which, in turn, is equivalent to (3) in Definition 2.1. It remains to show that (1) in Definition 2.1 is equivalent to (a) above. To this end, we note that (1) in Definition 2.1 is equivalent to

$$DV = V_x. \tag{3.2}$$

Substituting $D = P^{-1}Q$ and multiplying both sides of (3.2) by P from the left, demonstrates that (1) in Definition 2.1 is equivalent to

$$QV = PV_x. \tag{3.3}$$

We next split Q into its symmetric and anti-symmetric part, rewriting it as $Q = (Q+Q^T)/2+(Q-Q^T)/2$, where the symmetric and anti-symmetric part is given by $(Q+Q^T)/2 = B/2$ and $(Q-Q^T)/2 = S$, respectively. Substituting the resulting representation $Q = B/2 + S$ into (3.3) yields that (1) in Definition 2.1 is equivalent to

$$BV/2 + SV = PV_x. \tag{3.4}$$

Finally, (3.4) can be reformulated as

$$\underbrace{\begin{bmatrix} S & P \end{bmatrix}}_{=X} \underbrace{\begin{bmatrix} V \\ -V_x \end{bmatrix}}_{=W} = -BV/2, \tag{3.5}$$

which shows that (1) in Definition 2.1 is equivalent to (a) above. ■

Building upon Lemma 3.1, we can now identify FSBP operators as solutions to specific optimization problems. To this end, recall that an $\mathcal{F}$-exact diagonal-norm FSBP operator can be written as $D = P^{-1}(S + B/2)$, where P is a diagonal positive definite matrix and S is anti-symmetric. Furthermore, observe that (a) in Lemma 3.1 is equivalent to $\|XW + BV/2\|_2^2 = 0$. Hence, if $D = P^{-1}(S + B/2)$ is an $\mathcal{F}$-exact diagonal-norm FSBP operator then $X = [S, P]$ is the global minimizer of the constrained quadratic minimization problem

$$\min_{X\in\mathcal{X}} \left\| XW + BV/2 \right\|_2^2. \tag{3.6}$$

Here, the set of admissible solutions, or constraints, are

$$\mathcal{X} = \left\{ X = [S, P] \mid S^T = -S, P = \operatorname{diag}(p_1, \dots, p_N),\ p_i > 0, \sum_{n=1}^{N} p_n = x_R - x_L \right\}, \tag{3.7}$$

ensuring that S is anti-symmetric, P is diagonal positive definite, and P is exact for constants. The first two constraints correspond to (b) and (c) in Lemma 3.1, while the last one ensures that Requirement 2.3 holds. Conversely, if $X = [S, P]$ is a solution of (3.6) with $\left\| XW + BV/2 \right\|_2^2 = 0$, then $D = P^{-1}(S+B/2)$ is an $\mathcal{F}$-exact diagonal-norm FSBP operator. We summarize this characterization of FSBP operators as solutions of the constrained quadratic optimization problem (3.6) below in Lemma 3.2.

Lemma 3.2. *Let $\mathcal{F} \subset C^1([x_L, x_R])$. The operator $D = P^{-1}(S + B/2) \in \mathbb{R}^{N\times N}$ is an $\mathcal{F}$-exact diagonal-norm FSBP operator (see Definition 2.1) if and only if $X = [S, P] \in \mathbb{R}^{N\times 2N}$ solves the constraint quadratic minimization problem (3.6) that satisfies $\left\| XW + BV/2 \right\|_2^2 = 0$.*