Generalized Mean

Prototype for Python Development

If the goal is the simplest expression using the CoupledLogarithm than a better approach is to utilize dimension = 0 for the CoupledLogarithm. Then the power represents the risk_bias = $\frac{-\text{alpha kappa}}{1+\text{kappa}}$. Also included in this definition is a weight w_i on each input and the sum of the weights SumW for normalization

riskLogGM = $\frac{1}{\text{SumW}}$Sum[w_i CoupledLogarithm[x_i, r, 0], {i, 1, N}];
GM = CoupledExponential[riskLogGM, r, 0];

Note: its not necessary to computing r = 0 separately, since this is already part of the CoupledLogarithm and CoupledExponential expressions. Expanding the expression to double check that it simplifies.

$$\text{riskLogGM} = \frac{1}{\text{SumW}}\text{Sum}\left[\frac{w_i}{r}\left(x_i^r - 1\right),\ \{i,\ 1,\ N\}\right];$$

$$\text{GM} = \left(1 + \frac{r}{\text{SumW}}\text{Sum}\left[\frac{w_i}{r}\left(x_i^r - 1\right),\ \{i,\ 1,\ N\}\right]\right)^{\frac{1}{r}}$$

$$= \left(1 + r/r\left(\frac{1}{\text{SumW}}\,\text{Sum}[w_i\, x_i^r, \{\text{i, 1, N}\}] - \frac{\text{SumW}}{\text{SumW}}\right)\right)^{\frac{1}{r}}$$

$$= \left(\frac{1}{N}\,\text{Sum}[w_i\, x_i^r,\ \{i,\ 1,\ N\}]\right)^{\frac{1}{r}}$$

Alternative Definition

An alternative derivation which we will NOT plan to use.

One approach would be to utilize the structure of CoupledNormal. However, if its done this way then the powers are in terms of the coupling rather than the risk bias which will be very confusing. You would need to input the risk bias and translate that into values of alpha and kappa.
Utilizes the CoupledLogarithm[x, κ, 1] and CoupledExponential[x, κ, 1] in the form associated with CoupledNormalDistribution. Thus the summation of inputs (usually a probability) is

$$\kappa\text{LogGM} = \text{Sum}\left[\frac{1}{2}\,\text{CoupledLogarithm}[x_i^{-2},\ \kappa,\ 1],\ \{i,\ 1,\ N\}\right];$$

$$\text{GM} = \text{CoupledExponential}[2\ \kappa\text{LogGM},\ \kappa,\ 1]^{\frac{-1}{2}};$$

However; this approach seems unnecessarily complicated, so I don't recommend it at this time. I'm not sure what purpose it would serve, other than possibly that κLogGM may be a useful intermediate form, but this can be computed separately if necessary.