

Maxwell Bloch Solver

Equations Reference

Version 9.1.1

Vahid Anari

vahid.anari8@gmail.com

This document derives the dimensionless Maxwell–Bloch equations used by the solver, starting from the general two-level Hamiltonian and proceeding through the rotating-wave and slowly-varying-envelope approximations to the final dimensionless system.

Contents

1	Hamiltonian	2
2	Density-Matrix Equations of Motion	2
3	Maxwell Wave Equations	3
3.1	Maxwell's equations	3
3.2	General wave equations	3
3.3	Reduction to one dimension	3
4	Slowly Varying Envelope and Rotating-Wave Approximations	3
4.1	Envelope decomposition	4
4.2	Equations of motion	4
4.3	Relaxation and pumping	4
5	Retarded-Time Coordinates	5
6	Dimensionless Form	5
6.1	Scaled variables	5
6.2	Cooperation time T_R	5
6.3	Final dimensionless equations	6
6.4	Symbol table	6
7	Initial and Boundary Conditions	6
7.1	General case	6
7.2	Real-variable reduction	7
8	Recovering the Physical Fields	7
9	Magnetic-Dipole Case	8

1 Hamiltonian

Let $|2\rangle$ denote the upper state and $|1\rangle$ the lower state of a two-level system with transition frequency ω_0 . The free Hamiltonian is

$$\hat{H}_0 = \frac{\hbar\omega_0}{2}\hat{\sigma}_z, \quad \hat{\sigma}_z = |2\rangle\langle 2| - |1\rangle\langle 1|, \quad (1)$$

with raising and lowering operators

$$\hat{\sigma}_+ = |2\rangle\langle 1|, \quad \hat{\sigma}_- = |1\rangle\langle 2|. \quad (2)$$

The electric dipole operator is written in terms of the transition matrix elements $\mathbf{d}_{21} = \langle 2|\hat{\mathbf{d}}|1\rangle$ and $\mathbf{d}_{12} = \mathbf{d}_{21}^*$ as

$$\hat{\mathbf{d}} = \mathbf{d}_{21} \hat{\sigma}_+ + \mathbf{d}_{12} \hat{\sigma}_-. \quad (3)$$

Writing the electric field along polarization unit vector $\hat{\mathbf{e}}$ as $\mathbf{E} = E\hat{\mathbf{e}}$ and defining the scalar transition matrix element

$$d = \mathbf{d}_{21} \cdot \hat{\mathbf{e}}, \quad (4)$$

the interaction Hamiltonian takes the form

$$\hat{H}_{\text{int}} = -\hat{\mathbf{d}} \cdot \mathbf{E} = -(d\hat{\sigma}_+ + d^*\hat{\sigma}_-)E. \quad (5)$$

The total Hamiltonian is $\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}}$.

2 Density-Matrix Equations of Motion

The state of the medium is described by the density operator

$$\hat{\rho} = \sum_{m,n \in \{1,2\}} \rho_{mn} |m\rangle\langle n|, \quad (6)$$

with population inversion

$$w = \rho_{22} - \rho_{11}, \quad \rho_{21} = \rho_{12}^*. \quad (7)$$

From the Liouville–von Neumann equation $i\hbar\dot{\hat{\rho}} = [\hat{H}, \hat{\rho}]$, the matrix element $\langle 1|[\hat{H}, \hat{\rho}]|2\rangle$ gives

$$\begin{aligned} i\hbar\dot{\rho}_{12} &= \langle 1|[\hat{H}, \hat{\rho}]|2\rangle \\ &= -\hbar\omega_0\rho_{12} - d^*E(\rho_{22} - \rho_{11}), \end{aligned} \quad (8)$$

giving

$$\dot{\rho}_{12} = i\omega_0\rho_{12} + \frac{id^*}{\hbar} E w. \quad (9)$$

Since $\rho_{21} = \rho_{12}^*$, the complex conjugate gives immediately

$$\dot{\rho}_{21} = -i\omega_0\rho_{21} - \frac{id}{\hbar} E w. \quad (10)$$

For the inversion, taking $\langle 2|[\hat{H}, \hat{\rho}]|2\rangle - \langle 1|[\hat{H}, \hat{\rho}]|1\rangle$:

$$i\hbar\dot{w} = 2(d^*E\rho_{21} - dE\rho_{12}), \quad (11)$$

so that

$$\dot{w} = \frac{2i}{\hbar}(dE\rho_{12} - d^*E\rho_{21}). \quad (12)$$

3 Maxwell Wave Equations

3.1 Maxwell's equations

In a medium with no free charges or currents, Maxwell's equations are

$$\nabla \cdot \mathbf{D} = 0, \quad (13)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (14)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (15)$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}, \quad (16)$$

with constitutive relations

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}, \quad \mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M}). \quad (17)$$

3.2 General wave equations

Taking the curl of the curl equations and using the constitutive relations gives the general wave equations for $\mathbf{E}$ and $\mathbf{B}$:

$$\nabla^2 \mathbf{E} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = \mu_0 \frac{\partial^2 \mathbf{P}}{\partial t^2} + \mu_0 \nabla \times \frac{\partial \mathbf{M}}{\partial t}, \quad (18)$$

$$\nabla^2 \mathbf{B} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} = -\mu_0 \nabla \times \left(\frac{\partial \mathbf{P}}{\partial t} + \nabla \times \mathbf{M} \right). \quad (19)$$

3.3 Reduction to one dimension

We consider a plane wave propagating along z with fields transverse to z . For the **electric-dipole** case the medium has no magnetization ($\mathbf{M} = 0$), so the curl term in Eq. (18) vanishes and the wave equation reduces to

$$\frac{\partial^2 E}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} = \mu_0 \frac{\partial^2 P}{\partial t^2}. \quad (20)$$

The macroscopic polarization is related to the density operator through

$$P = n_0 \text{Tr}(\hat{\rho} \hat{\mathbf{d}} \cdot \hat{\mathbf{e}}) = n_0 (\rho_{12} d + \rho_{21} d^*). \quad (21)$$

4 Slowly Varying Envelope and Rotating-Wave Approximations

We apply two standard approximations: the **slowly varying envelope approximation** (SVEA) to the field, and the **rotating-wave approximation** (RWA) to the atom-field coupling.

4.1 Envelope decomposition

The field and coherence ρ_{21} are written as slowly varying envelopes modulated by the carrier $e^{i(kz-\omega t)}$:

$$E(z, t) = \frac{1}{2}A(z, t) e^{i(kz-\omega t)} + \text{c.c.}, \quad (22)$$

$$\rho_{21}(z, t) = \frac{1}{2}R(z, t) e^{i(kz-\omega t)}, \quad (23)$$

where $k = \omega/c$ is the carrier wave number, and A and R satisfy

$$\left| \frac{\partial A}{\partial t} \right| \ll \omega |A|, \quad \left| \frac{\partial A}{\partial z} \right| \ll k |A|, \quad (24)$$

and similarly for R . The factor of $\frac{1}{2}$ ensures that A equals the peak amplitude of the field.

4.2 Equations of motion

Substituting the envelope decomposition into Eq. (10), collecting terms at the resonant frequency $e^{i(kz-\omega t)}$, and dropping the fast counter-rotating terms (RWA). we define the detuning

$$\Delta = \omega - \omega_0, \quad (25)$$

the coherence equation becomes

$$\dot{R} = i\Delta R - \frac{id}{\hbar} A w. \quad (26)$$

Substituting into Eq. (12) and keeping only the two resonant products (one from each factor of the field and coherence):

$$\dot{w} = \frac{i}{2\hbar} (dAR^* - d^*A^*R). \quad (27)$$

Substituting into the wave equation (20), dropping second-order derivatives of A (SVEA), and using $k = \omega/c$:

$$\frac{\partial A}{\partial z} + \frac{1}{c} \frac{\partial A}{\partial t} = \frac{i\omega n_0 d^*}{2\epsilon_0 c} R. \quad (28)$$

4.3 Relaxation and pumping

In practice the two-level medium is coupled to its environment, leading to population relaxation on a timescale T_1 and dephasing on a timescale T_2 . These are added phenomenologically. A pump term Λ_n drives the inversion away from its equilibrium value w_0 , and the coherence relaxes toward its equilibrium value R_0 . We also set the detuning $\Delta = 0$, corresponding to exact resonance between the carrier and the atomic transition. The full Maxwell–Bloch system is:

$$\frac{\partial w}{\partial t} = \frac{i}{2\hbar} (dAR^* - d^*A^*R) - \frac{w - w_0}{T_1} + \Lambda_n, \quad (29)$$

$$\frac{\partial R}{\partial t} = -\frac{id}{\hbar} A w - \frac{R - R_0}{T_2}, \quad (30)$$

$$\frac{\partial A}{\partial z} + \frac{1}{c} \frac{\partial A}{\partial t} = \frac{i\omega n_0 d^*}{2\epsilon_0 c} R. \quad (31)$$

5 Retarded-Time Coordinates

We move to a frame co-propagating with the field by introducing the retarded time $\tau = t - z/c$, keeping z as the propagation coordinate. Under this transformation

$$\left. \frac{\partial}{\partial z} \right|_t = \left. \frac{\partial}{\partial z} \right|_\tau - \frac{1}{c} \left. \frac{\partial}{\partial \tau} \right|_z, \quad \left. \frac{\partial}{\partial t} \right|_z = \left. \frac{\partial}{\partial \tau} \right|_z, \quad (32)$$

so the operator $\partial_z|_t + c^{-1}\partial_t|_z$ reduces to $\partial_z|_\tau$, and the field equation becomes a pure spatial propagation equation. The full Maxwell–Bloch system in retarded-time coordinates is:

$$\frac{\partial w}{\partial \tau} = \frac{i}{2\hbar} (dAR^* - d^*A^*R) - \frac{w - w_0}{T_1} + \Lambda_n, \quad (33)$$

$$\frac{\partial R}{\partial \tau} = -\frac{id}{\hbar} Aw - \frac{R - R_0}{T_2}, \quad (34)$$

$$\frac{\partial A}{\partial z} = \frac{i\omega n_0 d^*}{2\epsilon_0 c} R. \quad (35)$$

6 Dimensionless Form

6.1 Scaled variables

The inversion w and coherence R are already dimensionless. We introduce dimensionless forms of the remaining variables by normalising lengths to the sample length L , times to the characteristic scale T_0 , and the field amplitude to $\hbar/dT_0$:

$$z' = \frac{z}{L}, \quad \tau' = \frac{\tau}{T_0}, \quad T'_1 = \frac{T_1}{T_0}, \quad T'_2 = \frac{T_2}{T_0}, \quad \Lambda'_n = T_0 \Lambda_n. \quad (36)$$

To absorb the imaginary unit in the field equation we define the dimensionless field amplitude

$$A' = \frac{A}{A_0}, \quad (37)$$

where the field amplitude scale is

$$A_0 = -i \frac{\hbar}{dT_0}. \quad (38)$$

6.2 Cooperation time T_R

Substituting Eqs. (37)–(38) into Eq. (35):

$$\frac{\partial A'}{\partial z'} = -\eta R, \quad (39)$$

where

$$\eta = \frac{T_0}{T_R}, \quad (40)$$

and the **cooperation time** is defined as

$$T_R = \frac{2\epsilon_0 c \hbar}{\omega |d|^2 n_0 L}. \quad (41)$$

6.3 Final dimensionless equations

Substituting Eq. (37) into Eqs. (33)–(35) and expressing all times in units of T_0 :

$$\frac{\partial w}{\partial \tau'} = \text{Re}(A'R^*) - \frac{w - w_0}{T'_1} + \Lambda'_n, \quad (42)$$

$$\frac{\partial R}{\partial \tau'} = -A'w - \frac{R - R_0}{T'_2}, \quad (43)$$

$$\frac{\partial A'}{\partial z'} = -\eta R. \quad (44)$$

Note

No explicit $\hbar$, d , or T_0 appear in Eqs. (42)–(44). All microscopic parameters are absorbed into η and the dimensionless field amplitude A' .

6.4 Symbol table

Symbol	Meaning	Range	Units
$A_0 = -i\hbar/dT_0$	Field amplitude scale	$\mathbb{C}$	(V/m)
$A' = A/A_0$	Dimensionless field envelope	$\mathbb{C}$	—
$R(z', \tau')$	Coherence envelope ($\rho_{21} = \frac{1}{2}R e^{i(kz - \omega t)}$)	$\mathbb{C}$	—
$w(z', \tau')$	Population inversion $\rho_{22} - \rho_{11}$	$\mathbb{R}$	—
$z' = z/L$	Normalised propagation distance	$[0, 1]$	—
$\tau' = \tau/T_0$	Normalised retarded time	$\mathbb{R}^+$	—
T_0	Physical reference time scale	$\mathbb{R}^+$	(s)
$T'_1 = T_1/T_0$	Longitudinal relaxation time (may vary with τ')	$\mathbb{R}^+$	—
$T'_2 = T_2/T_0$	Transverse dephasing time (may vary with τ')	$\mathbb{R}^+$	—
T_R	Cooperation time (Eq. 41)	$\mathbb{R}^+$	(s)
$\eta = T_0/T_R$	Dimensionless coupling parameter	$\mathbb{R}^+$	—
$\Lambda'_n = T_0\Lambda_n$	Dimensionless pump rate	$\mathbb{R}$	—
w_0	Equilibrium inversion	$\mathbb{R}$	—

7 Initial and Boundary Conditions

7.1 General case

The Maxwell–Bloch system (42)–(44) requires:

- an **initial condition** for R at $\tau' = 0$ for all z' ,
- a **boundary condition** for A' at $z' = 0$ for all τ' .

The inversion w is always real since ρ_{11} and ρ_{22} are real probabilities. In the general case R and A' are complex.

7.2 Real-variable reduction

We now show that under a natural choice of initial and boundary conditions, R and A' remain real for all τ' and z' , leading to a significant simplification.

Suppose the initial condition for the coherence satisfies

$$R(z', 0) \in \mathbb{R}, \quad (45)$$

and the boundary field is real:

$$A'(0, \tau') \in \mathbb{R}. \quad (46)$$

Note that A' real corresponds to A purely imaginary, since $A' = A/A_0$ with $A_0 = -i\hbar/dT_0$.

Inspecting the equations: if w , R , and A' are all real at some τ' , then the right-hand sides of Eqs. (42)–(44) are all real, so the variables remain real for all subsequent times and positions. Under these conditions $\text{Re}(A'R^*) = A'R$, and the Maxwell–Bloch system reduces to:

$$\frac{\partial w}{\partial \tau'} = A'R - \frac{w - w_0}{T_1'} + \Lambda_n', \quad (47)$$

$$\frac{\partial R}{\partial \tau'} = -A'w - \frac{R - R_0}{T_2'}, \quad (48)$$

$$\frac{\partial A'}{\partial z'} = -\eta R. \quad (49)$$

Note

Equations (47)–(49) are entirely real and require no complex arithmetic. This is the system solved by the application. The equations hold unchanged when the relaxation times depend on time, $T_1' = T_1'(\tau')$ and $T_2' = T_2'(\tau')$. How the boundary field, initial conditions, pump profile, and relaxation times are entered is described in the User Guide.

8 Recovering the Physical Fields

We define the field and polarization amplitude scales

$$E_0 = \frac{\hbar}{|d|T_0}, \quad P_0 = n_0 |d|. \quad (50)$$

Given the dimensionless solution $A'(z', \tau')$, the instantaneous electric field and macroscopic polarization are recovered as

$$E(z, t) = E_0 \text{Re}[-iA'(z', \tau') e^{i(kz - \omega t)}], \quad (51)$$

$$P(z, t) = P_0 \text{Re}[R(z', \tau') e^{i(kz - \omega t)}]. \quad (52)$$

Since the carrier $e^{i(kz - \omega t)}$ oscillates much faster than the envelopes A' and R , physical detectors respond only to the slowly varying amplitudes. The observable intensity is therefore proportional to $|A'|^2$.

In the real case (Subsection 7.2), and taking d real, A' and R are real, and the fields reduce to

$$E(z, t) = E_0 A'(z', \tau') \sin(kz - \omega t), \quad (53)$$

$$P(z, t) = P_0 R(z', \tau') \cos(kz - \omega t). \quad (54)$$

9 Magnetic-Dipole Case

All preceding sections were derived for the electric-dipole case. The magnetic-dipole (maser) case follows from the substitutions

$$E \rightarrow cB, \quad d \rightarrow \frac{m}{c}, \quad P \rightarrow \frac{M}{c}, \quad (55)$$

where $m = \mathbf{m}_{21} \cdot \hat{\mathbf{b}}$ is the magnetic transition dipole matrix element and $\hat{\mathbf{b}}$ is the magnetic field polarization unit vector. Under these replacements the coupling $dE = (m/c)(cB) = mB$ is preserved, and the dimensionless equations (47)–(49) hold unchanged in form.

The table below summarises the correspondence between the two cases at each stage of the derivation.

Quantity	Electric	Magnetic
<i>Microscopic quantities</i>		
Interaction Hamiltonian	$\hat{H}_{\text{int}} = -\hat{\mathbf{d}} \cdot \mathbf{E}$	$\hat{H}_{\text{int}} = -\hat{\mathbf{m}} \cdot \mathbf{B}$
Coupling constant	$d = \mathbf{d}_{21} \cdot \hat{\mathbf{e}}$	$m = \mathbf{m}_{21} \cdot \hat{\mathbf{b}}$
1D wave equation	$\frac{\partial^2 E}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} = \mu_0 \frac{\partial^2 P}{\partial t^2}$	$\frac{\partial^2 B}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 B}{\partial t^2} = \mu_0 \frac{\partial^2 M}{\partial z^2}$
Medium response	$P = n_0(\rho_{12} d + \rho_{21} d^*)$	$M = n_0(\rho_{12} m + \rho_{21} m^*)$
Field envelope	$E = \frac{1}{2} A e^{i(kz - \omega t)} + \text{c.c.}$	$B = \frac{1}{2} A e^{i(kz - \omega t)} + \text{c.c.}$
<i>Scaling quantities</i>		
Cooperation time	$T_R = \frac{2\epsilon_0 c \hbar}{\omega d ^2 n_0 L}$	$T_R = \frac{2c \hbar}{\omega \mu_0 m ^2 n_0 L}$
Field amplitude scale	$A_0 = -i \frac{\hbar}{d T_0}$	$A_0 = -i \frac{\hbar}{m T_0}$
Field scale	$E_0 = \frac{\hbar}{ d T_0}$	$B_0 = \frac{\hbar}{ m T_0}$
Medium response scale	$P_0 = n_0 d $	$M_0 = n_0 m $
Intensity scale	$I_0 = \frac{\epsilon_0 c E_0^2}{2}$	$I_0 = \frac{c B_0^2}{2\mu_0}$
<i>Physical observables (real case)</i>		
Physical field	$E_0 A' \sin(kz - \omega t)$	$B_0 A' \sin(kz - \omega t)$
Medium response	$P_0 R \cos(kz - \omega t)$	$M_0 R \cos(kz - \omega t)$
Intensity	$I_0 A' ^2 \sin^2(kz - \omega t)$	

In both cases the cooperation time can be expressed in terms of the single-atom spontaneous emission rate Γ . Using the standard relations

$$\Gamma = \frac{\omega^3 |d|^2}{3\pi\epsilon_0 \hbar c^3} \quad (\text{electric}), \quad \Gamma = \frac{\omega^3 |m|^2}{3\pi\epsilon_0 \hbar c^5} \quad (\text{magnetic}), \quad (56)$$

substituting into the respective expressions for T_R yields the same result in both cases:

$$T_R = \frac{2\omega^2}{3\pi c^2 n_0 L \Gamma}. \quad (57)$$

The cooperation time is therefore inversely proportional to n_0 , L , and Γ . For a sufficiently dense and inverted medium, $T_R \ll T_1$ and the system enters the superradiant regime. In the application, ν , Γ , n_0 , and L are entered in the **Sample** tab, from which T_R is computed internally.

Note

Because the dimensionless equations (47)–(49) depend only on η , T'_1 , T'_2 , w_0 , R_0 , Λ'_n , and the boundary field $A'(0, \tau')$, the same solver describes both electric-dipole and magnetic-dipole physics.