Evaluation of $v_e q_e$ at $y' = 0$

The question is the transformation of the last term in (B4), $(v_e q_e)_{y'=0}$, into (B5) in Huang and Nakamura (2016). Starting with the definitions

$$v_e(x, y', z, t) = v(x, y + y', z, t), \tag{1}$$
$$u_e(x, y, y', z, t) = u(x, y + y', z, t) - U_{\mathrm{REF}}(y, z, t), \tag{2}$$
$$\theta_e(x, y, y', z, t) = \theta(x, y + y', z, t) - \Theta_{\mathrm{REF}}(y, z, t), \tag{3}$$
$$q_e(x, y, y', z, t) = q(x, y + y', z, t) - Q_{\mathrm{REF}}(y, z, t), \tag{4}$$

where y' is the displacement coordinate relative to y. Here PV is defined as

$$q = f(y) + \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f e^{z/H} \frac{\partial}{\partial z}\left(\frac{e^{-z/H}\theta}{\partial\tilde{\theta}/\partial z}\right), \tag{5}$$

and

$$Q_{\mathrm{REF}} = f(y) - \frac{\partial U_{\mathrm{REF}}}{\partial y} + f e^{z/H} \frac{\partial}{\partial z}\left(\frac{e^{-z/H}\Theta_{\mathrm{REF}}}{\partial\tilde{\theta}/\partial z}\right). \tag{6}$$

Therefore

$$\begin{aligned} q_e &= \frac{\partial v_e}{\partial x} - \frac{\partial u_e}{\partial y} + f e^{z/H} \frac{\partial}{\partial z}\left(\frac{e^{-z/H}\theta_e}{\partial\tilde{\theta}/\partial z}\right) \\ &= \frac{\partial v_e}{\partial x} - \frac{\partial u_e}{\partial y'} + \frac{\partial U_{\mathrm{REF}}}{\partial y} + f e^{z/H} \frac{\partial}{\partial z}\left(\frac{e^{-z/H}\theta_e}{\partial\tilde{\theta}/\partial z}\right). \end{aligned} \tag{7}$$

In the above we used $\partial u_e/\partial y' = \partial u/\partial y = \partial u_e/\partial y + \partial U_{\mathrm{REF}}/\partial y$. Then multiplying by v_e and utilizing $\partial u_e/\partial x + \partial v_e/\partial y' = 0$, after arrangement,

$$v_e q_e = \frac{\partial}{\partial x}\left[\frac{1}{2}\left(v_e^2 - u_e^2 - \frac{R}{H}\frac{e^{-\kappa z/H}\theta_e^2}{\partial\tilde{\theta}/\partial z}\right)\right] - \frac{\partial}{\partial y'}(u_e v_e) + f e^{z/H}\frac{\partial}{\partial z}\left(\frac{e^{-z/H}v_e\theta_e}{\partial\tilde{\theta}/\partial z}\right) + v_e \frac{\partial U_{\mathrm{REF}}}{\partial y}. \tag{8}$$

So it seems that $\partial(u_e v_e)/\partial y$ in (B5) is equivalent to $\partial(u_e v_e)/\partial y' - v_e \partial U_{\mathrm{REF}}/\partial y$. Note that in the zonal average, Eq. (8) reduces to the usual Taylor identity

$$\overline{v'q'} = -\frac{\partial}{\partial y}\left(\overline{u'v'}\right) + f e^{z/H}\frac{\partial}{\partial z}\left(\frac{e^{-z/H}\overline{v'\theta'}}{\partial\tilde{\theta}/\partial z}\right). \tag{9}$$

In the subsequent papers we have used the spherical coordinate version of the budget (e.g. Huang and Nakamura 2017, Nakamura and Huang 2018, Valva and Nakamura 2020) – in those papers (and in Clare's GitHub), the momentum flux convergence is calculated with respect to the displacement latitude ϕ' but we might have neglected the correction term $(v_e/a\cos\phi)\partial(U_{\mathrm{REF}}\cos\phi)/\partial\phi$.