In this document we compute the double convolution between a regular coefficient function $C(x)$ and a splitting function $P(x)$ containing a regular, a singular and a local part, i.e.

$$P(x) = P^R(x) + P^S(x) + P^L \delta(x-1), \tag{1}$$

where P^R is regular in $x \to 1$, while P^S is singular in $x \to 1$. The convolution between $C(x)$ and $P(x)$ is

$$
(C \otimes P)(x) = \int_x^{x_\mathrm{m}} \frac{dz_2}{z_2} C(z_2) P^R \left(\frac{x}{z_2} \right) + \int_{\frac{x}{x_\mathrm{m}}}^1 dz_2 P^S(z_2) \left(\frac{C\left(\frac{x}{z_2}\right)}{z_2} - C(x) \right) - C(x) \int_0^{\frac{x}{x_\mathrm{m}}} dz_2 P^S(z_2) + C(x) P^L
$$

$$
= \int_x^{x_\mathrm{m}} \frac{dz_2}{z_2} C(z_2) P^R \left(\frac{x}{z_2} \right) + \int_{\frac{x}{x_\mathrm{m}}}^1 dz_2 P^S(z_2) \left(\frac{C\left(\frac{x}{z_2}\right)}{z_2} - C(x) \right) + C(x) \left(P^L - \mathcal{P}^S \left(\frac{x}{x_\mathrm{m}} \right) \right), \tag{2}
$$

where we defined

$$\mathcal{P}^S(x) = \int_0^x dz\, P^S(z),$$

and we used that $C(x)$ is defined for $0 < x < x_\mathrm{m}$ and zero outside. The double convolution is defined as

$$
(C \otimes P \otimes P)(x) = \int_x^{x_\mathrm{max}} \frac{dz_1}{z_1} (C \otimes P)(z_1) P^R \left(\frac{x}{z_1} \right)
$$

$$
+ \int_{\frac{x}{x_\mathrm{m}}}^1 dz_1 P^S(z_1) \left(\frac{(C \otimes P)\left(\frac{x}{z_1}\right)}{z_1} - (C \otimes P)(x) \right) \tag{3}
$$

$$
+ (C \otimes P)(x) \left(P^L - \mathcal{P}^S \left(\frac{x}{x_\mathrm{m}} \right) \right).
$$

Let's consider each piece separately.

1 Regular term

$$
\int_x^{x_\mathrm{max}} \frac{dz_1}{z_1} (C \otimes P)(z_1) P^R \left(\frac{x}{z_1} \right) =
$$

$$
\int_x^{x_\mathrm{m}} \frac{dz_1}{z_1} P^R \left(\frac{x}{z_1} \right) \left[\int_{z_1}^{x_\mathrm{m}} \frac{dz_2}{z_2} C(z_2) P^R \left(\frac{z_1}{z_2} \right) + \int_{\frac{z_1}{x_\mathrm{m}}}^1 dz_2 P^S(z_2) \left(\frac{C\left(\frac{z_1}{z_2}\right)}{z_2} - C(z_1) \right) + C(z_1) \left(P^L - \mathcal{P}^S \left(\frac{z_1}{x_\mathrm{m}} \right) \right) \right]
$$

$$
= \int_x^{x_\mathrm{m}} dz_1 \int_{z_1}^{x_\mathrm{m}} dz_2 \frac{1}{z_1 z_2} P^R \left(\frac{x}{z_1} \right) C(z_2) P^R \left(\frac{z_1}{z_2} \right) + \int_x^{x_\mathrm{m}} dz_1 \int_{\frac{z_1}{x_\mathrm{m}}}^1 dz_2 \frac{P^R \left(\frac{x}{z_1} \right) P^S(z_2)}{z_1} \left(\frac{C\left(\frac{z_1}{z_2}\right)}{z_2} - C(z_1) \right) \tag{4}
$$

$$
- \int_x^{x_\mathrm{m}} \frac{dz_1}{z_1} P^R \left(\frac{x}{z_1} \right) C(z_1) \mathcal{P}^S \left(\frac{z_1}{x_\mathrm{m}} \right) + P^L \int_x^{x_\mathrm{m}} \frac{dz_1}{z_1} P^R \left(\frac{x}{z_1} \right) C(z_1).
$$

In the second integral of the last line we can perform the change of variable $z_2 \to \frac{z_1}{z_2}$, so that we get

$$
= \int_x^{x_\mathrm{m}} dz_1 \int_{z_1}^{x_\mathrm{m}} dz_2 \frac{1}{z_1 z_2} P^R \left(\frac{x}{z_1} \right) C(z_2) P^R \left(\frac{z_1}{z_2} \right) + \int_x^{x_\mathrm{m}} dz_1 \int_{z_1}^{x_\mathrm{m}} \frac{dz_2}{z_2} P^R \left(\frac{x}{z_1} \right) P^S \left(\frac{z_1}{z_2} \right) \left(\frac{C(z_2)}{z_1} - \frac{C(z_1)}{z_2} \right)
$$

$$
- \int_x^{x_\mathrm{m}} dz_1 \frac{P^R \left(\frac{x}{z_1} \right) C(z_1) \mathcal{P}^S \left(\frac{z_1}{x_\mathrm{m}} \right)}{z_1} + P^L \int_x^{x_\mathrm{m}} \frac{dz_1}{z_1} P^R \left(\frac{x}{z_1} \right) C(z_1). \tag{5}
$$

Now the double integrals can be trasformed in integrals over rectangles, using the theta functions:

$$
= \int \int_x^{x_\mathrm{m}} dz_1 dz_2 \theta(z_2 - z_1) \frac{1}{z_1 z_2} P^R \left(\frac{x}{z_1} \right) C(z_2) P^R \left(\frac{z_1}{z_2} \right)
$$

$$
+ \int \int_x^{x_\mathrm{m}} dz_1 dz_2 \theta(z_2 - z_1) \frac{1}{z_2} P^R \left(\frac{x}{z_1} \right) P^S \left(\frac{z_1}{z_2} \right) \left(\frac{C(z_2)}{z_1} - \frac{C(z_1)}{z_2} \right) \tag{6}
$$

$$
- \int_x^{x_\mathrm{m}} dz_1 \frac{P^R \left(\frac{x}{z_1} \right) C(z_1) \mathcal{P}^S \left(\frac{z_1}{x_\mathrm{m}} \right)}{z_1} + P^L \int_x^{x_\mathrm{m}} \frac{dz_1}{z_1} P^R \left(\frac{x}{z_1} \right) C(z_1).
$$

The last equation is what has been implemented in the code with Monte Carlo integrals.

2 Singular term

$$\int_{\frac{x}{x_{\mathrm{m}}}}^{1} dz_1 P^{S}(z_1) \left\{ \frac{(C \otimes P)\left(\frac{x}{z_1}\right)}{z_1} - (C \otimes P)(x) \right\} =$$

$$\int_{\frac{x}{x_{\mathrm{m}}}}^{1} dz_1 P^{S}(z_1) \Bigg\{ \frac{1}{z_1} \left[\int_{\frac{x}{z_1}}^{x_{\mathrm{m}}} \frac{dz_2}{z_2} C(z_2) P^{R}\left(\frac{x}{z_1 z_2}\right) + \int_{\frac{x}{z_1 x_{\mathrm{m}}}}^{1} dz_2 P^{S}(z_2) \left(\frac{C\left(\frac{x}{z_1 z_2}\right)}{z_2} - C\left(\frac{x}{z_1}\right) \right) + C\left(\frac{x}{z_1}\right)\left(P^{L} - \mathcal{P}^{S}\left(\frac{x}{z_1 x_{\mathrm{m}}}\right) \right) \right]$$

$$- \left[\int_{x}^{x_{\mathrm{m}}} \frac{dz_2}{z_2} C(z_2) P^{R}\left(\frac{x}{z_2}\right) + \int_{\frac{x}{x_{\mathrm{m}}}}^{1} dz_2 P^{S}(z_2) \left(\frac{C\left(\frac{x}{z_2}\right)}{z_2} - C(x) \right) + C(x)\left(P^{L} - \mathcal{P}^{S}\left(\frac{x}{x_{\mathrm{m}}}\right) \right) \right] \Bigg\}$$

$$= \int_{\frac{x}{x_{\mathrm{m}}}}^{1} dz_1 \int_{\frac{x}{z_1}}^{x_{\mathrm{m}}} dz_2 \frac{1}{z_1 z_2} P^{S}(z_1) C(z_2) P^{R}\left(\frac{x}{z_1 z_2}\right) + \int_{\frac{x}{x_{\mathrm{m}}}}^{1} dz_1 \int_{\frac{x}{z_1 x_{\mathrm{m}}}}^{1} dz_2 \frac{P^{S}(z_1) P^{S}(z_2)}{z_1} \left(\frac{C\left(\frac{x}{z_1 z_2}\right)}{z_2} - C\left(\frac{x}{z_1}\right) \right)$$

$$- \int_{\frac{x}{x_{\mathrm{m}}}}^{1} dz_1 \frac{P^{S}(z_1) \mathcal{P}^{S}\left(\frac{x}{z_1 x_{\mathrm{m}}}\right)}{z_1} C\left(\frac{x}{z_1}\right) + P^{L} \int_{\frac{x}{x_{\mathrm{m}}}}^{1} \frac{dz_1}{z_1} P^{S}(z_1) C\left(\frac{x}{z_1}\right)$$

$$- \int_{\frac{x}{x_{\mathrm{m}}}}^{1} dz_1 \int_{x}^{x_{\mathrm{m}}} \frac{dz_2}{z_2} P^{S}(z_1) C(z_2) P^{R}\left(\frac{x}{z_2}\right) - \int_{\frac{x}{x_{\mathrm{m}}}}^{1} dz_1 \int_{\frac{x}{x_{\mathrm{m}}}}^{1} dz_2 P^{S}(z_1) P^{S}(z_2) \left(\frac{C\left(\frac{x}{z_2}\right)}{z_2} - C(x) \right) \tag{7}$$

$$+ \int_{\frac{x}{x_{\mathrm{m}}}}^{1} dz_1 P^{S}(z_1) C(x) \mathcal{P}^{S}\left(\frac{x}{x_{\mathrm{m}}}\right) - P^{L} \int_{\frac{x}{x_{\mathrm{m}}}}^{1} dz_1 P^{S}(z_1) C(x) \,.$$

Collecting the terms with same integration boundaries, and using theta functions, we get

$$= \int_{\frac{x}{x_{\mathrm{m}}}}^{1} dz_1 \int_{x}^{x_{\mathrm{m}}} dz_2 P^{S}(z_1) \left(\frac{P^{R}\left(\frac{x}{z_1 z_2}\right) \theta\left(z_2 - \frac{x}{z_1}\right)}{z_1} - P^{R}\left(\frac{x}{z_2}\right) \right) \frac{C(z_2)}{z_2}$$

$$+ \int_{\frac{x}{x_{\mathrm{m}}}}^{1} dz_1 \int_{\frac{x}{x_{\mathrm{m}}}}^{1} dz_2 P^{S}(z_1) \left[\frac{P^{S}(z_2)}{z_1} \left(\frac{C\left(\frac{x}{z_1 z_2}\right)}{z_2} - C\left(\frac{x}{z_1}\right) \right) \theta\left(z_2 - \frac{x}{z_1 x_{\mathrm{m}}}\right) - P^{S}(z_2) \left(\frac{C\left(\frac{x}{z_2}\right)}{z_2} - C(x) \right) \right] \tag{8}$$

$$- \int_{\frac{x}{x_{\mathrm{m}}}}^{1} dz_1 P^{S}(z_1) \left(\frac{C\left(\frac{x}{z_1}\right) \mathcal{P}^{S}\left(\frac{x}{z_1 x_{\mathrm{m}}}\right)}{z_1} - C(x) \mathcal{P}^{S}\left(\frac{x}{x_{\mathrm{m}}}\right) \right)$$

$$+ P^{L} \int_{\frac{x}{x_{\mathrm{m}}}}^{1} dz_1 P^{S}(z_1) \left(\frac{C\left(\frac{x}{z_1}\right)}{z_1} - C(x) \right) \,.$$

Observe that all these terms are finite since the singularity of P^{S} for $z \to 1$ is integrable since it always appears together with a term of the form $f(x/z)/z - f(x)$ so that the integral

$$\int_{x/x_{\mathrm{m}}}^{1} dz P^{S}(z) \left(\frac{f\left(\frac{x}{z}\right)}{z} - f(x) \right) \,,$$

is finite.