Dynamic behavior & inertia

author: stephane.ploix@grenoble-inp.fr

reviewer: mircea_stefan.simoiu@upb.ro

Intuitive approach of thermal dynamics

Let's consider an empty wooden house under a controlled outdoor temperature and a controlled solar irradiance: for the sake of clarity, each day, the weather is the same.

The house is 120m^2 well insulated with a 30cm external insulation in glass foam, and the windows are directed to the South with a surface of 18m^2.

We want to predict the behavior of such house.

Section I. Why using state-space models?

Suppose we know

- the current heat gain,
- the weather forecast,
- the heating schedule.

Can we predict the indoor temperature over the next 24 hours?
This is the objective of a **dynamic model**.

Why a model with derivative is required?

When the heating is switched on,

- the heater reacts immediately,
- but the indoor temperature changes slowly.

...because the building stores thermal energy in

- the walls,
- the floor,
- the ceiling,
- the furniture.

The future therefore depends on more than the current indoor temperature.

A building has an internal state

To predict the future, we must know the building's current internal condition e.g. the quantity of energy stored.
This internal condition is called the **state**.

> “ The state summarizes all the information required to predict the future behaviour of the system.
> The state represents the system's memory. ”

Inputs, states and outputs

Every dynamic system is described by three kinds of variables.

Variable	Meaning	Building example
Inputs U	External actions acting on the system	Heating power, outdoor temperature, solar radiation
States X	Internal condition of the system	Wall temperatures, indoor temperature
Outputs Y	Variables we predict	Indoor temperature

Example

For a simple building,

$$X = \begin{bmatrix} T_{wall} \\ T_{air} \end{bmatrix}$$

$$U = \begin{bmatrix} P_{heat} \\ T_{out} \\ G_{solar} \end{bmatrix}$$

$$Y = T_{air}$$

How does the state evolve?

The evolution of the state depends on

- the current state,
- the current inputs.

In its most general form,

$$\dot{X} = f(X, U)$$

The outputs (simulated variables) are computed from

$$Y = g(X, U)$$

Why using (slowly varying) linear models?

Because there are lots of tools to deal with. Nonlinear models are transformed into slowly varying linear models like:

From $\frac{d\,X(t)}{dt} = f(X(t), U(t))$ and $Y(t) = g(X(t), U(t))$, we try to get:

$$\frac{d\,X(t)}{dt} = A_t X(t) + B_t U(t)$$

$$Y(t) = C_t X(t) + D_t U(t)$$

where A_t, B_t, C_t and D_t are almost constant matrices (that will be simply denoted in the next A, B, C and D).

Physical interpretation of the matrices

Matrix	Physical meaning
A	Internal dynamics of the system
B	Effect of external inputs
C	Selection of the observed states
D	Direct influence of the inputs on the outputs

For a building,

- **A** describes heat exchanges between walls and air.
- **B** describes the influence of heating and weather.

More generally speaking

A system of **ordinary differential equations** (containing only linear equations with time derivatives) can always be rewritten as:

$$
\begin{aligned}
\frac{x_1}{dt} &= f_1(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_m) \\
\frac{x_2}{dt} &= f_2(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_m) \\
&\vdots \\
\frac{x_n}{dt} &= f_n(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_m)
\end{aligned}
$$

with:

$$
\begin{aligned}
y_1 &= g_1(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_m) \\
y_2 &= g_2(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_m) \\
&\vdots \\
y_n &= g_n(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_m)
\end{aligned}
$$

If $f_i()$ and $g_i()$ are (slowly varying) linear, the system such a system of equations can be rewritten as:

$$\frac{x_i}{dt} = a_1x_1 + a_2x_2 + \cdots + a_nx_n + b_1u_1 + b_2u_2 + \cdots + b_mu_m$$

$$y_i = c_1x_1 + c_2x_2 + \cdots + c_nx_n + d_1u_1 + d_2u_2 + \cdots + d_mu_m$$

This can be rewritten using a matrix formalism:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1m} \\ b_{21} & b_{22} & \dots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nm} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$$

and

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \dots & c_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} d_{11} & d_{12} & \dots & d_{1m} \\ d_{21} & d_{22} & \dots & d_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ d_{n1} & d_{n2} & \dots & d_{nm} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$$

Continuous and discrete models

Nature evolves continuously,

$$\frac{d\,X(t)}{dt} = A(t)X(t) + B(t)U(t)$$

but computers calculate periodically. After time discretization, we get (A_d and B_d different from A, B):

$$X_{k+1} = A_d X_k + B_d U_k$$

$$Y_k = C_d X_k + D_d U_k$$

State update principle for simulation

At every sampling period,

```
Current state X(k) → Read inputs U(k) → Compute X(k+1) → Compute outputs Y(k) → Repeat
```

The newly computed state becomes the starting point for the next time step.

Why this representation is so useful?

Once a state-space model is available, it can be used for

- dynamic simulation,
- state estimation,
- fault detection,
- optimal control,
- Model Predictive Control (MPC).

The state-space representation provides a unified mathematical framework for analyzing and controlling dynamic systems.

In short

The concise form of the state space model is:

$$\begin{cases} \frac{dX}{dt} = AX + BU \\ Y = CX + DU \end{cases}$$

It's a standard easy way to represent dynamic linear system where $Y(t)$ is the output, $X(t)$ the state vector, $U(t)$ the input vector and A, B, C, D the matrices of the system.

If we consider that $U(t)$ is invariant in time during a sample period T_s the state space model can be automatically integrated during a sample period T_s to yield the discrete-time state space model:

$$\begin{aligned} X_{k+1} &= A' X_k + B' U_k \\ Y_k &= C' X_k + D' U_k \end{aligned}$$

where $U_k = U(kT_s)$, $X_k = X(kT_s)$ and $Y_k = Y(kT_s)$.

Such a model is easy to simulated providing you know the initial state $X(0) = X_0$ and the input $U(t)$ for all $t \in [0, T_s]$. It is indeed a recurrent equation easy to solve:

$$
\begin{cases}
U_0, X_0 \rightarrow X_1 = A'X_0 + B'U_0 \\
U_0, X_0 \rightarrow Y_0 = C'X_0 + D'U_0 \\
U_1, X_1 \rightarrow X_2 = A'X_1 + B'U_1 \\
U_1, X_1 \rightarrow Y_1 = C'X_1 + D'U_1 \\
\ldots \\
U_{k-1}, X_{k-1} \rightarrow X_k = A'X_{k-1} + B'U_{k-1} \\
U_{k-1}, X_{k-1} \rightarrow Y_{k-1} = C'X_{k-1} + D'U_{k-1} \\
U_k, X_k \rightarrow Y_k = C'X_k + D'U_k
\end{cases}
$$

It is understood by simulation software like [Simulink] or [Dymola].

A same dynamic behavior can be represented by an infinite number of state space models. Indeed, the state space model is not unique. It is a way to represent the dynamic behavior of a system.

Let's choose a invertible P matrix such as $X_k = P\mathcal{X}_k$ where $\mathcal{X}_k$ is the new state vector. We get a new input-output equivalent state space model:

$$\mathcal{X}_{k+1} = (P^{-1}A'P)\mathcal{X}_k + (P^{-1}B')UY = (C'P)\mathcal{X} + D'U$$

A particular case is when P is such that $P^{-1}AP$ is a diagonal matrix i.e. it has 0 everywhere except on the diagonal. The dynamics, called modes of the system of equations, are uncoupled and appear in the diagonal

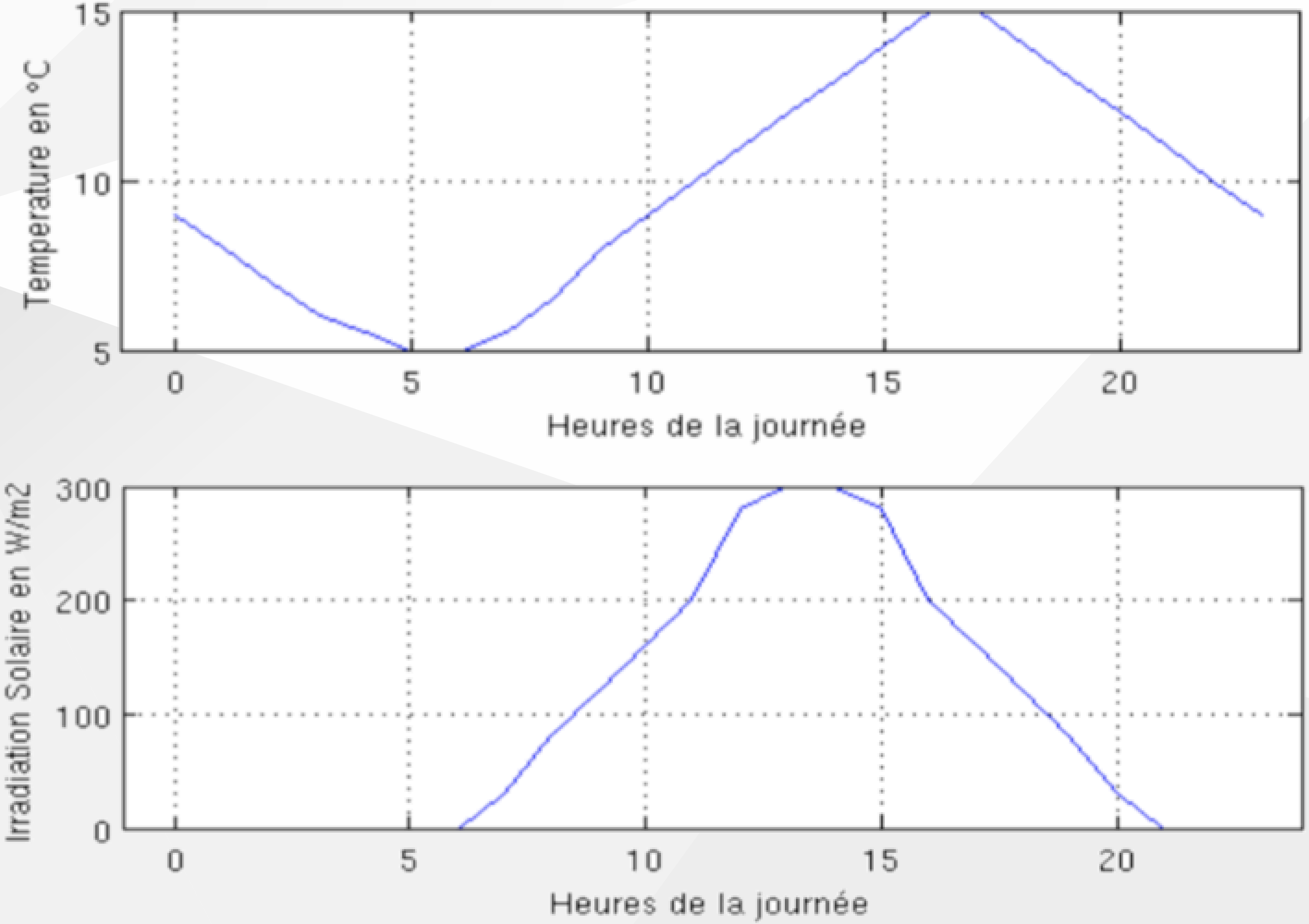
If you need further explanations, you can read this [doc/ssm.pdf](doc/ssm.pdf).

The state model of the swedish wooden house that has been identified is given by:

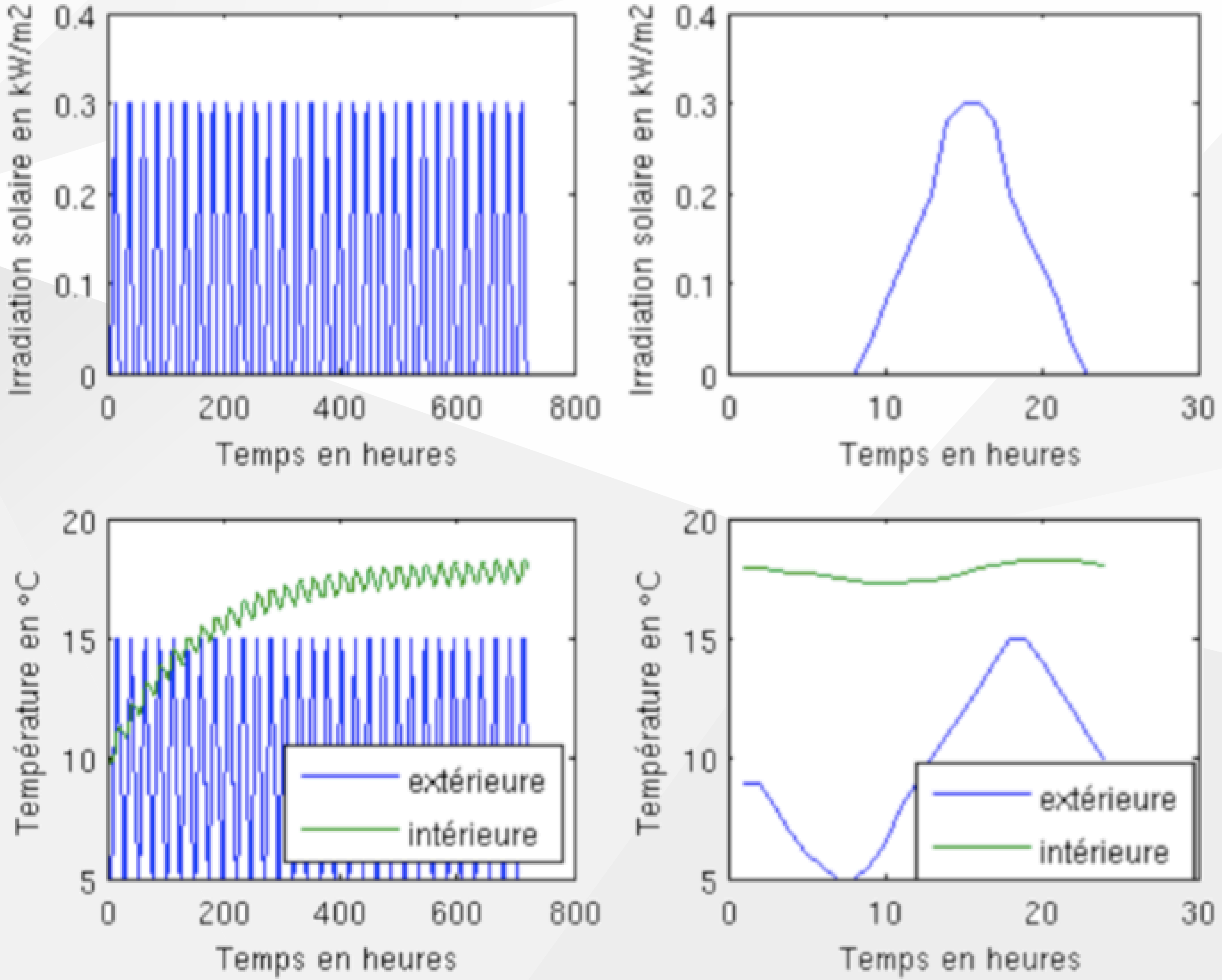
$$\frac{d}{dt}\begin{bmatrix} T_i(t) \\ T_w(t) \end{bmatrix} = \begin{bmatrix} \frac{-1}{r_i C_m} & \frac{1}{r_i C_m} \\ \frac{-1}{r_i C_i} & \frac{-1}{r_m C_i} + \frac{-1}{r_i C_i} \end{bmatrix} \begin{bmatrix} T_i(t) \\ T_w(t) \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{r_m C_i} & \frac{1}{C_i} & \frac{A_w}{r_m C_i} \end{bmatrix} \begin{bmatrix} T_o(t) \\ \Phi_r(t) \\ \Phi_s(t) \end{bmatrix}$$

- T_w and T_i, the wall and internal air temperatures (in °C)
- T_o, the outdoor temperature (in °C)
- Φ_r, the solar irradiance (in W/m^2)
- Φ_s, the solar irradiance absorbed by the wall (in W/m^2)
- $C_i = 11.837kWh/^\circ C, C_m = 3.987kWh/^\circ C, r_i = 0.4788^\circ C/kW$, $r_m = 29.38^\circ C/kW$ and $A_w = 2.845m^2$

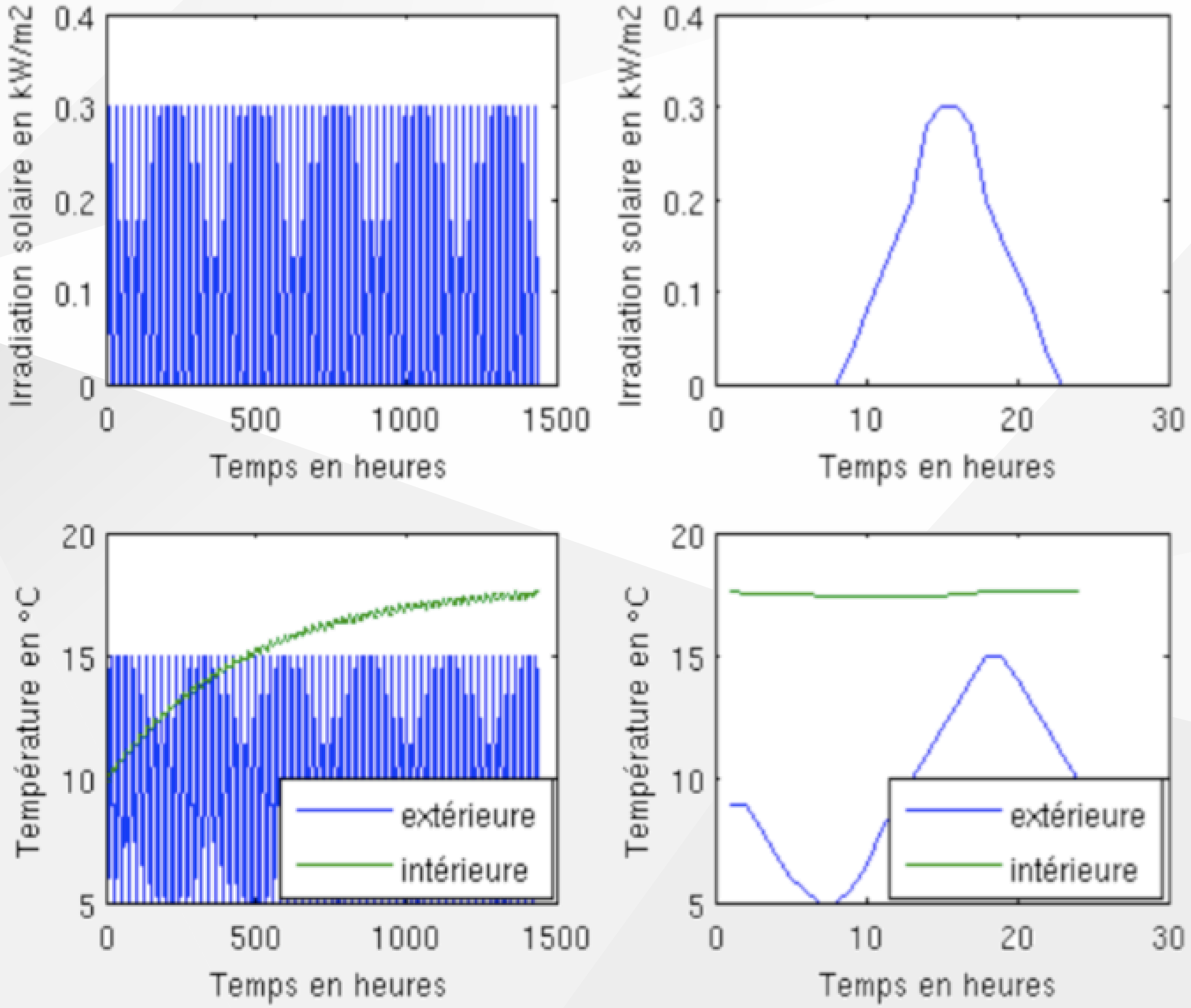
Here is the day used for the simulation:



Here is a 30 days reference simulation:



The capacity has been multiplied by 10 (60 days instead of 30):



Inertia in ThBAT (French standard to check a building performance)

- inertia is intervening in the Th-BCE performances, in the CEP and TIC.
- there are different kinds of inertia Th-I:
  - hourly inertia (in the Th-C), to compute HVAC intermittence
  - daily inertia (in Th-C and Th-E), absorption of the daily wave
  - sequential inertia (typ. 12 days in Th-E), absorption of the daily wave for summer

3 approaches are proposed:

- a simplified approach
- an approach based on points
- an approach based on calculations (ISO13786)

Diffusivity, thickness and inertia

Let's return to the heat equation applied to a wall (1D):

$$\frac{d^2T(X,t)}{dx^2} = \frac{\rho c_p}{\lambda}\frac{\partial T(X,t)}{\partial t}$$

with ρ, the density (kg/m^3), λ the conductivity (W/m.K), c_p the specific heat capacity (J/kg.K), T(X,t) the temperature (in K or °C) at position X and time t and $\varphi(X,t)$ the heat power.

$\alpha = \frac{\lambda}{\rho c_p}$ is the **thermal diffusivity**: the lower, the slower the heat will diffuse and therefore the heat will move slowly in the material and impact room temperature with delay.

The **volumetric heat storage capacity** is given by: ρc_p.

The **inertia** is due to materials having:

- **high values of heat storage capacity**
- **low values of thermal diffusivity**.

Let's recall a couple of values for diffusivity

material	conductivity (W/m.K)	specific heat capacity (J/kg.K)	density (kg/m3)	emmisivity coefficient	thermal diffusivity (mm^2/s)	volumetric heat capacity (J/m3/K)
immobile CO2	0.01	1,005	1.56	0	8.92	1570
immobile air	0.02	1,005	1.26	0	19.03	1261
polyurethane foam	0.03	1,000	30	0.82	0.97	30000
dry glass wool	0.05	840	25	0.97	2.38	21000
asbestos (amiante)	0.15	816	2,400	0.93	**0.08**	**1958400**
lightweight wood	0.04	1,300	400	0.95	0.08	520000
heavyweight wood	0.12	2,400	890	0.95	**0.06**	**2136000**
plaster	0.35	1,090	2,300	0.98	**0.14**	**2507000**
water	0.60	4,180	997	0.95	**0.14**	**4167460**
glass	1	753	2,600	0.94	**0.51**	**1957800**
full brick	1	840	2,000	0.93	**0.60**	**1680000**
concrete	0.5	880	2,000	0.85	**0.28**	**1760000**

material	conductivity (W/m.K)	specific heat capacity (J/kg.K)	density (kg/m3)	emmisivity coefficient	thermal diffusivity (mm ^{2}/s)	volumetric heat capacity (J/m3/K)
graphite	5	780	2,500	0.96	2.56	1950000
stainless steel	15	440	7,820	0.07	4.36	3440800
lead	35	129	11,350	0.43	23.90	1464150
iron	57	449	7,200	0.44	17.63	3232800
aluminium	203	897	2,700	0.07	83.82	2421900
copper	380	385	8,790	0.03	112.29	3384150
silver	410	235	10,500	0.03	166.16	2467500
wood	0.20	1,300	700	0.95	**0.22**	**910000**
PVC	0.25	1,046.00	1,395.00	0.92	**0.17**	**1459170**

For more materials, see `/buildingenergy/data/propertiesDB.xslx`

The depth of penetration of a temperature signal into a solid, such as a concrete wall, can be analyzed using principles of heat conduction. When the temperature signal is sinusoidal, it is governed by Fourier's law and can be described using the concept of thermal diffusivity.

For a temperature signal varying sinusoidally with time: $T(x,t) = T_0 e^{-x/\delta} \sin\left(2\pi \frac{t}{\tau}\right)$

The temperature wave will penetrate the material to a certain depth, with its amplitude decreasing exponentially. The penetration depth d_p is given by: $d_p = \sqrt{\frac{\alpha\tau}{\pi}}$.

- **Shallow Penetration:** Layers with smaller penetration depths (e.g., due to low diffusivity) contribute most to thermal inertia because they absorb and store heat near the surface.
- **Deep Penetration:** Layers with higher penetration depths contribute less because they allow heat to dissipate farther into the structure, reducing their ability to buffer surface temperature fluctuations.

The active wall layer thickness is given by:

$$d_a = min(d_{wall}, d_p) = min(d_{wall}, \sqrt{\frac{\alpha\tau}{\pi}}) = min(d_{wall}, \sqrt{\frac{\lambda\tau}{\pi\rho c_p}}).$$

Only the material within the penetration depth can absorb and release heat efficiently during a given time period.

Let's consider the 24h sine wave, that corresponds to the main frequency of the daily temperature variation, with a period $\tau = 24 \times 3600$s related to a 13cm concrete wall having $\lambda = 1.7$ W/m.K, $\rho = 2400$ kg/m^3 and $c_p = 880$ J/kg.K.

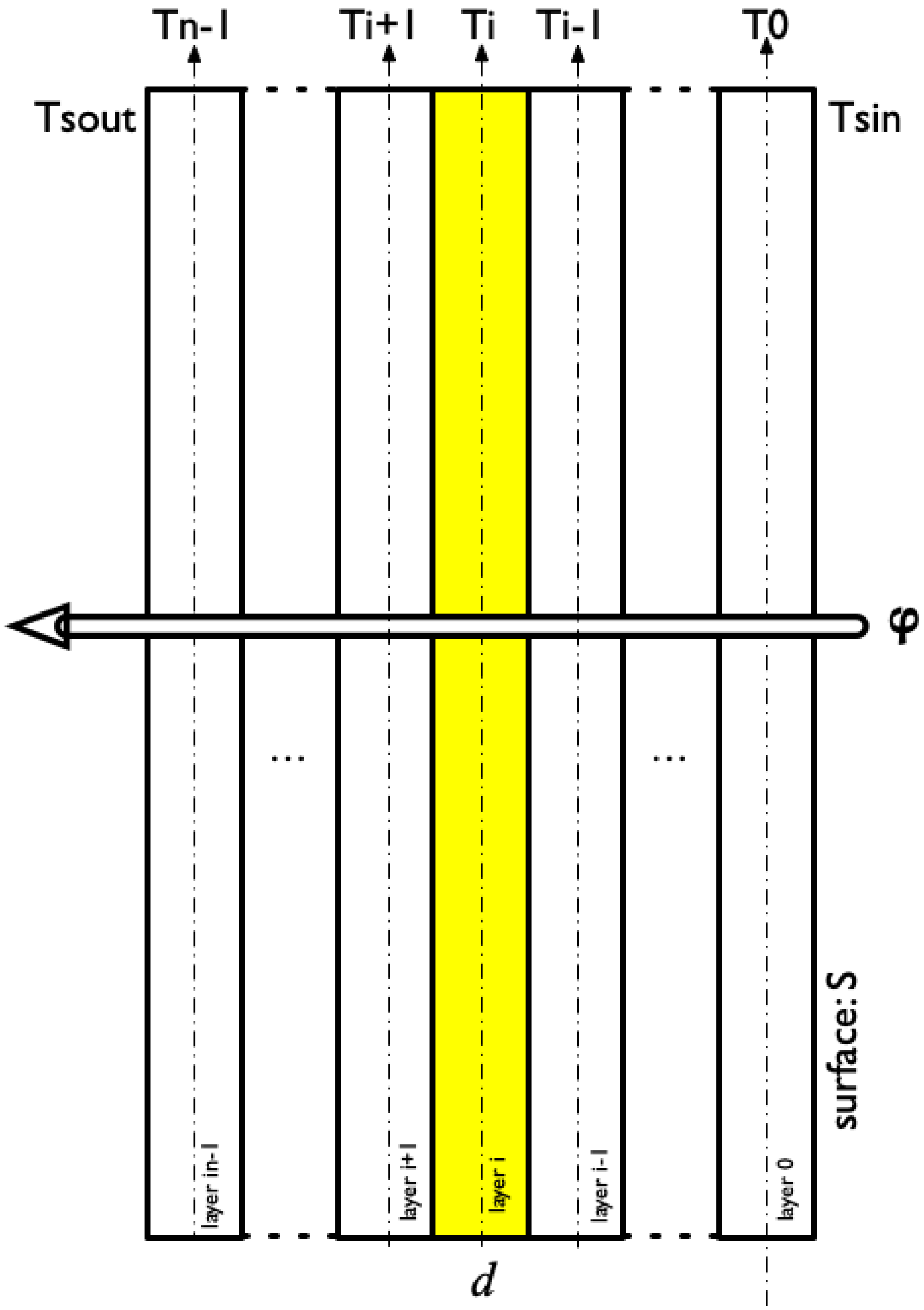
The resulting penetration depth is $d_p = 15$cm but the wall is 13cm thick. Therefore, the active wall layer thickness is $d_a = 13$cm.

We can calculate the thermal resistance and heat storage capacity of a fraction r_a of the active thickness of the wall i.e.: $R = r_a \frac{d_a}{\lambda S}$ and $C = r_a \rho c_p d_a S$ and, assimilating it to a first order system, the time constant is $RC = \frac{\rho c_p r_a^2 d_a^2}{\lambda}$.

$r_a \approx 0.9$ is a good approximation for a daily wave.

However, this approach is rather limited because r_a has still to be guessed and only a single wall with a unique homogeneous layer can be considered.

This approach is interesting for understanding the layer effect on inertia.



Tn-1
Ti+1
Ti
Ti-1
T0
Tsout
Tsin
φ
...
...
surface: S
layer in-1
layer i+1
layer i
layer i-1
layer 0
d

Each layer i has:

- a temperature T_i, a density ρ, a specific heat c_p, a conductivity λ
- a thickness d/n
- a surface S
- a mass $\frac{\rho d S}{n}$
- a stored energy $E_i = \frac{\rho c_p d S}{n} T_i$
- a heat flow (variation of energy) $\varphi = \frac{\rho c_p d S}{n} \frac{dT_i}{dt}$
- $\forall i / 1 \leq i \leq n-2, T_{i-1} - T_i = \frac{d}{\lambda S} \varphi$
- for extreme layer, $T_{sin} - T_0 = \frac{d}{2 \lambda S} \varphi$ and $T_{n-1} - T_{sout} = \frac{d}{2 \lambda S} \varphi$

State space model

A state space model can represent any linear set of ordinary differential equations with a set of matrices:

$$\frac{d}{dt}X(t) = AX(t) + BU(t)$$

$$Y(t) = CX(t) + DU(t)$$

with $X(t) \in \mathbb{R}^{n\times 1}$, the state vector, $U(t) \in \mathbb{R}^{m\times 1}$, the vector of inputs and $Y(t) \in \mathbb{R}^{p\times 1}$, the vector of outputs, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ and $D \in \mathbb{R}^{p\times m}$.

Let's apply this to a wall with n layers, the state space model can be written as:

Let $X(t) = [T_0(t), T_1(t), \ldots, T_{n-1}(t)]^T$ be the state vector of temperatures
Let $U(t) = [T_{sin}(t), T_{sout}(t)]^T$ be the input vector of boundary temperatures

The state matrix A is tridiagonal:

$$A = \frac{n\lambda}{\rho c_p d^2} \begin{bmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ 0 & 1 & -2 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & 0 & \cdots & 1 & -2 \end{bmatrix}$$

The input matrix B is:

$$B = \frac{2n\lambda}{\rho c_p d^2} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

For output, if we want to observe all layer temperatures:

$$C = I_{n\times n} \text{ and } D = 0_{n\times 2}$$

This model represents the heat diffusion through the wall layers, where:

- The diagonal terms in A represent heat loss to adjacent layers
- The off-diagonal terms in A represent heat gain from adjacent layers
- The B matrix connects the boundary temperatures to the first and last layers
- The scaling factor $\frac{n\lambda}{\rho c_p d^2}$ comes from the thermal diffusivity

A state model can be easily simulated, discretized in time and analyzed.
If the input values are composed of samples, constant during each time slot having a duration Δ each, the state space model can be discretized as:

$$X(t+\Delta) = \Phi X(t) + \Gamma U(t)$$

with $\Phi = e^{A\Delta}$ and $\Gamma = \int_0^{\Delta} e^{A\tau} B d\tau$= $(e^{A\Delta} - I)A^{-1}B$

Therefore, with such a recurrent model, simulating a wall amounts to iterate from time step to time step.

With Python, a recurrent state model can easily be obtained with:

```python
from scipy.signal import cont2discrete

A = ...
B = ...
C = ...
D = ...
sample_time = ...

Phi, Gamma, C, D = cont2discrete((A, B, C, D), sample_time, method='zoh')
```

If $U(t) = U_k + (U_{k+1} - U_k)\frac{t-k\Delta}{dt}$, the recurrent state model is obtained by using 'foh' (first order hold) instead of 'zoh' (zero order hold).

If the matrices of the state space model are slightly time varying, like for a time-varying ventilation, the state matrices has to recompute at each time step. A buffer can be used to store the matrices avoiding to recompute them at each time step.

Eigenvalues to analyze the dynamic behavior

A state space model is not unique: it can be modeled in different state spaces. Let's call P the projection matrix into the so-called canonical eigenvalues' model: $X = PX'$ such as $P^{-1}AP = \Lambda$ is a diagonal matrix. The state space model is then:

$$\frac{d}{dt}X'(t) = \Lambda X'(t) + P^{-1}BU(t)$$

$$Y(t) = CPX'(t) + DU(t)$$

The first equation can be developed as:

$$\forall i, \frac{dx'_i(t)}{dt} = \lambda_i x'_i(t) + b_i^T U(t)$$

The free regime is obtained for $U(t) = 0$ and the eigenvalues λ_i are the eigenvalues of A:

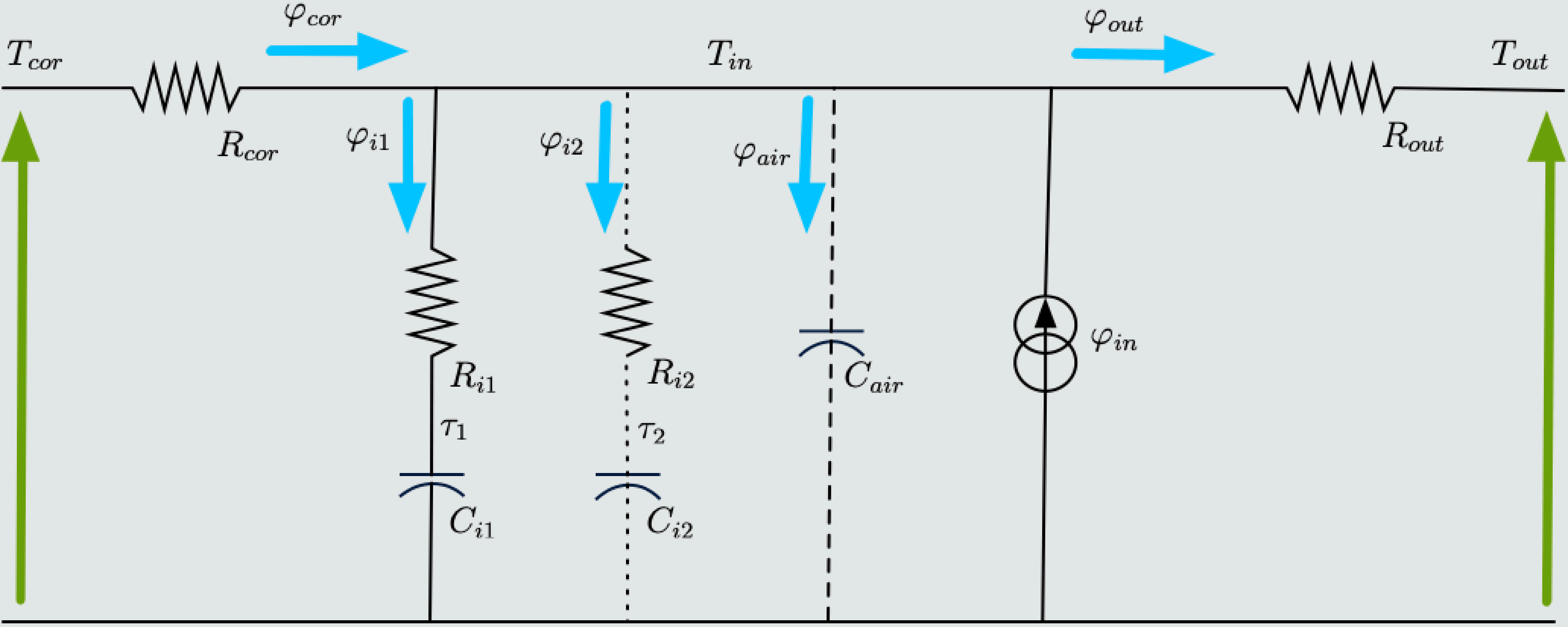
$$x'_i(t) = e^{\lambda_i t} x'_i(0)$$

For convergence, all λ_i have to be negative. The mode of $x'_i(t)$ will be attenuated by 99% after $t = \frac{ln(0.99)}{\lambda_i} = 0.01\tau_i$ with $\tau_i = -\frac{1}{\lambda_i}$ the time constant of the mode i.

Reduced 2nd/3rd order model of the thermal behavior of office H358

We are going to model the dynamic behavior of H358 office with reduced 2nd/3rd order state space model.

Here is a 3 capacitances model of the office. A state space model is going to be deduced as an illustrative example of the state space modeling.
There are 3 temperature nodes T_{cor}, T_{in} and T_{out} and 3 heat flows $\varphi_{cor}, \varphi_{in}$ and φ_{out}.


T_{cor}
φ_{cor}
T_{in}
φ_{out}
T_{out}
R_{cor}
φ_{i1}
φ_{i2}
φ_{air}
R_{out}
R_{i1}
R_{i2}
τ_1
τ_2
C_{i1}
C_{i2}
C_{air}
φ_{in}

Let's model the material storage capabilities by capacitances $C = \rho c_p V$, with ρ the density, c_p the specific heat and V the contributing wall volume. We will have to check wether this simplified way of modeling is an acceptable or not. The model with 3 thermal capacitances (2 for the dynamics of the wall and one for the air) is given by:

$$\varphi_{cor} + \varphi_{in} = \varphi_{i1} + \varphi_{i2} + \varphi_{air} + \varphi_{out}$$

$$T_{cor} - T_{in} = R_{cor}\varphi_{cor}$$

$$T_{in} - T_{out} = R_{out}\varphi_{out}$$

$$T_{in} - \tau_1 = R_{i1}\varphi_{i1}$$

$$T_{in} - \tau_2 = R_{i2}\varphi_{i2}$$

$$C_{i1}\frac{d\tau_1}{dt} = \varphi_{i1}$$

$$C_{i2}\frac{d\tau_2}{dt} = \varphi_{i2}$$

$$C_{air}\frac{dT_{in}}{dt} = \varphi_{air}$$

It can be reformulated as:

$$C_{i1}\frac{d\tau_1}{dt} = -\frac{\tau_1}{R_{i1}} + \frac{T_{in}}{R_{i1}}$$

$$C_{i2}\frac{d\tau_2}{dt} = -\frac{\tau_2}{R_{i2}} + \frac{T_{in}}{R_{i2}}$$

$$C_{air}\frac{dT_{in}}{dt} = -\frac{1}{R}T_{in} + \frac{\tau_1}{R_{i1}} + \frac{\tau_2}{R_{i2}} + \frac{T_{out}}{R_{out}} + \frac{T_{cor}}{R_{cor}} + \varphi_{in}$$

$$\frac{1}{R} = \frac{1}{R_{cor}} + \frac{1}{R_{i1}} + \frac{1}{R_{i2}} + \frac{1}{R_{out}}$$

and the resulting state space model is given by:

$$\frac{d}{dt}\begin{bmatrix}\tau_1\\ \tau_2\\ T_{in}\end{bmatrix} = \begin{bmatrix}-\frac{1}{R_{i1}C_{i1}} & 0 & \frac{1}{R_{i1}C_{i1}}\\ 0 & -\frac{1}{R_{i2}C_{i2}} & \frac{1}{R_{i2}C_{i2}}\\ \frac{1}{R_{i1}C_{air}} & \frac{1}{R_{i2}C_{air}} & -\frac{1}{RC_{air}}\end{bmatrix}\begin{bmatrix}\tau_1\\ \tau_2\\ T_{in}\end{bmatrix} + \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ \frac{1}{R_{cor}C_{air}} & \frac{1}{R_{out}C_{air}} & \frac{1}{C_{air}}\end{bmatrix}\begin{bmatrix}T_{cor}\\ T_{out}\\ \varphi_{in}\end{bmatrix}$$

However, the air capacitance yields fast dynamics which are much faster than the 1 hour sample time, that is commonly used for building simulation and for generating energy strategies. Let's neglect the air capacitance and re-calculate the model. We consider that RC_{air} is small and consequently that $\frac{dT_{in}}{dt} = 0$. It leads to a 2nd order model (2 derivatives):

$$C_{i1}\frac{d\tau_1}{dt} = -\frac{\tau_1}{R_{i1}} + \frac{T_{in}}{R_{i1}}$$

$$C_{i2}\frac{d\tau_2}{dt} = -\frac{\tau_2}{R_{i2}} + \frac{T_{in}}{R_{i2}}$$

$$T_{in} = \frac{R}{R_{i1}}\tau_1 + \frac{R}{R_{i2}}\tau_2 + \frac{R}{R_{cor}}T_{cor} + \frac{R}{R_{out}}T_{out} + R\varphi_{in}$$

that can be rewritten as:

$$C_{i1}\frac{d\tau_1}{dt} = \frac{R - R_{i1}}{R_{i1}^2}\tau_1 + \frac{R}{R_{i1}R_{i2}}\tau_2 + \frac{R}{R_{i1}R_{cor}}T_{cor} + \frac{R}{R_{i1}R_{out}}T_{out} + \frac{R}{R_{i1}}\varphi_{in}$$

$$C_{i2}\frac{d\tau_2}{dt} = \frac{R}{R_{i1}R_{i2}}\tau_1 + \frac{R - R_{i2}}{R_{i2}^2}\tau_2 + \frac{R}{R_{i2}R_{cor}}T_{cor} + \frac{R}{R_{i2}R_{out}}T_{out} + \frac{R}{R_{i2}}\varphi_{in}$$

$$T_{in} = \frac{R}{R_{i1}}\tau_1 + \frac{R}{R_{i2}}\tau_2 + \frac{R}{R_{cor}}T_{cor} + \frac{R}{R_{out}}T_{out} + R\varphi_{in}$$

and reformulated as a state space model:

$$\frac{d}{dt}\begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix} = \begin{bmatrix}\frac{R-R_{i1}}{R_{i1}^2 C_{i1}} & \frac{R}{R_{i1}R_{i2}C_{i1}}\\ \frac{R}{R_{i1}R_{i2}C_{i2}} & \frac{R-R_{i2}}{R_{i2}^2 C_{i2}}\end{bmatrix}\begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix} + \begin{bmatrix}\frac{R}{R_{i1}R_{cor}C_{i1}} & \frac{R}{R_{i1}R_{out}C_{i1}} & \frac{R}{R_{i1}C_{i1}}\\ \frac{R}{R_{i2}R_{cor}C_{i2}} & \frac{R}{R_{i2}R_{out}C_{i2}} & \frac{R}{R_{i2}C_{i2}}\end{bmatrix}\begin{bmatrix}T_{cor}\\ T_{out}\\ \varphi_{in}\end{bmatrix}$$

$$T_{in} = \begin{bmatrix}\frac{R}{R_{i1}} & \frac{R}{R_{i2}}\end{bmatrix}\begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix} + \begin{bmatrix}\frac{R}{R_{cor}} & \frac{R}{R_{out}} & R\end{bmatrix}\begin{bmatrix}T_{cor}\\ T_{out}\\ \varphi_{in}\end{bmatrix}$$

Handling time-varying state models

The natural ventilation represents an advection phenomena i.e. a transportation of heat with the air. The state space model has to be time-varying.

Model reduction with balanced truncature

Most common approach for model reduction is named Balanced Truncature, which inherits from the Moore approach (see (doc/modelreduction.pdf)). The principle is to truncate a balanced state representation, which is a projection of the state space representation where state variables are ordered according to their impact on controllability and observability following a descending order.

It relies on 2 steps:

- computation of an equivalent state space representation where state variables are ordered according to their impact on controllability and observability following the descending order using a base change
- truncate the state space representation in order to keep the r first state variables, where r is the reduction order (the pymor library can be used in Python).

Self-check questionnaire

12 questions — dynamic behavior and thermal inertia

Choose the best answer each time.

(In class: discuss in pairs; answers on the following slides.)

Q1 — State-space form

The continuous-time state-space model on the slides is:

- A. $\dfrac{dX}{dt} = AX + BU,\ Y = CX + DU$
- B. $Y = AX + BU$ only, with no state vector
- C. $\dfrac{dX}{dt} = U$ regardless of A
- D. $X_{k+1} = X_k + U_k$ with no matrices

Q2 — Discrete-time model

When the input is piecewise constant over each sample period T_s, the discrete model is:

- A. $X_{k+1} = A'X_k + B'U_k,\ Y_k = C'X_k + D'U_k$
- B. $X_k = A'U_k$ with no recurrence on past states
- C. Identical to the continuous model without discretization
- D. Valid only for nonlinear systems

Q3 — Non-uniqueness of state space

The slides stress that a given input–output behavior can correspond to:

- A. Infinitely many equivalent state-space realizations (change of state coordinates, e.g. $X = P\mathcal{X}$)
- B. Exactly one unique set of (A, B, C, D) matrices forever
- C. Only models with a single state variable
- D. Models without any eigenvalues

Q4 — Swedish house (reference simulation)

In the reference unheated house simulation, the average indoor temperature is higher than outdoor mainly because:

- A. Solar gains warm the building despite no heating system
- B. The model has no thermal resistance to outdoors
- C. Outdoor temperature is higher than indoor every hour
- D. Capacitance was multiplied by 10

Q5 — Halving insulation

When insulation is divided by 2, the lecture notes that:

- A. Steady state is reached faster (~200 h vs ~300 h) and the average indoor temperature falls (more heat loss), while daily swing stays ~1°C
- B. Daily swing becomes 10°C and steady state never changes
- C. The house behaves like a cave with zero variation
- D. Solar gains disappear

Q6 — Multiplying thermal capacity by 10

With capacity ×10 (“very heavy house”), the slides report:

- A. Very slow dynamics (~1400 h to steady state) and almost no daily indoor temperature swing (cave-like buffering)
- B. Instant response and ±10°C daily swings
- C. No effect on time constants
- D. Higher average indoor temperature than the reference solely from insulation change

Q7 — ThBAT inertia (Th-I)

In ThBAT, inertia appears in performance indicators and includes, among others:

- A. Hourly, daily, and sequential (e.g. ~12-day) inertia for different regulatory uses
- B. Only acoustic inertia in walls
- C. Only wind speed in the Reynolds number
- D. PMV and PPD only

Q8 — Thermal diffusivity and inertia

Thermal diffusivity is $\alpha = \lambda/(\rho c_p)$. For strong thermal inertia in a material layer, the slides emphasize:

- A. High volumetric heat capacity ρc_p and low α (heat moves slowly through the layer)
- B. Low ρc_p and very high α only
- C. Zero conductivity and zero density
- D. α must equal U in W/m²·K

Q9 — Penetration depth (daily wave)

For a sinusoidal surface temperature with period τ, the penetration depth is:

- A. $d_p = \sqrt{\dfrac{\alpha\tau}{\pi}}$
- B. $d_p = \lambda/U$
- C. $d_p = \tau/T_s$ with no material properties
- D. Always equal to the full wall thickness regardless of period

Q10 — Active layer thickness

The active thickness used for inertia of a wall is:

- A. $d_a = \min(d_{\text{wall}},\, d_p)$
- B. Always the full building height
- C. $d_a = U \times R$ only
- D. Independent of α and τ

Q11 — RC nodal model (H358)

In the 3-capacitance office model, thermal storage in walls is represented by:

- A. Capacitances $C = \rho c_p V$ and resistances R linking temperatures, giving ODEs $C\,dT/dt = \sum \varphi$
- B. Only steady-state U-values with no C
- C. Electrical inductors only
- D. A single outdoor temperature with no states

Q12 — Reducing to 2nd order (hourly simulation)

For hourly building simulation, the slides neglect air capacitance (C_{air}) because:

- A. Air dynamics are much faster than the 1 h time step → quasi-steady indoor air temperature → 2nd-order wall-state model
- B. Air has no thermal mass in any building
- C. It eliminates all solar gains
- D. Eigenvalues must all be positive for stability

Answer key (1/2)

Q	Answer	Short rationale
1	A	Standard $\dot{X} = AX + BU, Y = CX + DU$.
2	A	Discrete recurrence $X_{k+1} = A'X_k + B'U_k$.
3	A	Similarity transform; non-unique state coordinates.
4	A	Solar gains raise mean indoor T without heating.
5	A	Less insulation → faster steady state, lower mean T_i.
6	A	Large C → slow dynamics, damped daily swing.

Answer key (2/2)

Q	Answer	Short rationale
7	A	Th-I: hourly / daily / sequential inertia.
8	A	High ρc_p, low α → inertia.
9	A	$d_p = \sqrt{\alpha\tau/\pi}$.
10	A	Active layer: $\min(d_{\text{wall}}, d_p)$.
11	A	RC network: $C = \rho c_p V$, $R\varphi$ balances.
12	A	Fast air C → algebraic T_{in}; 2 states in walls.