Topics in Statistics: Markov chain Monte Carlo
Problem Set 1

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Exercise 1 Solution

Part 1

Let $Y \sim \text{Exp}(\lambda)$ and $X = Y \mid Y \geq a$ where $a > 0$.
CDF of X:

$$F_X(x) = P(X \leq x) = P(Y \leq x \mid Y \geq a) = \frac{P(a \leq Y \leq x)}{P(Y \geq a)} \tag{1}$$

Since $F_Y(y) = 1 - e^{-\lambda y}$ for $y \geq 0$:

- For $x < a$: $F_X(x) = 0$
- For $x \geq a$:

$$F_X(x) = \frac{F_Y(x) - F_Y(a)}{1 - F_Y(a)} = \frac{(1 - e^{-\lambda x}) - (1 - e^{-\lambda a})}{e^{-\lambda a}} = 1 - e^{-\lambda(x-a)} \tag{2}$$

Therefore:

$$F_X(x) = \begin{cases} 0 & \text{if } x < a \\ 1 - e^{-\lambda(x-a)} & \text{if } x \geq a \end{cases} \tag{3}$$

Quantile function: Solving $u = F_X(x) = 1 - e^{-\lambda(x-a)}$:

$$F_X^{-1}(u) = a - \frac{1}{\lambda} \ln(1-u) \tag{4}$$

Algorithm:

1. Generate $U \sim \mathcal{U}[0,1]$
2. Return $X = a - \frac{1}{\lambda} \ln(1-U)$

Part 2

We need to show that $X = F_Y^{-1}(F_Y(a)(1-U) + F_Y(b)U)$ has distribution $Y \mid a \leq Y \leq b$.
Let $\alpha = F_Y(a)$ and $\beta = F_Y(b)$. Then:

$$X = F_Y^{-1}(\alpha + U(\beta - \alpha)) \tag{5}$$

Since $U \sim \mathcal{U}[0,1]$, we have $V = \alpha + U(\beta - \alpha) \sim \mathcal{U}[\alpha, \beta]$.
For $a \leq x \leq b$:

$$P(X \leq x) = P(F_Y^{-1}(V) \leq x) = P(V \leq F_Y(x)) \tag{6}$$

$$= \frac{F_Y(x) - \alpha}{\beta - \alpha} = \frac{F_Y(x) - F_Y(a)}{F_Y(b) - F_Y(a)} \tag{7}$$

This equals $P(Y \le x \mid a \le Y \le b)$ by definition. ✓
Application to exponential: Taking $b \to \infty$:

$$X = F_Y^{-1}((1 - e^{-\lambda a})(1 - U) + U) \tag{8}$$
$$= -\frac{1}{\lambda} \ln(e^{-\lambda a}(1 - U)) \tag{9}$$
$$= a - \frac{1}{\lambda} \ln(1 - U) \tag{10}$$

This matches Part 1. ✓

Part 3

The rejection algorithm simulates from:

- **Proposal:** $q(x) = \lambda e^{-\lambda x} \mathbf{1}_{[0,\infty)}(x)$ (standard exponential)
- **Target:** $\pi(x) = \lambda e^{-\lambda(x-a)} \mathbf{1}_{[a,\infty)}(x)$ (truncated exponential)

The ratio for $x \ge a$:

$$\frac{\pi(x)}{q(x)} = \frac{\lambda e^{-\lambda(x-a)}}{\lambda e^{-\lambda x}} = e^{\lambda a} \tag{11}$$

Therefore:

$$M = e^{\lambda a} \tag{12}$$

The acceptance probability is $\frac{1}{M} = e^{-\lambda a} = P(Y > a)$.
Expected number of trials:

$$E[\text{trials}] = e^{\lambda a} \tag{13}$$

Why inversion is preferred for $a \gg 1/\lambda$:
When $a \gg 1/\lambda$, we have $\lambda a \gg 1$, so $e^{\lambda a} \gg 1$. This means:

- Rejection algorithm: Expected $e^{\lambda a}$ trials (grows exponentially with a)
- Inversion method: Exactly 1 step regardless of a

Therefore, inversion is much more efficient for large values of a.