MATH4408 Notes

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Definition 1 (Ring)**.** *A set $\mathcal{R}$ is called a **ring** if is is non-empty, and $A \cup B$ and $A \setminus B$ are in $\mathcal{R}$ whenever $A, B \in \mathcal{R}$.*

Definition 2 (Algebra)**.** *A set $\mathcal{R}$ is an **algebra** if it is a ring and contains a universal set R.*

Definition 3 (σ-ring)**.** *A ring $\mathcal{R}$ is a σ-ring if*

$$\bigcup_{n=1}^{\infty} A_n \in \mathcal{R} \quad \textit{whenever} \quad (A_n)_{n=1}^{\infty} \subset \mathcal{R}.$$

Definition 4 (σ-algebra)**.** *A σ-algebra is a σ-ring containing the universal set R.*

Theorem 1 (Ring generated by a collection)**.** *If $\mathscr{U}$ is a non-empty collection of subsets of R, then there exists a unique ring $\mathcal{R}(\mathscr{U})$ containing $\mathscr{U}$ and contained in every other ring that contains $\mathscr{U}$.*

Proof. Let Σ be the class of all rings that contain $\mathscr{U}$. Since $2^R \in \Sigma$, the class is non-empty. Define

$$\mathcal{R}(\mathscr{U}) = \bigcap_{r \in \Sigma} r.$$

By construction it is contained within all rings that contain $\mathscr{U}$. It is easy to see that it is a ring. □

Definition 5 (Ring generated by a collection)**.** *If $\mathcal{R}$ is a ring containing $\mathscr{U}$ and is contained by all rings which contain $\mathscr{U}$, then it is called the **ring generated by** $\mathscr{U}$, denoted $\mathcal{R}(\mathscr{U})$.*

Definition 6 (Measure)**.** *Let $\mathcal{R}$ be an algebra of sets. A function $\mu : \mathcal{R} \to [0, \infty]$ is a **measure** if:*

1. $\mu(\emptyset) = 0$
2. ***σ-additivity.*** *If $(A_n)_{n=1}^{\infty} \subset \mathcal{R}$ are such that $A_i \cap A_j = \emptyset$ for $i \neq j$, and $\bigcup_{n=1}^{\infty} A_n \in \mathcal{R}$, then*

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

Definition 7 (Outer measure)**.** *Let μ be a measure on an algebra $\mathcal{R} \subset 2^R$. The* ***outer measure*** *associated with μ is the function $\mu^\star : 2^R \to \mathbb{R}$ given by*

$$\mu^\star(A) = \inf \sum_{i=1}^{n(\infty)} \mu(E_i)$$

where the infimum is taken over all possible coverings of A by sets $E_1, ..., E_n, ...$ from $\mathcal{R}$. The coverings may or may not be finite.

Theorem 2. *If $A \in \mathcal{R}$ then $\mu^\star(A) = \mu(A)$.*

Proof. Since $A \in \mathcal{R}$ and it covers itself, we have $\mu^\star(A) \leq \mu(A)$. Choose $\epsilon > 0$. There exists a cover $(E_i)_{i=1}^\infty \subseteq \mathcal{R}$ such that

$$\mu^\star(A) > \left(\sum_{n=1}^\infty \mu(E_i) \right) - \epsilon.$$

Now, $A = A \cap (\bigcup_{i=1}^\infty E_i) = \bigcup_{i=1}^\infty (A \cap E_i)$ so

$$\mu(A) \leq \sum_{i=1}^\infty \mu(A \cap E_i) \leq \sum_{i=1}^\infty \mu(E_i) \leq \mu^\star(A) + \epsilon.$$

As $\epsilon \to 0$, we must have $\mu(A) \leq \mu^\star(A)$. □

Theorem 3. *The outer measure $\mu^\star$ is σ-semi-additive, i.e. given $(A_n)_{n=1}^\infty \subset 2^R$ we have*

$$\mu^\star \left(\bigcup_{n=1}^\infty A_n \right) \leq \sum_{n=1}^\infty \mu^\star(A_n).$$

Proof. For each n and $\epsilon > 0$, there exists a cover $(E_{in})_{i=1}^\infty \subset \mathcal{R}$ such that $A_n \subset \bigcup_{n=1}^\infty E_{in}$ and

$$\mu^\star(A_n) > \sum_{n=1}^\infty \mu(E_{in}) - \frac{\epsilon}{2^n}.$$

Clearly,

$$\bigcup_{n=1}^\infty A_n \subset \bigcup_{n=1}^\infty E_{in}$$

and

$$\begin{aligned} \mu^\star \left(\bigcup_{n=1}^\infty A_n \right) &< \sum_{i,n=1}^\infty \mu(E_{in}) \\ &\leq \sum_{n=1}^\infty \mu^\star(A_n) + \frac{\epsilon}{2^n} \end{aligned}$$

□

Definition 8 (Measureable)**.** *A set $A \subset R$ is* ***measurable*** *if*

$$\mu^\star = \mu^\star(A \cap E) + \mu^\star(\bar{A} \cap E) \tag{0.1}$$

for all $E \subset R$.

Theorem 4. *The class $\tilde{\mathcal{R}}$ is a σ-algebra containing $\mathcal{R}$. The restriction $\tilde{\mu}$ is a measure on $\tilde{\mathcal{R}}$*

Definition 9 (Complete)**.** *A measure μ on an algebra $\mathcal{R}$ is called* ***complete*** *if every subset of a set $A \in \mathcal{R}$ such that $\mu(A) = 0$ lies in $\mathcal{R}$.*

Theorem 5. *If μ is a measure on an algebra $\mathcal{R}$ with outer measure $\mu^\star$, and $\mu^\star(A) = 0$ for some $A \subset R$ then A is measurable.*

Proof. We need

$$\mu^\star(E) \geq \mu^\star(E \cap A) + \mu^\star(E \cap \bar{A})$$

for $E \subset R$. We know

$$\mu^\star(E \cap A) \leq \mu^\star(A) = 0$$

and $\mu^\star(E \cap \bar{A}) \leq \mu^\star(E)$ by monotonicity. □