

WORD-COMPATIBLE GRAPHS AND GROVE POLYNOMIALS

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ABSTRACT.

1. WC GRAPHS

Definition 1.1. A *WC graph* is simply a finite subset $W \subseteq \mathbb{N} \times \mathbb{N}$ with no additional restrictions. We impose a total ordering on these pairs by declaring that $(a, b) \leq (c, d)$ if $a < c$ or $a = c$ and $b \geq d$. We define the *weight* of a WC graph W to be the sequence

$$\mathbf{wt}(W) = (i_1, \dots, i_n)$$

where i_j is the number of elements of W in row j . We define the *permutation* of a WC graph W to be the permutation

$$\mathbf{perm}(W) = s(w_1) \bullet \dots \bullet s(w_n)$$

where

$$w_i = (a_i, b_i)$$

is the i th element in the imposed ordering, and

$$s(a_i, b_i) = s_{a_i+b_i-1}$$

with $\bullet$ being the Hecke product. The *word* of W is

$$(s(w_1), \dots, s(w_n))$$

Define $\mathcal{WC}(w)$ to be the set of all WC graphs W such that $\mathbf{perm}(W) = w$.

The following is well-known.

Theorem 1.2. *We have*

$$\mathfrak{G}_w^{(\beta)}(x) = \sum_{W \in \mathcal{WC}(w)} \beta^{|W|-\ell(w)} x^{\mathbf{wt}(W)}$$

2. GROVE POLYNOMIALS AND WC GRAPHS

Definition 2.1. A labeled indexed forest (F, P) is an indexed forest F with a function (labeling) $P : \text{IN}(F) \rightarrow \mathbb{N}$ satisfying the following properties:

- (1) $P(\text{left}(u)) < P(u)$ for all $u \in \text{IN}(F)$.
- (2) $P(u) \leq P(\text{right}(u))$ for all $u \in \text{IN}(F)$.
- (3) $P(u) \geq \rho_F(u)$ for all $u \in \text{IN}(F)$.
- (4) $P(u) \leq \lambda_F(u)$ for all $u \in \text{IN}(F)$.

The set $\text{Komp}_P(F)$ is the set of functions $\kappa : \text{IN}(F) \rightarrow 2^{\mathbb{N}}$ such that

- (1) $\max(\kappa(u)) \leq P(u)$ for all $u \in \text{IN}(F)$.
- (2) $\max(\kappa(u)) \leq \min(\kappa(\text{left}(u)))$ for all $u \in \text{IN}(F)$.
- (3) $\max(\kappa(u)) < \min(\kappa(\text{right}(u)))$ for all $u \in \text{IN}(F)$.

We have that

$$\mathbf{wt}_i(\kappa) = \#\{u \in \text{in}(F) \mid i \in \kappa(u)\}$$

Theorem 2.2. Suppose a labeled indexed forest (F, P) is such that there is a $\kappa \in \text{Komp}_P(F)$ with

$$\mathbf{wt}(\kappa) = \mathbf{c}(F).$$

Then

$$\tilde{\mathfrak{P}}_F(x) = \sum_{\kappa \in \text{Komp}_P(F)} \beta^{|\kappa| - |F|} x^{\mathbf{wt}(\kappa)}$$

Definition 2.3. Let $W \in \mathcal{WC}(w)$ be a WC graph for w . Define via the Nadeau-Tewari insertion algorithm

$$(P(W), Q(W)) = (P(\text{word}(W)), Q(\text{word}(W)))$$

and let

$$F(W) = \text{shape}(P(W))$$

Define $\mathbf{c}(F)$ to be the code of the indexed forest F .

Definition 2.4. Let (F, P) be a labeled indexed forest and let $\kappa \in \text{Komp}_P(F)$. Consider the set

$$A = \bigcup_{v \in \text{IN}(F)} \kappa(v) \times \{P(v)\}$$

Define

$$\mathcal{WC}(\kappa) = \{(a_i, b_i + 1 - a_i) \mid (a_i, b_i) \in A\}$$

Theorem 2.5. For a permutation w , let $\mathcal{WC}_{\mathfrak{P}}(w)$ be the set of $W \in \mathcal{WC}(w)$ such that $\mathbf{wt}(W) = \mathbf{c}(F(W))$. Then we have that $\mathcal{WC}(w)$ is the disjoint union of $\mathcal{WC}(\text{Komp}_{P(W)}(F(W)))$ for $W \in \mathcal{WC}_{\mathfrak{P}}(w)$.

Corollary 2.6. The Grothendieck polynomial $\mathfrak{G}_w^{(\beta)}(x)$ satisfies

$$\mathfrak{G}_w^{(\beta)}(x) = \sum_{F \in \text{IF}} \beta^{|F| - \ell(w)} a_w^F \tilde{\mathfrak{P}}_F(x)$$

where

$$a_w^F = \#\{W \in \mathcal{WC}_{\mathfrak{P}}(w) \mid F(W) = F\}$$

3. WC GRAPH RING

Proposition 3.1. The linear map

$$\rho : \mathcal{BWC} \rightarrow \mathcal{BRC}$$

such that

$$\rho(W) = \begin{cases} W & \text{if } W \text{ is an RC graph} \\ 0 & \text{otherwise} \end{cases}$$

is a surjective ring homomorphism.

Definition 3.2. This homomorphism has a section. Specifically, for each $n > 0$ define a ring Plac_n^K to be the algebra spanned by the n -Grassmannian WC graphs with multiplication given by the squash product. Then we have a surjective ring homomorphism

$$\text{Plac}_n^K \rightarrow \text{Plac}_n$$

obtained on elementary symmetric WC graphs by sending $E_{p,k}^\beta$ with $|\beta| > p$ to $E_{|\beta|,k}^\beta$. Define

$$\mathcal{M}_n^K = \bigotimes_{i=1}^n \text{Plac}_i^K$$

There is an evident homomorphism $\mathcal{M}_n^K \rightarrow \mathcal{M}_n$ obtained by applying the above homomorphism to each factor. We have a commutative diagram

$$\begin{array}{ccc} \mathcal{M}_n^K & \longrightarrow & \mathcal{M}_n \\ \downarrow & & \downarrow \\ \mathcal{BWC}_n & \dashrightarrow & \mathcal{BRC}_n \end{array}$$

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REFERENCES

- [1] SAMUEL, M. J. A Molev-Sagan type formula for double Schubert polynomials. *J. Pure Appl. Algebra* 228, 7 (2024), 107636.