Maximum Coupled Entropy Principle

Maximum Coupled Entropy

Use of Lagrange Method to determine one-dimensional Maximum Entropy distribution

In[65]:=
```
$Assumptions = n ∈ PositiveIntegers && {κ, α, σ, ZP, λ, W} ∈ Reals &&
    0 < κ < ∞ && 0 < σ < ∞ && 0 < ZP < ∞ && 0 < λ < ∞ && 0 < W < ∞;
```

$$\text{(*P}[\kappa\_,\alpha\_,n\_]:=\frac{\text{Table}\left[p_i^{1+\frac{\alpha\,\kappa}{1+\kappa}},\{i,1,n\}\right]}{\sum_{i=1}^{n} p_i^{1+\frac{\alpha\,\kappa}{1+\kappa}}};\text{*)}$$

$$\text{P}[\kappa\_,\,\alpha\_,\,n\_]\;:=\;\frac{\text{Table}\left[p_i^{1+\frac{\alpha\,\kappa}{1+\kappa}},\,\{i,\,1,\,n\}\right]}{\text{ZP}};$$

Investigation clarified that α is determined by the highest power of the constraint. Given a requirement that the coupled entropy converge to the Shannon entropy for $\kappa \to 0$ and all α a multiplier by $\frac{1}{\alpha}$ is needed for both the entropy and the constraint. This investigation is limited to just two constraints, the sum of probabilities is one, and one of the coupled moments.

In[10]:=
$$\phi[\kappa\_,\,\alpha\_,\,\sigma\_,\,n\_]\;:=$$
$$\frac{1}{\alpha\,\kappa}\left(\sum_{i=1}^{n}\text{P}[\kappa,\,\alpha,\,n]_{[\![i]\!]}\left((p_i)^{\frac{-\alpha\,\kappa}{1+\kappa}}-1\right)\right)+\sum_{i=1}^{n}p_i-\frac{1}{\sigma}\sum_{i=1}^{n}\text{P}[\kappa,\,\alpha,\,n]_{[\![i]\!]}\,x_i$$

In[11]:=
$$\text{Solve}[\text{D}[\phi[\kappa,\,1,\,\sigma,\,3],\,p_1]==0,\,p_1]$$

••• Solve: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information.

Out[11]=
$$\left\{\left\{p_1\to\left(\frac{(1+\kappa)\,(1+\text{ZP}\,\kappa)\,\sigma}{(1+2\,\kappa)\,(\sigma+\kappa\,x_1)}\right)^{\frac{1+\kappa}{\kappa}}\right\}\right\}$$

$$\left\{\left\{p_1\to\left(\frac{(1+2\,\kappa)\,\left(1+\kappa\,\frac{x_1}{\sigma}\right)}{(1+\kappa)\,(1+2\,\text{ZP}\,\kappa)}\right)^{\frac{1+\kappa}{-\kappa}}\right\}\right\}$$

If $x_i = i - 1$, then ZP is (however, unable to solve since ZP is part of solution)

In[•]:=
$$\text{ZP}\;=\;\sum_{i=1}^{3}\left(\left(\frac{(1+2\,\kappa)\,\left(1+\kappa\,\frac{i-1}{\sigma}\right)}{(1+\kappa)\,(1+2\,\text{ZP}\,\kappa)}\right)^{\frac{1+\kappa}{-\kappa}}\right)^{1+\frac{\kappa}{1+\kappa}}\quad\text{// FullSimplify}$$

••• $RecursionLimit: Recursion depth of 1024 exceeded during evaluation of $\left(\frac{1+2\,\kappa}{(1+\kappa)\,(1+2\,\text{ZP}\,\kappa)}\right)^{-2-\frac{1}{\kappa}}$.

Out[•]=
$$\text{Hold}\left[\left(\frac{1+2\,\kappa}{(1+\kappa)\,(1+2\,\text{ZP}\,\kappa)}\right)^{-2-\frac{1}{\kappa}}+\left(\frac{(1+2\,\kappa)\,(\kappa+\sigma)}{(1+\kappa)\,(1+2\,\text{ZP}\,\kappa)\,\sigma}\right)^{-2-\frac{1}{\kappa}}+\left(\frac{(1+2\,\kappa)\,(2\,\kappa+\sigma)}{(1+\kappa)\,(1+2\,\text{ZP}\,\kappa)\,\sigma}\right)^{-2-\frac{1}{\kappa}}\right]$$

But given the structure of the solution, can determine the normalization

In[6]:= $\sum_{i=1}^{3}\left(1+\kappa\,\frac{i-1}{\sigma}\right)^{\frac{1+\kappa}{-\kappa}}$ // FullSimplify

Out[6]= $1+\left(1+\frac{2\,\kappa}{\sigma}\right)^{-\frac{1+\kappa}{\kappa}}+\left(\frac{\kappa+\sigma}{\sigma}\right)^{-\frac{1+\kappa}{\kappa}}$

In[9]:= $\left(1+\left(1+\frac{2\,\kappa}{\sigma}\right)^{-\frac{1+\kappa}{\kappa}}+\left(\frac{\kappa+\sigma}{\sigma}\right)^{-\frac{1+\kappa}{\kappa}}\right)^{-1}\left(1+\kappa\,\frac{x_i}{\sigma}\right)^{\frac{1+\kappa}{-\kappa}}$ // FullSimplify

Out[9]= $\frac{\left(1+\frac{\kappa\,x_i}{\sigma}\right)^{-\frac{1+\kappa}{\kappa}}}{1+\left(1+\frac{2\,\kappa}{\sigma}\right)^{-\frac{1+\kappa}{\kappa}}+\left(\frac{\kappa+\sigma}{\sigma}\right)^{-\frac{1+\kappa}{\kappa}}}$

Solution for $\alpha = 2$ with Coupled Variance Constraint

In order to ensure that the correct form of the probability is achieved, either
1) the $\frac{1}{2}$needs to be removed from the coupled entropy, or
2) a $\frac{1}{2}$needs to multiply the coupled variance constraint term

In[12]:= D[ϕ[κ, 2, σ, 3]] // FullSimplify

Out[12]= $\frac{1}{2\,\text{ZP}\,\kappa\,\sigma}\left(\left(\sigma+2\,\text{ZP}\,\kappa\,\sigma\right)p_1+\left(\sigma+2\,\text{ZP}\,\kappa\,\sigma\right)p_2-p_1^{1+\frac{2\,\kappa}{1+\kappa}}\left(\sigma+2\,\kappa\,x_1\right)-p_2^{1+\frac{2\,\kappa}{1+\kappa}}\left(\sigma+2\,\kappa\,x_2\right)+p_3\left(\sigma+2\,\text{ZP}\,\kappa\,\sigma-p_3^{\frac{2\,\kappa}{1+\kappa}}\left(\sigma+2\,\kappa\,x_3\right)\right)\right)$

In[13]:= Solve[D[ϕ[κ, 2, σ, 3], p₁] == 0, p₁]

Solve: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information.

Out[13]= $\left\{\left\{p_1\to\left(-\frac{\sqrt{1+\kappa}\,\sqrt{1+2\,\text{ZP}\,\kappa}\,\sqrt{\sigma}}{\sqrt{\sigma+3\,\kappa\,\sigma+2\,\kappa\,x_1+6\,\kappa^2\,x_1}}\right)^{\frac{1+\kappa}{\kappa}}\right\},\left\{p_1\to\left(\frac{\sqrt{1+\kappa}\,\sqrt{1+2\,\text{ZP}\,\kappa}\,\sqrt{\sigma}}{\sqrt{\sigma+3\,\kappa\,\sigma+2\,\kappa\,x_1+6\,\kappa^2\,x_1}}\right)^{\frac{1+\kappa}{\kappa}}\right\}\right\}$

In[14]:= $1\Big/\left(\frac{\sqrt{1+\kappa}\,\sqrt{1+\text{ZP}\,\kappa\,\sigma}}{\sqrt{\sigma^2+3\,\kappa\,\sigma^2+\kappa\,x_1^2+3\,\kappa^2\,x_1^2}}\right)$ // FullSimplify

Out[14]= $\frac{\sqrt{\frac{(1+3\,\kappa)\,\left(\sigma^2+\kappa\,x_1^2\right)}{(1+\kappa)\,(1+\text{ZP}\,\kappa)}}}{\sigma}$

$$p_i\to\frac{1}{Z}\left(1+\kappa\,\frac{x_i^2}{\sigma^2}\right)^{\frac{1+\kappa}{-2\,\kappa}}$$

So indeed the factor of 2 division causes a problem in correctly specifying the coupled variance constraint; however, I next need to examine whether the entropy function should change or the constraint should change.

Solution for α with α constraint

In[∘]:=
```
ϕAlpha[κ_, α_, σ_, n_] :=
```
$$\frac{1}{\alpha\,\kappa}\left(\sum_{i=1}^{n} P[\kappa,\alpha,n][[i]]\left((p_i)^{\frac{-\alpha\,\kappa}{1+\kappa}}-1\right)\right)+\sum_{i=1}^{n} p_i-\frac{1}{\alpha\,\sigma^{\alpha}}\sum_{i=1}^{n} P[\kappa,\alpha,n][[i]]\,x_i{}^{\alpha};$$

In[∘]:=
```
D[ϕAlpha[κ, α, σ, 3], p1] // FullSimplify
```

Out[∘]=
$$\frac{\sigma^{-\alpha}\left((1+\kappa)\,(1+\mathrm{ZP}\,\alpha\,\kappa)\,\sigma^{\alpha}-(1+\kappa+\alpha\,\kappa)\,p_1^{\frac{\alpha\,\kappa}{1+\kappa}}\,(\sigma^{\alpha}+\kappa\,x_1^{\alpha})\right)}{\mathrm{ZP}\,\alpha\,\kappa\,(1+\kappa)}$$

In[∘]:=
$$\mathrm{Solve}\left[\frac{\sigma^{-\alpha}\left((1+\kappa)\,(1+\mathrm{ZP}\,\alpha\,\kappa)\,\sigma^{\alpha}-(1+\kappa+\alpha\,\kappa)\,p_1^{\frac{\alpha\,\kappa}{1+\kappa}}\,(\sigma^{\alpha}+\kappa\,x_1^{\alpha})\right)}{\mathrm{ZP}\,\alpha\,\kappa\,(1+\kappa)}==0,\ p_1\right]$$

Out[∘]=
$$\left\{\left\{p_1\to\left(\frac{(1+\kappa)\,(1+\mathrm{ZP}\,\alpha\,\kappa)\,\sigma^{\alpha}}{(1+\kappa+\alpha\,\kappa)\,(\sigma^{\alpha}+\kappa\,x_1^{\alpha})}\right)^{\frac{1+\kappa}{\alpha\,\kappa}}\right\}\right\}$$

$$p_1=\frac{1}{Z}\left(1+\kappa\,\frac{x_1^{\alpha}}{\sigma^{\alpha}}\right)^{\frac{1+\kappa}{-\alpha\,\kappa}}$$

Scaling Properties of Coupled Entropy

Hanel, Thurner have written a series of papers on the broadest generalization of entropies relaxing the assumption of additivity. They formulated a *c,d*-entropy in which *c* and *d* relate to different scaling properties of the generalized entropy. The parameter c is equal to Tsallis' q parameter, and thus $c=q=1+\frac{\alpha\,\kappa}{1+d\,\kappa}$. The parameter d relates to raising the generalized logarithm to a power. Following the scaling derivation in R. Hanel and S. Thurner, "A classification of complex statistical systems in terms of their stability and a thermodynamical derivation of their entropy and distribution functions," 2010., we have the following properties for the Coupled Entropy.

I. ASYMPTOTIC PROPERTIES OF NON-ADDITIVE ENTROPIES

We now discuss 2 scaling properties of generalized entropies of the form $S = \sum_i g(p_i)$ assuming the validity of the first 3 Khinchin axioms.

The first asymptotic property is found from the scaling relation

$$\frac{S_g(\lambda W)}{S_g(W)} = \lambda \frac{g(\frac{1}{\lambda W})}{g(\frac{1}{W})} \quad , \tag{3}$$

in the limit $W \to \infty$, i.e. by defining the scaling function

$$f(z) \equiv \lim_{x \to 0} \frac{g(zx)}{g(x)} \qquad (0 < z < 1) \quad . \tag{4}$$

The scaling function f for systems satisfying K1,K2, K3, but not K4, can only be a power $f(z) = z^c$, with $0 < c \leq 1$, given f being continuous. This is shown in the SI (Theorem 1). Inserting Eq. (4) in Eq. (3) gives the first asymptotic law

$$\lim_{W \to \infty} \frac{S_g(\lambda W)}{S_g(W)} = \lambda^{1-c} \quad . \tag{5}$$

From this it is clear that

In[84]:= $\text{FullSimplify}\left[\lambda \frac{\frac{1}{\lambda W}\left(\frac{1}{-2}\right)\text{CoupledLogarithm}\left[\left(\frac{1}{\lambda W}\right)^{-2}, \kappa, 1\right]}{\frac{1}{W}\left(\frac{1}{-2}\right)\text{CoupledLogarithm}\left[\left(\frac{1}{W}\right)^{-2}, \kappa, 1\right]}\right]$

Out[84]= $\frac{-1 + (W\lambda)^{\frac{2\kappa}{1+\kappa}}}{-1 + W^{\frac{2\kappa}{1+\kappa}}}$

In[85]:= $\text{Limit}\left[\frac{-1 + (W\lambda)^{\frac{2\kappa}{1+\kappa}}}{-1 + W^{\frac{2\kappa}{1+\kappa}}}, W \to \infty\right]$

Out[85]= $\lambda^{\frac{2\kappa}{1+\kappa}}$

As $0 \leq \kappa \leq \infty$, the scaling ranges from $0 \leq \frac{2\kappa}{1+\kappa} \leq 1$ which is consistent with the specifications given by Hanel, Thurner. Also note, that $-1 \leq \kappa < 0$ the scaling is faster than exponential and could be governed by the $\kappa = 0$ case. If we solve for c, we have $c = 1 - \frac{2\kappa}{1+\kappa}$, which is not q. Rather $c = 1 - (-1 + q) = 2 + q$ and $q = c - 2$. This seems inconsistent in that for $0 < c \leq 1$, then $-2 < q \leq -1$. There must be another transformation, for instance the $Q = 2 - q$, which switches the domains of q. Substituting $q \to 2 - Q$, gives $c = 4 - Q$, which is not helpful, but if $c \to 2 - Q$, then have $q = -Q$. If we take the relationship to be $c = 2 - Q$, then for $0 < c \leq 1$, $2 > Q \geq 1$. This leaves out the domain $2 < Q < 3$; however, it is closer to the intent of the heavy-tail scaling.

Examination of the Coupled Entropy with a Root

Investigation of Alternative Forms of Coupled Entropy and its Constraints