Old and revised versions of eddy momentum flux convergence calculation

In the old version, the eddy momentum fluxes in the displacement coordinate were calculated as

$$\mathrm{ep2} = (u(\phi+\Delta\phi) - u_{\mathrm{REF}}(\phi))v(\phi+\Delta\phi)\cos^2(\phi+\Delta\phi), \tag{1}$$
$$\mathrm{ep3} = (u(\phi-\Delta\phi) - u_{\mathrm{REF}}(\phi))v(\phi-\Delta\phi)\cos^2(\phi-\Delta\phi), \tag{2}$$
$$\nabla\cdot\mathbf{F}_{\mathrm{old}} = \frac{1}{2a\cos\phi\Delta\phi}\left(\mathrm{ep2}-\mathrm{ep3}\right). \tag{3}$$

In the revised (correct) version,

$$\mathrm{ep2a} = (u(\phi+\Delta\phi) - u_{\mathrm{REF}}(\phi+\Delta\phi))v(\phi+\Delta\phi)\cos^2(\phi+\Delta\phi), \tag{4}$$
$$\mathrm{ep3a} = (u(\phi-\Delta\phi) - u_{\mathrm{REF}}(\phi-\Delta\phi))v(\phi-\Delta\phi)\cos^2(\phi-\Delta\phi), \tag{5}$$
$$\nabla\cdot\mathbf{F}_{\mathrm{new}} = \frac{1}{2a\cos\phi\Delta\phi}\left(\mathrm{ep2a}-\mathrm{ep3a}\right). \tag{6}$$

Then

$$\begin{aligned}\nabla\cdot\mathbf{F}_{\mathrm{new}} - \nabla\cdot\mathbf{F}_{\mathrm{old}} = \frac{1}{2a\cos\phi\Delta\phi}\{&[u_{\mathrm{REF}}(\phi) - u_{\mathrm{REF}}(\phi+\Delta\phi)]\,v(\phi+\Delta\phi)\cos^2(\phi+\Delta\phi)\\ &+[u_{\mathrm{REF}}(\phi-\Delta\phi) - u_{\mathrm{REF}}(\phi)]\,v(\phi-\Delta\phi)\cos^2(\phi-\Delta\phi)\}\\ &\to -\frac{v(\phi)\cos\phi}{a}\frac{\partial u_{\mathrm{REF}}}{\partial\phi} \quad \text{as} \quad \Delta\phi\to 0.\end{aligned} \tag{7}$$

So it looks like the necessary correction term should be

$$-\frac{v(\phi)\cos\phi}{a}\frac{\partial u_{\mathrm{REF}}}{\partial\phi} \tag{8}$$

instead of

$$-\frac{v(\phi)}{a}\frac{\partial(u_{\mathrm{REF}}\cos\phi)}{\partial\phi}. \tag{9}$$

Apparently the difference arises from the fact that Eqs. (1)-(3) are actually NOT the correct representation of

$$\frac{1}{a\cos\phi}\frac{\partial[u_e v_e\cos^2(\phi+\phi')]}{\partial\phi'} \tag{10}$$

at $\phi'=0$. To be consistent with Eq. (10), Eqs. (1)-(3) should have been

$$\mathrm{ep2b} = [u(\phi+\Delta\phi)\cos(\phi+\Delta\phi) - u_{\mathrm{REF}}(\phi)\cos(\phi)]v(\phi+\Delta\phi)\cos(\phi+\Delta\phi), \tag{11}$$
$$\mathrm{ep3b} = [u(\phi-\Delta\phi)\cos(\phi-\Delta\phi) - u_{\mathrm{REF}}(\phi)\cos(\phi)]v(\phi-\Delta\phi)\cos(\phi-\Delta\phi), \tag{12}$$
$$\nabla\cdot\mathbf{F}_{\mathrm{old}} = \frac{1}{2a\cos\phi\Delta\phi}\left(\mathrm{ep2b}-\mathrm{ep3b}\right). \tag{13}$$

If we re-evaluate Eq. (7) using Eq. (13),

$$\nabla \cdot \mathbf{F}_{\rm new} - \nabla \cdot \mathbf{F}_{\rm old} = \frac{1}{2a\cos\phi\Delta\phi}\{[u_{\rm REF}(\phi)\cos\phi - u_{\rm REF}(\phi+\Delta\phi)\cos(\phi+\Delta\phi)]\, v(\phi+\Delta\phi)\cos(\phi+\Delta\phi) + [u_{\rm REF}(\phi-\Delta\phi)\cos(\phi-\Delta\phi) - u_{\rm REF}(\phi)\cos\phi]\, v(\phi-\Delta\phi)\cos(\phi-\Delta\phi)\} \rightarrow -\frac{v(\phi)}{a}\frac{\partial(u_{\rm REF}\cos\phi)}{\partial\phi} \quad \text{as} \quad \Delta\phi \rightarrow 0, \quad (14)$$

as expected. So the old code was doubly incorrect, and the new code should use Eqs. (4)-(6). Rest assured that other parts of the budget need no modification.