

Quantization Error in 3D Vertex Predictions: A Probabilistic Analysis

1 Setup and Notation

Let $p \in \mathbb{R}^3$ be a predicted vertex position and $g \in \mathbb{R}^3$ be the ground truth position. Define the *prediction error* as $e = p - g$.

Quantization with step size $\Delta > 0$ maps each coordinate to the nearest multiple of Δ :

$$q(x) = \Delta \cdot \text{round}(x/\Delta)$$

The *quantization error* for a single coordinate is:

$$u = q(x) - x \in [-\Delta/2, \Delta/2]$$

Assumption. For a coordinate x not already aligned to the quantization grid, the quantization error u is approximately uniformly distributed:

$$u \sim \text{Uniform}(-\Delta/2, \Delta/2)$$

This holds when the fractional part of x/Δ is uniformly distributed, which is reasonable for real-world vertex coordinates.

For 3D vertices, let $\mathbf{u} = (u_1, u_2, u_3)$ be the vector of independent quantization errors per coordinate.

2 Effect on Mean Error

Theorem 1 (Unbiasedness of Quantized Predictions). *The expected signed error is preserved under quantization:*

$$\mathbb{E}[q(p) - g] = \mathbb{E}[p - g]$$

Proof.

$$\begin{aligned} \mathbb{E}[q(p) - g] &= \mathbb{E}[(p + \mathbf{u}) - g] \\ &= \mathbb{E}[p - g] + \mathbb{E}[\mathbf{u}] \\ &= \mathbb{E}[p - g] + \mathbf{0} \end{aligned}$$

since $\mathbb{E}[u_i] = 0$ for uniform $u_i \sim \text{Uniform}(-\Delta/2, \Delta/2)$. □

Corollary 1 (Mean Vertex Error Convergence). *When averaging over many vertices, the contribution of quantization error to the mean signed error converges to zero by the law of large numbers.*

3 Effect on Mean Euclidean Distance

The Euclidean distance metric is more nuanced. Define:

$$\begin{aligned} d &= \|p - g\| = \|e\| \quad (\text{original error}) \\ d' &= \|q(p) - g\| = \|e + \mathbf{u}\| \quad (\text{quantized error}) \end{aligned}$$

Theorem 2 (Expected Squared Distance).

$$\mathbb{E}[d'^2] = d^2 + \mathbb{E}[\|\mathbf{u}\|^2] = d^2 + \frac{\Delta^2}{4}$$

where $\frac{\Delta^2}{4} = 3 \cdot \frac{\Delta^2}{12}$ is the expected squared magnitude of the 3D quantization error.

Proof.

$$\begin{aligned} \mathbb{E}[\|e + \mathbf{u}\|^2] &= \mathbb{E}[(e + \mathbf{u})^\top (e + \mathbf{u})] \\ &= \mathbb{E}[\|e\|^2 + 2e^\top \mathbf{u} + \|\mathbf{u}\|^2] \\ &= \|e\|^2 + 2e^\top \mathbb{E}[\mathbf{u}] + \mathbb{E}[\|\mathbf{u}\|^2] \\ &= d^2 + 0 + \sum_{i=1}^3 \text{Var}(u_i) \\ &= d^2 + 3 \cdot \frac{\Delta^2}{12} = d^2 + \frac{\Delta^2}{4} \end{aligned}$$

using $\text{Var}(u_i) = \frac{\Delta^2}{12}$ for $u_i \sim \text{Uniform}(-\Delta/2, \Delta/2)$. □

4 The Asymptotic Regime: Large Prediction Error

When $d \gg \Delta$ (prediction error much larger than quantization step), we can analyze the expected Euclidean distance.

Theorem 3 (Large Error Regime). *When $d \gg \Delta$, the expected distance after quantization satisfies:*

$$\mathbb{E}[d'] \approx d + O\left(\frac{\Delta^2}{d}\right)$$

The relative change vanishes as $d/\Delta \rightarrow \infty$.

Proof. Using the Taylor expansion $\sqrt{a+b} \approx \sqrt{a} + \frac{b}{2\sqrt{a}}$ for $b \ll a$:

$$\mathbb{E}[d'] = \mathbb{E}[\sqrt{d^2 + 2e^\top \mathbf{u} + \|\mathbf{u}\|^2}]$$

Let $X = 2e^\top \mathbf{u} + \|\mathbf{u}\|^2$. Note that $\mathbb{E}[X] = \frac{\Delta^2}{4}$ and $|X| = O(\Delta \cdot d + \Delta^2)$.

For $d \gg \Delta$, expand around d^2 :

$$\mathbb{E}[d'] \approx d \cdot \mathbb{E}\left[\sqrt{1 + \frac{X}{d^2}}\right] \approx d + \frac{\mathbb{E}[X]}{2d} = d + \frac{\Delta^2}{8d}$$

The relative error is $\frac{\Delta^2}{8d^2} \rightarrow 0$ as $d/\Delta \rightarrow \infty$. □

Corollary 2. *For typical pose estimation with $d \approx 50\text{mm}$ and $\Delta = 0.5\text{mm}$:*

$$\frac{\Delta^2}{8d^2} = \frac{0.25}{8 \cdot 2500} \approx 0.00001 = 0.001\%$$

The quantization effect on mean Euclidean error is negligible.

5 The Problematic Regime: Small Prediction Error

When $d \ll \Delta$, quantization has a fundamentally different character.

Theorem 4 (Small Error Regime). *When $d = 0$ (perfect prediction), quantization can only increase the error:*

$$p = g \implies \mathbb{E}[\|q(p) - g\|] = \mathbb{E}[\|\mathbf{u}\|] = \frac{\Delta}{2} \sqrt{\frac{3}{2}} \approx 0.61\Delta$$

Proof. When $p = g$, we have $q(p) - g = \mathbf{u}$, so $d' = \|\mathbf{u}\|$.

For $\mathbf{u}$ with independent uniform components on $[-\Delta/2, \Delta/2]$:

$$\mathbb{E}[\|\mathbf{u}\|] = \mathbb{E}\left[\sqrt{u_1^2 + u_2^2 + u_3^2}\right]$$

This equals $\frac{\Delta}{2} \sqrt{\frac{3}{2}}$ (derived via the distribution of the norm of a uniform random vector in a cube). $\square$

Proposition 1 (Overshoot Phenomenon). *For small $d > 0$, quantization can “overshoot” past the ground truth. Specifically, if $0 < d < \Delta/2$, there is positive probability that the quantization moves p to the opposite side of g .*

Proof. Consider the 1D case for simplicity. Let $p = g + \epsilon$ with $0 < \epsilon < \Delta/2$. The quantized value is $q(p) = q(g + \epsilon)$.

If g is at position $k\Delta + \delta$ for some integer k and $|\delta| < \Delta/2$:

- If $\delta + \epsilon > \Delta/2$, then $q(p) = (k + 1)\Delta$, overshooting by $\Delta - \delta - \epsilon$ on the opposite side of g .

The probability of overshoot depends on the position of g relative to the grid, but is non-zero for any $\epsilon < \Delta/2$. $\square$

6 Averaging Over Many Vertices

Theorem 5 (Concentration of Mean Error). *Let $d_1, \dots, d_n$ be the original errors for n vertices, and $d'_1, \dots, d'_n$ be the errors after quantization. Define:*

$$\bar{d} = \frac{1}{n} \sum_{i=1}^n d_i, \quad \bar{d}' = \frac{1}{n} \sum_{i=1}^n d'_i$$

In the large error regime ($d_i \gg \Delta$ for all i):

$$\mathbb{E}[\bar{d}' - \bar{d}] = O\left(\frac{\Delta^2}{\min_i d_i}\right)$$

and by concentration inequalities:

$$|\bar{d}' - \bar{d}| = O\left(\frac{\Delta}{\sqrt{n}}\right) \quad \text{with high probability}$$

Proof. For each vertex i , let $\delta_i = d'_i - d_i$. We have shown $\mathbb{E}[\delta_i] = O(\Delta^2/d_i)$.

The variance of δ_i is bounded: $\text{Var}(\delta_i) \leq \mathbb{E}[\delta_i^2] = O(\Delta^2)$ since $|\delta_i| \leq \|\mathbf{u}_i\| \leq \frac{\sqrt{3}}{2}\Delta$.

By independence across vertices:

$$\text{Var}(\bar{d}' - \bar{d}) = \frac{1}{n^2} \sum_{i=1}^n \text{Var}(\delta_i) = O\left(\frac{\Delta^2}{n}\right)$$

By Chebyshev's inequality, $|\bar{d}' - \bar{d}| = O(\Delta/\sqrt{n})$ with high probability. $\square$

7 Practical Implications

1. **For evaluation metrics:** When prediction errors are much larger than quantization step (typical case), quantization has negligible effect on mean vertex error.
2. **Recommended quantization:** A step of $\Delta = 0.5\text{mm}$ introduces P99 Euclidean error of $\sim 0.4\text{mm}$, but contributes $< 0.001\text{mm}$ to mean error when averaging over meshes with typical prediction errors.
3. **The caveat:** For near-perfect predictions ($d < \Delta$), quantization becomes the dominant error source and cannot be averaged away.

8 Summary

Regime	Effect on $\mathbb{E}[d']$	Practical Impact
$d \gg \Delta$	$d + O(\Delta^2/d)$	Negligible
$d \sim \Delta$	Moderate increase	Noticeable
$d \ll \Delta$	$\sim 0.6\Delta$	Quantization-dominated
$d = 0$	$\sim 0.6\Delta$	Lower bound on error