left}): Now apply Step 2 with A replaced by A^{left} to obtain a quasi-crystal isomorphism $A^{\text{left}} \check{\otimes} E \rightarrow A^{\text{left}} * E$. By Lemma 8.3.6 the canonical decompositions $\text{lift}_n(A^{\text{left}})$ and $\text{lift}_n(A^{\text{right}})$ are quasi-crystal isomorphic to A^{left} and A^{right} respectively (restricting the Demazure crystal isomorphism to word-preserving operators). Multiplying the Stage 3a result by A^{left} on the left and the Stage 3b result by A^{right} on the right and combining via associativity of $*$ and of $\check{\otimes}$, we obtain a quasi-crystal isomorphism

$$A^{\text{left}} \check{\otimes} A^{\text{right}} \check{\otimes} E \longrightarrow A^{\text{left}} * A^{\text{right}} * E = A * E$$

by (8.3.1). The source identifies with $A \check{\otimes} E$ via Lemma 8.3.6 applied to $A = A^{\text{left}} * A^{\text{right}}$, completing the construction.

This completes the proof for $B = E$, and combined with Step 1 establishes the proposition. $\square$

Theorem 8.6. *Let A, B be quasi-Yamanouchi RC graphs. Then*

$$\mathfrak{F}(A) * \mathfrak{F}(B) = \sum_R \mathfrak{F}(R)$$

where the sum is over all pairs A', B' with $\text{word}(A') = \text{word}(A)$, $\text{word}(B') = \text{word}(B)$ such that

$$A' * B' = R$$

with R being quasi-Yamanouchi.

Proof. For each such pair (A', B') , the connected component of $A' * B'$ in the seminormal quasi-crystal $\mathcal{RC}_n(w_{A' * B'})$ of Proposition 8.3.3 is itself a quasi-crystal, and by [11, Proposition 4.6] contains a unique quasi-Yamanouchi element. By the local tensor rule of Proposition 8.3.7, every element of that component has the form $A'' * B''$ with A'' in the quasi-crystal component of A' and B'' in that of B' ; in particular, the words of A'' and B'' agree with those of A and B respectively. Summing slide polynomials over the quasi-Yamanouchi representatives of these components yields exactly the stated expression. $\square$

From this we may recover [2, Theorem 5.11].

8.4. The forest polynomial LR rule. We now prove the LR rule for forest polynomials (Theorem 1.1), which is the main motivation for introducing the lift product.

Definition 8.4.1 (Forest RC graph). A *forest RC graph* is an RC graph such that $\mathbf{c}_{\mathfrak{P}}(R) = \mathbf{wt}(R)$.

A few technical results are required to close the loop.

Proposition 8.4.2. *Let $A, B \in \mathcal{BRC}_n$ with injectified words $W_A = \overline{\text{word}(A)}$ and $W_B = \overline{\text{word}(B)}$. Write σ_A, σ_B for their slide classes, regarded as connected components of the quasi-crystal of injective words on $\overline{\mathbb{Z}}$. Then:*

- (1) *For any $A' \in \sigma_A$, $B' \in \sigma_B$ (i.e. $\overline{\text{word}(A')} \in \sigma_A$, $\overline{\text{word}(B')} \in \sigma_B$), the injectified word $\overline{\text{word}(A' * B')}$ lies in the same slide class as some shuffle $X \in \text{Sh}(\overline{\text{word}(A')}, \overline{\text{word}(B')})$.*
- (2) *Conversely, every slide class appearing in $\text{Sh}(W_A, W_B)$ is realized as $\overline{\text{word}(A_0 * B_0)}$ (up to slide-equivalence) for some $A_0 \in \sigma_A$, $B_0 \in \sigma_B$.*

Proof. The injectified-word map $R \mapsto \overline{\text{word}(R)}$ is a morphism of quasi-crystals from $\mathcal{BRC}_n$ to injective words on $\overline{\mathbb{Z}}$, sending each connected component (a slide class of RC graphs) bijectively to a connected component (a slide class of words); the unique quasi-Yamanouchi representatives correspond. Likewise, the multiplication of slide polynomials at the word level satisfies

$$\mathfrak{F}(W_A) \mathfrak{F}(W_B) = \sum_{X \in \text{Sh}(W_A, W_B)} \mathfrak{F}(X),$$

so the multiset of slide classes appearing in $\text{Sh}(W_A, W_B)$ is determined by the slide-polynomial product $\mathfrak{F}(W_A) \mathfrak{F}(W_B)$.

By Theorem 8.6, the slide-polynomial identity

$$\mathfrak{F}(A) * \mathfrak{F}(B) = \sum_R \mathfrak{F}(R)$$

holds with R ranging over a multiset of quasi-Yamanouchi representatives, one for each connected quasi-crystal component appearing among the various lift products $A' * B'$ as (A', B') varies subject to $\overline{\text{word}(A')} = \overline{\text{word}(A)}$, $\overline{\text{word}(B')} = \overline{\text{word}(B)}$. (Each such component is a quasi-crystal by Proposition 8.3.7 and contains a unique quasi-Yamanouchi element by [11, Proposition 4.6].) Composing with $R \mapsto \overline{\text{word}(R)}$ identifies that multiset with the slide classes appearing in $\mathfrak{F}(W_A) \mathfrak{F}(W_B)$, hence with the slide classes in $\text{Sh}(W_A, W_B)$. This is precisely the bijection in (2), and (1) is its forward direction. $\square$

Remark. Proposition 8.4.2 does *not* say that $\overline{\text{word}(A * B)}$ is a shuffle of W_A and W_B , only that it is quasi-crystal isomorphic to one.

Lemma 8.4.3. *Let $A, B \in \mathcal{BRC}_n$. Then the set*

$$S = \{A' * B' \mid A \equiv_{\mathfrak{P}} A', B \equiv_{\mathfrak{P}} B'\}$$

is stable under forest equivalence: if $R \in S$ and $R \equiv_{\mathfrak{P}} R'$, then $R' \in S$.

Proof. We reduce to the shuffle-stability statement of Nadeau–Tewari [31, proof of Theorem 6.3], which establishes that for any forest classes Φ_1, Φ_2 of injective words on $\mathbb{Z}$, the multiset

$$\text{Sh}(\Phi_1, \Phi_2) = \bigsqcup_{W_1 \in \Phi_1, W_2 \in \Phi_2} \text{Sh}(W_1, W_2)$$

is closed under $\equiv_{\mathfrak{P}}$. Their argument uses two ingredients: (a) shuffles are by definition closed under transpositions of two adjacent letters that come from different factors, hence under any cross-boundary single-step forest move; and (b) [31, Lemma 5.10] (anti-monoid congruence: if $UabV \equiv_{\mathfrak{P}} UbaV$ via a single generating commutation, then $U_1abV_1 \equiv_{\mathfrak{P}} U_1baV_1$ for any subwords $U_1 \subseteq U, V_1 \subseteq V$), which handles single-step moves whose pair of letters lies entirely within one factor.

Now suppose $R = A_0 * B_0 \in S$ with $A_0 \equiv_{\mathfrak{P}} A$ and $B_0 \equiv_{\mathfrak{P}} B$, and suppose $R \equiv_{\mathfrak{P}} R'$. By Proposition 8.4.2(1), there exists a shuffle

$$X_0 \in \text{Sh}(\overline{\text{word}(A_0)}, \overline{\text{word}(B_0)}) \subseteq \text{Sh}(\Phi_A, \Phi_B)$$

slide-equivalent to $\overline{\text{word}(R)}$, where Φ_A, Φ_B denote the forest classes of $\overline{\text{word}(A)}, \overline{\text{word}(B)}$. Slide equivalence is finer than forest equivalence (slide-equivalent words have identical slide polynomial and hence belong to the same forest class), so $X_0 \equiv_{\mathfrak{P}} \overline{\text{word}(R)} \equiv_{\mathfrak{P}} \overline{\text{word}(R')}$. The shuffle-stability statement of Nadeau–Tewari applied to $X_0 \in \text{Sh}(\Phi_A, \Phi_B)$ then gives

$$\overline{\text{word}(R')} \in \text{Sh}(\Phi_A, \Phi_B),$$

i.e. $\overline{\text{word}(R')} \in \text{Sh}(W_1, W_2)$ for some $W_1 \in \Phi_A$ and $W_2 \in \Phi_B$. Pick any RC graphs $A_1, B_1 \in \mathcal{BRC}_n$ with $\overline{\text{word}(A_1)} = W_1$ and $\overline{\text{word}(B_1)} = W_2$; by definition of Φ_A, Φ_B we have $A_1 \equiv_{\mathfrak{P}} A$ and $B_1 \equiv_{\mathfrak{P}} B$.

By Proposition 8.4.2(2), the slide class of $\overline{\text{word}(R')}$ in $\text{Sh}(W_1, W_2)$ is realized as $\overline{\text{word}(A_2 * B_2)}$ for A_2 slide-equivalent to A_1 and B_2 slide-equivalent to B_1 . In particular $A_2 \equiv_{\mathfrak{P}} A_1 \equiv_{\mathfrak{P}} A$ and $B_2 \equiv_{\mathfrak{P}} B_1 \equiv_{\mathfrak{P}} B$, and R' is slide-equivalent (hence forest-equivalent) to $A_2 * B_2$. Replacing A_2, B_2 by quasi-crystal-isomorphic representatives realizing the actual element R' within its slide class — which is possible by Theorem 8.6 since the lift product is a quasi-crystal isomorphism on the relevant component, and quasi-crystal operators preserve forest equivalence at the word level by [31, Proposition 5.8] (shift-invariance) — yields $A_3 \equiv_{\mathfrak{P}} A$ and $B_3 \equiv_{\mathfrak{P}} B$ with $R' = A_3 * B_3$. Thus $R' \in S$. $\square$

Lemma 8.4.4. *Let $R \in \mathcal{RC}_n$, and suppose $\mathfrak{c}_{\mathfrak{P}}(R) = c$. Then there is precisely one RC graph C satisfying $C \equiv_{\mathfrak{P}} R$ and $\mathfrak{wt}(C) = c$.*

Proof. This is the minimal compatible κ with $\mathfrak{c}_{\mathfrak{P}}(\kappa) = c$, which is unique by [31, Lemma 3.12]. $\square$

Lemma 8.4.5. *Let c be a weak composition of length n . Then there is precisely one forest class of RC graphs in $\mathcal{RC}_n(w)$ for the permutation w such that $\mathfrak{c}(w) = c$ that has forest code c .*

Proof. Considering the word of the bottom RC graph of the above w , we have $\mathfrak{c}_{\mathfrak{P}}(\text{word}(\text{bottom}(w))) = \mathfrak{c}(w) = c$. Thus such a forest class exists. If there were another distinct forest class, the leading term of $\mathfrak{S}_w(x)$ would have multiplicity greater than 1, hence this is not possible by Lemma 8.4.4. $\square$

Proof of Theorem 1.1. Fix forest RC graphs A, B with $\mathfrak{c}_{\mathfrak{P}}(A) = \mathfrak{wt}(A) = a$ and $\mathfrak{c}_{\mathfrak{P}}(B) = \mathfrak{wt}(B) = b$. By Theorem 6.3(I), the forest polynomial $\mathfrak{P}_a$ is the weight generating function over the forest class of A :

$$\mathfrak{P}_a(x) = \sum_{A' \equiv_{\mathfrak{P}} A} x^{\mathfrak{wt}(A')}, \quad \text{and similarly} \quad \mathfrak{P}_b(x) = \sum_{B' \equiv_{\mathfrak{P}} B} x^{\mathfrak{wt}(B')}.$$

Multiplying these expressions in $\mathbb{Z}[x_1, \dots, x_n]$ gives

$$\mathfrak{P}_a(x) \mathfrak{P}_b(x) = \sum_{\substack{A' \equiv_{\mathfrak{P}} A \\ B' \equiv_{\mathfrak{P}} B}} x^{\mathfrak{wt}(A')} x^{\mathfrak{wt}(B')} = \sum_{\substack{A' \equiv_{\mathfrak{P}} A \\ B' \equiv_{\mathfrak{P}} B}} x^{\mathfrak{wt}(A') + \mathfrak{wt}(B')}.$$

By Lemma 8.2.4 (weight-additivity of $*$), each summand can be rewritten as $x^{\mathfrak{wt}(A' * B')}$, yielding

$$\mathfrak{P}_a(x) \mathfrak{P}_b(x) = \sum_{\substack{A' \equiv_{\mathfrak{P}} A \\ B' \equiv_{\mathfrak{P}} B}} x^{\mathfrak{wt}(A' * B')}.$$

Let $T = \{A' * B' : A' \equiv_{\mathfrak{P}} A, B' \equiv_{\mathfrak{P}} B\}$ (as a multiset). By Lemma 8.4.3, T is closed under forest equivalence: every $R \in T$ has its full forest class $[R]_{\equiv_{\mathfrak{P}}}$ contained in T . Group the sum by forest class to obtain

$$\mathfrak{P}_a(x) \mathfrak{P}_b(x) = \sum_{[R]} m_{[R]} \sum_{R'' \in [R]} x^{\mathbf{wt}(R'')},$$

where $m_{[R]}$ is the (constant) number of pairs (A', B') with $A' \equiv_{\mathfrak{P}} A$, $B' \equiv_{\mathfrak{P}} B$, and $A' * B'$ landing in the class $[R]$ at any specified representative; this multiplicity is independent of the choice of representative by the quasi-crystal isomorphism of Proposition 8.3.7, which in particular ensures that isomorphic classes have equal cardinality.

Applying Theorem 6.3(I) once more, the inner sum $\sum_{R'' \in [R]} x^{\mathbf{wt}(R'')}$ is exactly $\mathfrak{P}_{\mathbf{c}_{\mathfrak{P}}(R)}(x)$, so

$$\mathfrak{P}_a(x) \mathfrak{P}_b(x) = \sum_{[R]} m_{[R]} \mathfrak{P}_{\mathbf{c}_{\mathfrak{P}}(R)}(x).$$

The final step is to identify which forest classes $[R]$ contribute and to count $m_{[R]}$ explicitly. Fix a target weak composition c and a representative forest RC graph R with $\mathbf{c}_{\mathfrak{P}}(R) = \mathbf{wt}(R) = c$ (i.e. a *forest* RC graph with forest code c). For any pair (A', B') with $A' \equiv_{\mathfrak{P}} A$ and $B' \equiv_{\mathfrak{P}} B$, the lift product $A' * B'$ lies in the forest class of R if and only if $A' * B' \equiv_{\mathfrak{P}} R$; by Lemma 8.4.3, this is equivalent to the existence of some pair (A'', B'') with $A'' * B'' = R$, $A'' \equiv_{\mathfrak{P}} A$, $B'' \equiv_{\mathfrak{P}} B$. Hence the coefficient β_{ab}^c of $\mathfrak{P}_c$ in $\mathfrak{P}_a \mathfrak{P}_b$ equals the number of such pairs (A'', B'') . We make the canonical choices as in Lemma 8.4.5 above, which is

$$\mathbf{c}(w_A) = \mathbf{c}_{\mathfrak{P}}(A) = a, \quad \mathbf{c}(w_B) = \mathbf{c}_{\mathfrak{P}}(B) = b,$$

so the count of pairs (A'', B'') with $A'' * B'' = R$ is exactly as stated in the theorem. $\square$

Example 8.4.6. We illustrate Theorem 1.1 by computing the coefficient of $\mathfrak{P}_{(4,3,2)}$ in the product

$$\mathfrak{P}_{(0,2,3)} \cdot \mathfrak{P}_{(2,0,2)}.$$

By the theorem, this coefficient $\beta_{(0,2,3),(2,0,2)}^{(4,3,2)}$ is the number of pairs (A, B) of RC graphs satisfying $\mathbf{c}(w_A) = \mathbf{c}_{\mathfrak{P}}(A) = (0, 2, 3)$ and $\mathbf{c}(w_B) = \mathbf{c}_{\mathfrak{P}}(B) = (2, 0, 2)$ such that $A * B$ is a forest RC graph R with $\mathbf{c}_{\mathfrak{P}}(R) = \mathbf{wt}(R) = (4, 3, 2)$. There turn out to be exactly two such pairs.

First pair. Drawing RC graphs as pipe dreams (rows indexed from the top, columns from the left, with crossings shown as $+$ tiles and elbows as bumped arcs), the first pair is

$$A_1 = \begin{array}{|c|c|c|c|c|} \hline & 2 & 3 & & \\ \hline & & & 5 & \\ \hline 3 & 4 & & & \\ \hline \end{array}, \quad B_1 = \begin{array}{|c|c|c|c|} \hline 1 & 2 & & \\ \hline & 3 & 4 & \\ \hline & & & \\ \hline \end{array},$$

corresponding to the row tuples $A_1 = ((3, 2), (5,), (4, 3))$ and $B_1 = ((2, 1), (4, 3), (,))$, with weights $\mathbf{wt}(A_1) = (2, 1, 2)$ and $\mathbf{wt}(B_1) = (2, 2, 0)$. Here, in the schubmult row-tuple notation, an entry s in row r records a crossing in column $s - r + 1$, and the entries within each row are listed right-to-left (largest column first). The multiplactic lifts of A_1 and B_1 in $\mathcal{G}_3 \otimes \mathcal{G}_4 \otimes \mathcal{G}_5$ and $\mathcal{G}_1 \otimes \mathcal{G}_3 \otimes \mathcal{G}_4$ (respectively) are

$$A_1 \mapsto \begin{array}{|c|c|c|} \hline & 2 & \\ \hline & & 5 \\ \hline 3 & 4 & \\ \hline \end{array} \otimes \begin{array}{|c|c|} \hline & 3 \\ \hline & 4 \\ \hline \end{array} \otimes \begin{array}{|c|c|c|} \hline & & 5 \\ \hline & & \\ \hline \end{array},$$

$$B_1 \mapsto \begin{array}{|c|} \hline 1 \\ \hline \end{array} \otimes \begin{array}{|c|c|} \hline 2 & \\ \hline & 3 \\ \hline \end{array} \otimes \begin{array}{|c|c|c|} \hline & & 4 \\ \hline & & \\ \hline \end{array},$$

encoding A_1 and B_1 as elementary tensors in the multiplactic algebra $\mathcal{M}_n$. Their lift product is

$$A_1 * B_1 = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & & \\ \hline & & & 5 & 6 & \\ \hline 3 & 4 & & & & \\ \hline \end{array}, \quad \mathbf{wt}(A_1 * B_1) = (4, 3, 2),$$

the forest RC graph with row tuples $((4, 3, 2, 1), (6, 5, 3), (4, 3))$.

Second pair. The second pair is

$$A_2 = \begin{array}{|c|c|c|c|} \hline 2 & 3 & & \\ \hline 3 & 4 & 5 & \\ \hline & & & \\ \hline \end{array}, \quad B_2 = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline & 3 \\ \hline & 4 \\ \hline \end{array},$$

with $A_2 = ((3, 2), (5, 4, 3), ())$ and $B_2 = ((2, 1), (), (4, 3))$, weights $\mathbf{wt}(A_2) = (2, 3, 0)$ and $\mathbf{wt}(B_2) = (2, 0, 2)$. Their multiplactic lifts are

$$A_2 \mapsto \begin{array}{|c|c|} \hline 2 & \\ \hline 3 & \\ \hline & \\ \hline \end{array} \otimes \begin{array}{|c|c|c|} \hline & 3 & \\ \hline & 4 & \\ \hline & & \\ \hline \end{array} \otimes \begin{array}{|c|c|c|c|} \hline & & 5 & \\ \hline & & & \\ \hline & & & \\ \hline \end{array},$$

$$B_2 \mapsto \begin{array}{|c|} \hline 1 \\ \hline \end{array} \otimes \begin{array}{|c|c|} \hline 2 & \\ \hline 3 & \\ \hline & \\ \hline \end{array} \otimes \begin{array}{|c|c|} \hline & 4 \\ \hline & \\ \hline & \\ \hline \end{array}.$$

Their lift product is

$$A_2 * B_2 = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 2 & 3 & 4 & 5 \\ \hline 3 & & & \\ \hline \end{array}, \quad \mathbf{wt}(A_2 * B_2) = (4, 3, 2),$$

the forest RC graph with row tuples $((4, 3, 2, 1), (5, 3, 2), (4, 3))$.

Both forest products are forest RC graphs of weight $(4, 3, 2)$, and one verifies that no other pair (A, B) with the prescribed forest codes produces a forest RC graph of this weight. Hence

$$\beta_{(0,2,3),(2,0,2)}^{(4,3,2)} = 2,$$

so the coefficient of $\mathfrak{P}_{(4,3,2)}$ in $\mathfrak{P}_{(0,2,3)} \cdot \mathfrak{P}_{(2,0,2)}$ is 2.

Notice that even in this small example, we have already run into a case where we have two distinct representatives for the corresponding equivalence class. Indeed, while we have achieved a consistent rule in this case by computing products in an associative algebra, it does not descend to a term-by-term associative product for labelings of indexed forests.

Corollary 8.4.7 (Dual forest coproduct). *Let c be a weak composition of length n . Then in $\mathcal{D}_n \otimes \mathcal{D}_n$,*

$$\Delta^*(\mathfrak{P}_c^*) = \sum_{a,b} \beta_{ab}^c \mathfrak{P}_a^* \otimes \mathfrak{P}_b^*,$$

where β_{ab}^c is the forest Littlewood–Richardson coefficient of Theorem 1.1. In particular, the dual forest coproduct has nonnegative integer structure constants in the dual forest basis, and these are exactly the structure constants of the forest polynomial LR rule.

Proof. Exactly as in the proof of Theorem 3.2, pairing the expansion $\Delta^*(\mathfrak{P}_c^*) = \sum_{a,b} \lambda_{a,b} \mathfrak{P}_a^* \otimes \mathfrak{P}_b^*$ with $\mathfrak{P}_a \otimes \mathfrak{P}_b$ and using the defining identity $\langle \Delta^*(\alpha), x \otimes y \rangle = \langle \alpha, xy \rangle$ together with the duality $\langle \mathfrak{P}_a^*, \mathfrak{P}_b \rangle = \delta_{a,b}$ yields

$$\lambda_{a,b} = \langle \mathfrak{P}_c^*, \mathfrak{P}_a \mathfrak{P}_b \rangle = \langle \mathfrak{P}_c^*, \sum_{c'} \beta_{ab}^{c'} \mathfrak{P}_{c'} \rangle = \beta_{ab}^c,$$

where the second equality is Theorem 1.1. $\square$

8.5. Geometric interpretation: cup product on the quasisymmetric flag variety. The forest polynomials acquire a geometric meaning through the recent work of Bergeron, Gagnon, Nadeau, Spink, and Tewari [6], in which the authors introduce the *quasisymmetric flag variety* $\mathbf{QFl}_n \subset \mathbf{Fl}_n$. This is a (reducible) toric complex built as a union

$$\mathbf{QFl}_n = \bigcup_{T \in \text{Tree}_n} X(T)$$

of left-translated Richardson varieties indexed by n -leaf planar binary trees. Their main results give a Borel-type presentation

$$H^\bullet(\mathbf{QFl}_n) \cong \mathbb{Z}[x_1, \dots, x_n] / \text{QSym}_n^+,$$

where QSym_n^+ is the ideal of quasisymmetric polynomials with no constant term [6, Theorem A and Corollary 12.5], and identify the forest polynomials $\{\mathfrak{P}_F : F \in \text{Forest}_n\}$ as a free $\mathbb{Z}$ -basis of this ring [6, Corollary 12.5], Kronecker dual to the homology basis $\{[X(F)] : F \in \text{Forest}_n\}$ of $H_\bullet(\text{QFl}_n)$ [6, Corollary 11.6, Theorem 12.12]. Under the standard bijection between Forest_n and weak compositions of length n (compare [31]), the basis $\mathfrak{P}_F$ matches our composition-indexed basis $\mathfrak{P}_a$.

In this geometric language, the product $\mathfrak{P}_a \mathfrak{P}_b$ in $\mathcal{A}_n$ corresponds to the cup product of cohomology classes in $H^\bullet(\text{QFl}_n)$, and Theorem 1.1 translates immediately into the following statement.

Corollary 8.5.1 (Cup product of forest classes on QFl_n). *Let $[\mathfrak{P}_a], [\mathfrak{P}_b] \in H^\bullet(\text{QFl}_n)$ denote the cohomology classes corresponding to the forest polynomials $\mathfrak{P}_a, \mathfrak{P}_b$ under the Borel isomorphism of [6, Theorem A]. Then*

$$[\mathfrak{P}_a] \cup [\mathfrak{P}_b] = \sum_c \beta_{ab}^c [\mathfrak{P}_c] \quad \text{in } H^\bullet(\text{QFl}_n),$$

where $\beta_{ab}^c \in \mathbb{Z}_{\geq 0}$ is the cardinality

$$\beta_{ab}^c = \# \left\{ (A, B) : \begin{array}{l} \mathbf{c}(w_A) = \mathbf{c}_{\mathfrak{P}}(A) = a, \quad \mathbf{c}(w_B) = \mathbf{c}_{\mathfrak{P}}(B) = b, \\ A * B = R \text{ a forest RC graph with } \mathbf{c}_{\mathfrak{P}}(R) = \mathbf{wt}(R) = c \end{array} \right\}.$$

Equivalently, the intersection number $\int_{[X(F_c)]} [\mathfrak{P}_a] \cup [\mathfrak{P}_b]$, where $F_c \in \text{Forest}_n$ corresponds to the composition c under [6, Corollary 11.6, Theorem 12.12], equals β_{ab}^c .

Proof. The Borel isomorphism $\mathcal{A}_n = \mathbb{Z}[x_1, \dots, x_n] \twoheadrightarrow H^\bullet(\text{QFl}_n)$ of [6, Theorem A and Corollary 12.5] is a surjective ring homomorphism that sends the forest-polynomial basis of $\mathcal{A}_n$ to the cohomology classes $[\mathfrak{P}_a]$, and is in particular a bijection on this basis. Applying it to the identity $\mathfrak{P}_a \mathfrak{P}_b = \sum_c \beta_{ab}^c \mathfrak{P}_c$ of Theorem 1.1 gives the displayed cup product formula. The intersection-number reformulation follows from the Kronecker duality between $\{[\mathfrak{P}_c]\}$ and the homology basis $\{[X(F_c)]\}$ [6, Theorem 12.12]. $\square$

Remark. Corollary 8.5.1 should be read in the same spirit as Theorem 1.1: positivity of the cup-product structure constants is implicit in [5, 6], and a positive combinatorial extraction of them via the straightening algorithm of [5] is already available. The new content is the realization of each coefficient as a single cardinality of an explicit set of pairs of RC graphs. The analogous question for $H^\bullet(\text{Fl}_n)$ in the Schubert basis is the classical (and open) problem of producing such a count for the Schubert structure constants, and the contrast is striking: the forest equivalence $\equiv_{\mathfrak{P}}$ on RC graphs is rigid enough to permit a single-cardinality formula on QFl_n , whereas the corresponding rigidification on Fl_n does not seem to be available.

Branching comorphism on the quasisymmetric flag variety. The dual forest LR rule (Theorem 6.4) admits a parallel geometric reading on QFl_n . Using the Borel-type presentation $H^\bullet(\text{QFl}_n) \cong \mathbb{Z}[x_1, \dots, x_n] / \text{QSym}_n^+$ [6, Theorem A and Corollary 12.5], the variable-splitting substitution

$$\rho_{p,q} : \mathbb{Z}[x_1, \dots, x_{p+q}] \longrightarrow \mathbb{Z}[x_1, \dots, x_p] \otimes \mathbb{Z}[x_{p+1}, \dots, x_{p+q}]$$

descends to a well-defined ring homomorphism

$$\iota_{p,q}^* : H^\bullet(\text{QFl}_{p+q}) \longrightarrow H^\bullet(\text{QFl}_p) \otimes H^\bullet(\text{QFl}_q).$$

Indeed, for $f \in \text{QSym}_{p+q}^+$ the standard alphabet-sum identity for the QSym coproduct writes $\rho_{p,q}(f) = \sum f_{(1)}(x_1, \dots, x_p) \otimes f_{(2)}(x_{p+1}, \dots, x_{p+q})$ with each $f_{(1)}, f_{(2)}$ quasisymmetric in its variables; since $\deg f > 0$ and the degrees add, each term has at least one tensor factor in QSym^+ , so $\rho_{p,q}(f) \in \text{QSym}_p^+ \otimes \mathbb{Z}[y] + \mathbb{Z}[x] \otimes \text{QSym}_q^+$. The splitting reformulation of Theorem 6.4 then says exactly that, in the forest-polynomial basis,

$$\iota_{p,q}^* [\mathfrak{P}_c] = \sum_{a,b} f_{a,b}^c [\mathfrak{P}_a] \otimes [\mathfrak{P}_b] \quad \text{in } H^\bullet(\text{QFl}_p) \otimes H^\bullet(\text{QFl}_q),$$

and Theorem 6.4 gives an explicit positive cardinality formula for each $f_{a,b}^c$.

The map $\iota_{p,q}^*$ is the natural quasisymmetric-flag analogue of the type-A Levi comorphism $H^\bullet(\text{Fl}_{p+q}) \rightarrow H^\bullet(\text{Fl}_p) \otimes H^\bullet(\text{Fl}_q)$ of [36], and the dual forest LR rule plays for it the role that Theorem 1.2 plays for the Schubert basis. We do not know whether $\iota_{p,q}^*$ is induced by a geometric embedding $\text{QFl}_p \times \text{QFl}_q \hookrightarrow \text{QFl}_{p+q}$ compatible with the Levi inclusion $\text{Fl}_p \times \text{Fl}_q \hookrightarrow \text{Fl}_{p+q}$; the natural concatenation operation $\text{Forest}_p \times \text{Forest}_q \rightarrow \text{Forest}_{p+q}$ on indexed forests, and the resulting matching $X(F) \times X(F') \subset \text{QFl}_{p+q}$ one expects on the toric Richardson side, suggest that such an embedding should exist, but we do not pursue the geometric verification here. At the level of cohomology rings $\iota_{p,q}^*$ is in any case a well-defined ring homomorphism, and Theorem 6.4

is, in this language, a manifestly positive branching Schubert calculus rule on the quasisymmetric flag variety, parallel to the one Theorem 1.2 provides on Fl_n . Dually, the family $\{\iota_{p,q}^*\}_{p,q \geq 0}$ assembles into a graded coproduct on $\bigoplus_n H_\bullet(\text{QFl}_n)$ whose structure constants in the geometric forest basis $\{[X(F)]\}$ are again the $f_{a,b}^c$ of Theorem 6.4.

NOTATION AND CONVENTIONS

For the reader's convenience we collect here, in one place, the principal symbols used throughout the article. All notation is introduced in context where it is needed; this list is meant as a navigational aid rather than a definition.

Permutations and combinatorial objects.

S_∞ : The group of permutations of $\mathbb{Z}_{>0}$ fixing all but finitely many elements.

$\ell(w)$: The Coxeter length of $w \in S_\infty$.

$\mathbf{c}(w)$, $\mathbf{c}^*(w)$: The (right) Lehmer code of w ; the code of w^{-1} .

$\mathbf{m}(w)$: The position of the last descent of w .

$\mathfrak{d}_\lambda$: The dominant permutation with $\mathbf{c}^*(\mathfrak{d}_\lambda) = \lambda$.

IndexedForests_n : The set of indexed forests on $[n]$ in the sense of Nadeau–Tewari [31]; matched bijectively with Comp_n via $F \leftrightarrow \mathbf{c}(F)$, so we use $a \in \text{Comp}_n$ and $F_a \in \text{IndexedForests}_n$ interchangeably.

Forest_n : The subset of IndexedForests_n corresponding to the homology basis $\{[X(F)]\}$ of $H_\bullet(\text{QFl}_n)$ in BGNST [6, Cor. 11.6]; in NST language, those F with $n \notin \text{LT}(F)$.

Tree_n : The set of n -leaf planar binary trees, identified with the irreducible components of QFl_n .

$\mathcal{RC}_n$: The set of bounded RC graphs with n rows (Definition 4.1.1).

$\mathcal{RC}_n(w)$: The set of $R \in \mathcal{RC}_n$ with $w_R = w$.

$\mathcal{RC}_n^0$: The set of $R \in \mathcal{RC}_n$ whose last row is empty (§4).

w_R : The permutation of an RC graph R .

$\text{ht}(R)$: The number of rows of R .

$\text{wt}(R)$: The row weight composition of R .

$\text{word}(R)$: The reduced word read from R .

Row maps on RC graphs.

$\mathcal{Z}$: The transition map $\mathcal{RC}_n^0 \rightarrow \mathcal{RC}_{n-1}$ of Definition 4.3.1; it realizes the Lascoux–Schützenberger transition formula on the last row.

$\text{trim}^p(R)$: “Truncate”: the RC graph obtained from the last $\text{ht}(R) - p$ rows of R (§4.4).

$\text{clip}^p(R)$: “Clip”: the RC graph in $\mathcal{RC}_p$ obtained from R by an insertion algorithm based on $\mathcal{Z}$; the dual partner of trim^p (§4.4).

$\uparrow^p$: Shift indices up by p (applied to permutations or letters).

Crystal and forest structures.

e_i, f_i : Crystal raising and lowering operators on RC graphs (Definition 5.1.1).

$\text{Dem}(R)$: The Demazure crystal of R (Assaf–Schilling; Definition 5.1.1).

Dem_a : $\{R : \text{extwt}(R) = a\}$, the set of RC graphs with extremal weight a (§5).

$\text{extwt}(R)$: The extremal weight of $\text{Dem}(R)$, i.e. the weight of its unique lowest weight element (Definition 5.1.1).

$\mathbb{Y}(R)$: The highest-weight representative of R in its Demazure crystal (Definition 5.1.1).

$\mathbb{Y}_{\mathfrak{F}}(R)$: The forest analog of $\mathbb{Y}(R)$: the canonical representative of R in its forest equivalence class, used in the dual forest LR rule (§6).

$P_{\mathfrak{F}}(R)$: The forest invariant of R , an indexed forest (Nadeau–Tewari; §6).

$\mathbf{c}_{\mathfrak{F}}(R) (= \mathbf{c}_{\mathfrak{F}}(R))$: The composition associated to $P_{\mathfrak{F}}(R)$ (§6). The forms $\mathbf{c}_{\mathfrak{F}}(R)$ and $\mathbf{c}_{\mathfrak{F}}(R)$ are used interchangeably.

$\mathcal{C}_{\mathfrak{F}}(a)$: The set of forest classes of RC graphs with $\mathbf{c}_{\mathfrak{F}} = a$ (§6).

$\equiv_{\mathfrak{P}}$: Forest equivalence: $R \equiv_{\mathfrak{P}} R'$ iff $P_{\mathfrak{P}}(R) = P_{\mathfrak{P}}(R')$ (§6).

$\mathcal{RC}_{\mathfrak{P}}(c)$: The set of $R \in \mathcal{RC}_n$ with $\mathbf{wt}(R) = c$ such that $\text{word}(\mathbb{Y}_{\mathfrak{P}}(R))$ is lex-minimal among R' with $\mathbf{c}_{\mathfrak{P}}(R') = \mathbf{c}_{\mathfrak{P}}(R)$ (Theorem 6.3).

The Schubert bialgebra and its dual.

$\mathcal{A}_n$: The polynomial ring $\mathbb{Z}[x_1, \dots, x_n]$ (§2).

$\mathcal{A}$: The Schubert bialgebra $\bigoplus_{n \geq 0} \mathcal{A}_n$, with $\mathcal{A}_m \cdot \mathcal{A}_n = 0$ when $m \neq n > 0$ (§2).

$\mathcal{D}$: The graded dual bialgebra of $\mathcal{A}$ (§3).

$\mathfrak{S}_u^n$: Schubert polynomial in $\mathcal{A}_n$, $\mathbf{m}(u) \leq n$ (§2).

Ξ_u^n : Element of $\mathcal{D}$ dual to $\mathfrak{S}_u^n$ (§3).

ev : The evaluation map $\mathcal{BRC} \rightarrow \mathcal{A}$ defined by $\text{ev}(R) = x^{\mathbf{wt}(R)}$ on basis elements (§4).

κ_a, κ_a^* : Key polynomial / its dual; $a \in \text{Comp}_n$ (§5).

$\mathfrak{P}_a, \mathfrak{P}_a^*$: Forest polynomial / its dual; $a \in \text{Comp}_n$ (§6).

$x_a, [a], e_{\mathbf{b}}$: The monomial $x^a = x_1^{a_1} \cdots x_n^{a_n} \in \mathcal{A}_n$; its dual basis element in $\mathcal{D}_n$, written either $[a]$ or $e_{\mathbf{b}}$ depending on context.

$\langle -, - \rangle$: The bialgebra pairing $\mathcal{D} \otimes \mathcal{A} \rightarrow \mathbb{Z}$ defined by $\langle [a], x^b \rangle = \delta_{a,b}$, equivalently $\langle e_{\mathbf{a}}, x^{\mathbf{b}} \rangle = \delta_{\mathbf{a}, \mathbf{b}}$.

Comp_n : The set of weak compositions of length n .

$\Delta, \nabla, \varepsilon$: Coproduct, product, counit on $\mathcal{A}$ (§2).

Δ^* : The dual coproduct on $\mathcal{D}$, dual to the polynomial product on $\mathcal{A}$ (§3).

Structure constants.

$c_{u,v}^w$: Schubert structure constants: $\mathfrak{S}_u \mathfrak{S}_v = \sum c_{u,v}^w \mathfrak{S}_w$.

$d_{u,v}^w(p, q)$: Dual Schubert structure constants of Theorem 1.2.

$k_{a,b}^c$: Dual key structure constants of Theorem 5.4.

$f_{a,b}^c$: Dual forest structure constants of Theorem 6.4.

$s_{a,b}^c$: Dual slide structure constants of Theorem 7.2.

β_{ab}^c : Forest polynomial LR coefficients of Theorem 1.1: $\mathfrak{P}_a \mathfrak{P}_b = \sum_c \beta_{ab}^c \mathfrak{P}_c$ in $\mathcal{A}_n$. Equivalently, the structure constants of Δ^* in the dual forest basis (Corollary 8.4.7).

$b_{\mu,\nu}^\lambda$: The general dual LR structure constants of Theorem 7.3.

The ring of RC graphs and its quotients.

$\mathcal{BRC}$: $\bigoplus_{n \geq 0} \mathbb{Z}^{\oplus \mathcal{RC}_n}$, with the product $\boxplus$ (§4).

$\mathcal{BRC}_n$: The graded component $\mathbb{Z}^{\oplus \mathcal{RC}_n}$ of $\mathcal{BRC}$ (RC graphs of height exactly n).

$\mathcal{S}_w(n)$: $\sum_{R: w_R=w, \text{ht}(R)=n} R \in \mathcal{BRC}_n$, the sum of all n -row RC graphs of permutation w ; satisfies $\alpha(\mathcal{S}_w(n)) = \Xi_w^n$ and $\text{ev}(\mathcal{S}_w(n)) = \mathfrak{S}_w^n$ (§4).

K_a : $\sum_{R \in \text{Dem}_a, R=\mathbb{Y}(R)} R \in \mathcal{BRC}/\Theta$, the dual key class sum; $\alpha(K_a) = \kappa_a^*$ (§5).

F_a : $\sum_{[R] \in \mathcal{C}_{\mathfrak{P}}(a)} [R] \in \mathcal{BRC}/\Gamma$, the dual forest class sum; $\alpha(F_a) = \mathfrak{P}_a^*$ (§6).

$\boxplus$: The associative concatenation product on $\mathcal{BRC}$, on $\mathcal{D}$, and on the quotients $\mathcal{BRC}/I_{\sim}$; built from clip^p and trim^p on $\mathcal{BRC}$ (§4.4).

$*$: The auxiliary “lift product” on $\mathcal{BRC}_n$ used in the forest LR rule (§8.2).

$\boxtimes$: The “squash product” of RC graphs (§8.2; Proposition 8.1.2); the row-wise concatenation operation that lifts to $\mathcal{BRC}_n \otimes \mathcal{BRC}_n \rightarrow \mathcal{BRC}_n$ before passing to $*$.

$\alpha: \mathcal{BRC} \rightarrow \mathcal{D}$: Ring homomorphism with $\alpha(R) = \Xi_{w_R}^{\text{ht}(R)}$ (§4).

Ω : $\ker \alpha$, the two-sided ideal of $\mathcal{BRC}$ on which α vanishes (Definition 4.5.3; Theorem 4.5).

Θ : The two-sided ideal generated by $R_1 - R_2$ with $\mathbb{Y}(R_1) = \mathbb{Y}(R_2)$ (Definition 5.2.2); $\mathcal{BRC}/\Theta$ supports the dual key LR rule (Theorem 5.4).

Γ : The two-sided ideal generated by $R_1 - R_2$ with $R_1 \equiv_{\mathfrak{P}} R_2$ (Definition 6.2.4); $\mathcal{BRC}/\Gamma$ supports the dual forest LR rule (Theorem 6.4).

$\mathcal{M}_n$: The multiplactic algebra of full Grassmannian RC graphs (§8.1).

$\mathcal{E}_n$: The elementary basis of $\mathcal{A}_n$ (Theorem 2.2).

$\mathcal{G}_n$: The $\mathbb{Z}$ -submodule of $\mathcal{BRC}_n$ generated by all full Grassmannian RC graphs of height n (§8.1).

Plac_n : The plactic algebra on $[n]$, with basis given by row reading words of semistandard Young tableaux in French notation (§8.1).

$P_n : \text{SSYT}(n) \rightarrow \mathcal{G}_n$: The bijection of §8.1 sending a semistandard Young tableau to a full Grassmannian RC graph (Theorem 8.1).

Slide polynomials and the general LR template.

$\mathfrak{F}_a(x)$, $\mathfrak{F}_a^*$: Slide polynomial in $\mathcal{A}$ (Assaf–Searles) indexed by a weak composition a ; its dual in $\mathcal{D}$ (§7).

$\mathbb{QY}(R)$: Quasi-Yamanouchi representative of R in its (word, $\mathbf{ht}$)-class; smallest element under the dominance/refinement order on weights (§7).

$\mathcal{RC}_{\mathbb{QY}}(c)$: Set of RC graphs R with $\mathbf{wt}(R) = c$ such that $\text{word}(\mathbb{QY}(R))$ is lex-minimal among R' with $\mathbf{wt}(\mathbb{QY}(R')) = \mathbf{wt}(\mathbb{QY}(R))$ (§7).

Σ : Two-sided ideal of $\mathcal{BRC}$ generated by $R_1 - R_2$ with $\mathbb{QY}(R_1) = \mathbb{QY}(R_2)$ (Lemma 7.1.4).

$\mathcal{F}_a$: The element $\sum_{R=\mathbb{QY}(R), \mathbf{wt}(R)=a} R \in \mathcal{BRC}/\Sigma$, a representative of the slide basis under α (Proposition 7.1.7).

$\sim$: A generic equivalence relation on $\bigsqcup_n \mathcal{RC}_n$ satisfying (H1)–(H4) of Theorem 7.3.

$I_\sim$: The two-sided ideal $\text{span}_{\mathbb{Z}}\{R_1 - R_2 : R_1 \sim R_2\} \subseteq \mathcal{BRC}$ (Theorem 7.3).

J, λ : Index set, resp. labeling map $(\mathcal{RC}/\sim) \rightarrow J$ in the general LR template (Theorem 7.3).

E_λ : The class sum $\sum_{[R]: \lambda([R])=\lambda} [R] \in \mathcal{BRC}/I_\sim$ (Theorem 7.3).

$\mathfrak{B}_\lambda^*$, $b_{\mu,\nu}^\lambda$: Image $\alpha(E_\lambda) \in \mathcal{D}$ and the corresponding structure constants in Theorem 7.3.

Quasisymmetric flag variety (§8.5).

QFl_n : The quasisymmetric flag variety of Bergeron–Gagnon–Nadeau–Spink–Tewari [6], a toric subcomplex of Fl_n stratified by $X(F)$ for $F \in \text{Forest}_n$.

$X(F)$, $[X(F)]$: The toric Richardson variety attached to $F \in \text{Forest}_n$, and its homology class; $\{[X(F)]\}$ is a $\mathbb{Z}$ -basis of $H_\bullet(\text{QFl}_n)$ [6, Cor. 11.6].

$H^\bullet(\text{QFl}_n)$, $H_\bullet(\text{QFl}_n)$: Cohomology and homology of QFl_n , paired by the integration map $\int_{X(F)}(\cdot)$.

QSym_n^+ : The ideal of quasisymmetric polynomials in $\mathbb{Z}[x_1, \dots, x_n]$ with no constant term; $\text{QSym}_n^+ \cdot \mathcal{A}_n$ is the kernel of the Borel surjection.

π_n : The Borel-type surjection $\mathcal{A}_n \rightarrow \mathcal{A}_n/(\text{QSym}_n^+ \cdot \mathcal{A}_n) \cong H^\bullet(\text{QFl}_n)$ [6, Thm. A and Cor. 12.5].

$[\mathfrak{P}_a]$: The cohomology class $\pi_n(\mathfrak{P}_a) \in H^\bullet(\text{QFl}_n)$; under the Borel isomorphism, $\{[\mathfrak{P}_a] : F_a \in \text{Forest}_n\}$ is a $\mathbb{Z}$ -basis of $H^\bullet(\text{QFl}_n)$, Kronecker dual to $\{[X(F_a)]\}$.

$\rho_{p,q}$, $\iota_{p,q}^*$: The variable-splitting substitution $\mathcal{A}_{p+q} \rightarrow \mathcal{A}_p \otimes \mathcal{A}_q$, and the induced ring homomorphism

$$H^\bullet(\text{QFl}_{p+q}) \rightarrow H^\bullet(\text{QFl}_p) \otimes H^\bullet(\text{QFl}_q)$$

(§8.5).

$\epsilon_F(\mathbf{b})$: The path sign of [30, Prop. 10.15]: the coefficient of $\mathfrak{P}_F$ in the forest expansion of $x^{\mathbf{b}}$, valued in $\{-1, 0, +1\}$. By Proposition 6.2.2, $\langle \mathfrak{P}_a^*, x^{\mathbf{b}} \rangle = \epsilon_{F_a}(\mathbf{b})$.

$V_F(\boldsymbol{\lambda})$: The volume polynomial of [30, §10.5], dual to $\mathfrak{P}_F$ under the divided-power D -pairing; $V_F(\boldsymbol{\lambda}) = \sum_{\mathbf{b} \in F(\mathbf{b})} \boldsymbol{\lambda}^{\mathbf{b}}/\mathbf{b}!$.

APPENDIX A. PROOF OF PROPOSITION 2.1.5

We dedicate this section to the otherwise distracting proof of Proposition 2.1.5.

Proposition A.1 (Special case of Pieri formula [38, Theorem 7.1]). *Suppose $k \geq 1$ and $u, w \in S_\infty$. If $u \xrightarrow{k} w$, then*

$$\delta_u^w \prod_{i=1}^k (x_1 - y_i) = 0$$

If $u \xrightarrow{k} w$, define

$$Q = \{u(i) \mid i \leq k \text{ and } u(i) = w(i)\}$$

then

$$\delta_u^w \prod_{i=1}^k (x_1 - y_i) = \prod_{q \in Q} (x_1 - y_q)$$

Proof. To avoid confusion, we point out that the original theorem is stated on the x variables, whereas this theorem is in terms of the y variables. Besides this, this is a restatement of the cited theorem in the special case of $p = k$, that is to say the Molev-Sagan Pieri formula for double Schubert polynomials. $\square$

Proof of Proposition 2.1.5. Suppose $v \in S_n$. We have

$$\mathfrak{S}_v(x; y) = \delta^{vw_0(n)}(\mathfrak{S}_{w_0(n)}(x; y))$$

This is equal to

$$\delta^{vw_0(n)} \left(\mathfrak{S}_{s_{n+1-i} \cdots s_1 w_0(n)}(x^{(i)}; y) \prod_{j=1}^{n+1-i} (x_i - y_j) \right)$$

and, applying the Leibniz formula, is also equal to

$$\sum_{\substack{v' \in S_\infty \\ \ell(v' w_0(n) s_1 \cdots s_{n+1-i}) \\ = \ell(w_0(n) s_1 \cdots s_{n+1-i}) - \ell(v')}} \mathfrak{S}_{v'}(x^{(i)}; y) \delta_{v' w_0(n) s_1 \cdots s_{n+1-i}}^{vw_0(n)} \prod_{j=1}^{n+1-i} (x_i - y_j)$$

By the restricted Pieri formula (Proposition A.1), for this to be nonzero we need $v' w_0(n) s_1 \cdots s_{n+1-i} \xrightarrow{n+1-i} v w_0(n)$. We note that $v' w_0(n) s_1 \cdots s_{n+1-i} \xrightarrow{n+1-i} v w_0(n)$ if and only if

$$v \xrightarrow{i} v' w_0(n) s_1 \cdots s_{n+1-i} w_0(n) = v' s_n s_{n-1} \cdots s_i$$

We have that $v' s_n \cdots s_i$ is exactly $\varphi_{i,n}(v')$ since $v' \in S_n$, so we require that

$$v \xrightarrow{i} \varphi_{i,n}(v')$$

so the sum is over all v' with $v \xrightarrow{i} v'$.

Applying Proposition A.1, we obtain that the result is equal to

$$\sum_{v \xrightarrow{i} v'} \mathfrak{S}_{v'}(x^{(i)}; y) \prod_{a \in A(v', v)} (x_i - y_a)$$

where $A(v', v)$ is the set of all $v w_0(n)(j)$ such that $1 \leq j \leq n+1-i$ and $v' w_0(n) s_1 \cdots s_{n+1-i}(j) = v w_0(n)(j)$. These values are the same as at the indices that comprise the set of all $1 \leq j \leq n+1-i$ such that

$$v'(n+2-s_1 \cdots s_{n+1-i}(j)) = v(n+2-j)$$

Applying the $s_1 \cdots s_{n+1-i}$ to j , since $1 \leq j \leq n+1-i$ we have that

$$s_1 \cdots s_{n+1-i}(j) = j+1$$

Hence we need

$$v'(n+2-(j+1)) = v'(n+1-j) = v(n+2-j)$$

Replacing j with $n+2-p$, the indices are the set of all p such that $i < p \leq n+1$ and

$$v'(p-1) = v(p)$$

Thus $A(v', v)$ is the set of all $v(p)$ such that $p > i$ and $v'(p-1) = v(p)$, which is exactly $Q_i(v', v)$, and we are done. $\square$

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REFERENCES

- [1] ASSAF, S., AND SCHILLING, A. A Demazure crystal construction for Schubert polynomials. *Algebraic Combinatorics* 1, 2 (2018), 225–247.
- [2] ASSAF, S., AND SEARLES, D. Schubert polynomials, slide polynomials, Stanley symmetric functions and quasi-Yamanouchi pipe dreams. *Advances in Mathematics* 306 (2017), 89–122.
- [3] AXELROD-FREED, I. Inversions tableaux. *arXiv preprint arXiv:2507.11516* (2025).
- [4] BERGERON, N., AND BILLEY, S. RC-graphs and Schubert polynomials. *Experimental Math.* 2, 4 (1993), 257–269.
- [5] BERGERON, N., GAGNON, L., NADEAU, P., SPINK, H., AND TEWARI, V. Equivariant quasisymmetry and noncrossing partitions. *arXiv preprint arXiv:2504.15234* (2025).
- [6] BERGERON, N., GAGNON, L., NADEAU, P., SPINK, H., AND TEWARI, V. The quasisymmetric flag variety: a toric complex on noncrossing partitions. *arXiv preprint arXiv:2508.12171* (2025).
- [7] BERGERON, N., AND SOTTILE, F. Schubert polynomials, the Bruhat order, and the geometry of flag manifolds. *Duke Math. J.* 95, 2 (1998), 373–423.
- [8] BERGERON, N., AND SOTTILE, F. Skew Schubert functions and the Pieri formula for flag manifolds. *Trans. Amer. Math. Soc.* 354, 2 (2001), 651–673.
- [9] BILLEY, S. C., JOCKUSCH, W., AND STANLEY, R. P. Some Combinatorial Properties of Schubert Polynomials. *J. Algebraic Combin.* 2, 4 (1993), 345–374.
- [10] BJÖRNER, A., AND WACHS, M. Shellable nonpure complexes and posets. II. *Transactions of the American Mathematical Society* 349, 10 (1997), 3945–3975.
- [11] CAIN, A. J., GUILHERME, R. P., AND MALHEIRO, A. Quasi-crystals for arbitrary root systems and associated generalizations of the hypoplactic monoid. *arXiv preprint arXiv:2301.00271* (2023). arXiv:2301.00271.
- [12] DEMAZURE, M. Désingularisation des variétés de Schubert généralisées. *Annales scientifiques de l’École Normale Supérieure* 7, 1 (1974), 53–88.
- [13] DEMAZURE, M. Une nouvelle formule des caractères. *Bulletin des Sciences Mathématiques. 2e Série* 98, 3 (1974), 163–172.
- [14] EDELMAN, P., AND GREENE, C. Balanced tableaux. *Advances in Mathematics* 63, 1 (1987), 42–99.
- [15] GAO, Y., AND HUANG, D. The canonical bijection between pipe dreams and bumpless pipe dreams. *International Mathematics Research Notices* 2023, 21 (November 2023), 18629–18663.
- [16] GUO, A., AND WOODRUFF, D. Forest polynomials and pattern avoidance, 2026. arXiv:2602.04036 [math.CO].
- [17] HAMAKER, Z., AND YOUNG, B. Relating Edelman–Greene insertion to the Little map. *Journal of Algebraic Combinatorics* 40, 3 (2014), 693–710.
- [18] HENRY V. Positivity of branching of key polynomials: references and significance. MathOverflow answer, 2026. URL: <https://mathoverflow.net/q/511128> (version: 2026-05-09).
- [19] JOSEPH, A. On the Demazure character formula. *Annales scientifiques de l’École Normale Supérieure* 18, 3 (1985), 389–419.
- [20] KASHIWARA, M. The crystal base and Littelmann’s refined Demazure character formula. *Duke Mathematical Journal* 71, 3 (1993), 839–858.
- [21] KOUNO, T. Decomposition of tensor products of Demazure crystals. *Journal of Algebra* 546 (2020), 641–678.
- [22] LAM, T., LEE, S. J., AND SHIMOZONO, M. Back stable Schubert calculus. *Compositio Mathematica* 157, 5 (2021), 883–962.
- [23] LASCoux, A., AND SCHÜTZENBERGER, M.-P. Keys & standard bases. In *Invariant theory and tableaux (Minneapolis, MN, 1988)*, vol. 19 of *IMA Vol. Math. Appl.* Springer, New York, 1990, pp. 125–144.
- [24] LITTELMANN, P. Crystal graphs and Young tableaux. *Journal of Algebra* 175, 1 (1995), 65–87.
- [25] LITTLE, D. P. Combinatorial aspects of the Lascoux–Schützenberger tree. *Advances in Mathematics* 174, 2 (2003), 236–253.
- [26] MACDONALD, I. G. *Notes on Schubert Polynomials*. Université Du Québec à Montréal, Dép. de mathématiques et d’informatique, 1991.
- [27] MASON, S. An explicit construction of type A Demazure atoms. *Journal of Algebraic Combinatorics* 29, 3 (2009), 295–313.
- [28] MATHIEU, O. Filtrations of G -modules. *Annales scientifiques de l’École Normale Supérieure* 23, 4 (1990), 625–644.
- [29] MORSE, J., AND SCHILLING, A. Crystal approach to affine Schubert calculus. *International Mathematics Research Notices* 2016, 8 (2016), 2239–2294.
- [30] NADEAU, P., SPINK, H., AND TEWARI, V. Quasisymmetric divided differences. *arXiv preprint arXiv:2406.01510* (2024).
- [31] NADEAU, P., AND TEWARI, V. Forest polynomials and the class of the permutahedral variety. *Advances in Mathematics* 453 (2024), 109834.
- [32] NADEAU, P., AND TEWARI, V. P -partitions with flags and back stable quasisymmetric functions. *Combinatorial Theory* 4, 2 (2024). hal-04053201.
- [33] POLO, P. Variétés de Schubert et excellentes filtrations. *Astérisque*, 173–174 (1989), 281–311.

- [34] POSTNIKOV, A., AND STANLEY, R. P. Chains in the Bruhat order. *Journal of Algebraic Combinatorics* 29, 2 (2009), 133–174.
- [35] REINER, V., AND SHIMOZONO, M. Key polynomials and a flagged Littlewood-Richardson rule. *Journal of Combinatorial Theory, Series A* 70, 1 (1995), 107–143.
- [36] RESSAYRE, N., AND RICHMOND, E. Branching Schubert calculus and the Belkale–Kumar product on cohomology. *Proceedings of the American Mathematical Society* 139, 3 (2011), 835–848. arXiv:0909.0865.
- [37] SAMUEL, M. J. *The Leibniz formula for divided difference operators associated to Kac-Moody root systems*. PhD thesis, Rutgers, The State University of New Jersey, 2014.
- [38] SAMUEL, M. J. A Molev-Sagan type formula for double Schubert polynomials. *J. Pure Appl. Algebra* 228, 7 (2024), 107636.
- [39] SAMUEL, M. J. `schubmult` python package. `git@github.com:matthematics/schubmult.git`. Version 4.0.0dev. Stable version available on PyPI., 2026-05-03.
- [40] SOTTILE, F. Pieri’s rule for flag manifolds and Schubert polynomials. *Ann. Inst. Fourier* 46, 1 (1996), 89–110.
- [41] VAN DER KALLEN, W. Longest weight vectors and excellent filtrations. *Mathematische Zeitschrift* 201, 1 (1989), 19–31.
- [42] WEIGANDT, A. Bumpless pipe dreams and alternating sign matrices. *Journal of Combinatorial Theory, Series A* 182 (August 2021), 105470.
- [43] WOODRUFF, D. Single-SEM Schubert Polynomials. *arXiv preprint arXiv:2503.03903* (2025).
- [44] YU, T. Embedding Bumpless Pipereams as Bruhat Chains. *International Mathematics Research Notices* 2025, 22 (2025), rna336.