Orthogonal Transformations

- spatial relationships in 3D

Def – An $n \times n$ matrix A is an orthogonal transformation I.F.F (if and only if)

- It has n mutually perpendicular rows or columns with unit length
  - $\perp$ rows must be independent (can't be multiples of each other)

    ex $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \rightarrow$ linearly dependent

    $\begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix} \rightarrow$ independent but not $\perp$
  - to be perpendicular, the dot product must be 0

dot product: $x \cdot y = \sum_{i=1}^{n} x_i y_i$

$x \cdot y = 0 \longleftrightarrow x \perp y$ (perp.)

- rows/colums must have unit length

  $\hookrightarrow \|x\| = \sqrt{\sum_i x_i^2} = \sqrt{x \cdot x}$

- The rows or columns of A form an orthonormal basis of $\mathbb{R}^n$
  - basic for space – set of vectors that can combine to create any vector in a space
  - basically first point with more words
- $AA^T = A^T A = I$ $\rightsquigarrow$ transpose $[\,]^T$ switches the rows and columns
- $A^{-1} = A^T$

  ex: $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

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more about $AA^T = A^TA = I$ & $A^{-1} = A^T$

basically the same info

mtx mult:

$$\underbrace{\underset{m\times n}{A}\ \underset{n\times k}{B}} = \underbrace{\underset{m\times k}{C}}$$

So what about $(AB)^T \overset{?}{=} C^T$

$$\underset{m\times n\ \ n\times k}{AB} \neq \underset{n\times m\ \ k\times n}{A^T B^T}$$

$\neq$

so $(AB)^T = \underset{k\times m}{C^T} = \underset{k\times n}{B^T}\ \underset{n\times m}{A^T}$ ✓

identity matrix

$$I = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & \ddots & 1 \end{bmatrix}$$

so $IA = A$

$Ix = x$

example of an orthogonal transformation:

ex • 2×2 Rotation matrix:

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Want to show perpendicular columns:

$$\vec{u} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

Need to show ~~3 things~~...

① are they perpendicular?

take $\vec{u}\cdot\vec{v} = -\cos\theta\sin\theta + \sin\theta\cos\theta = 0$

so we know $\vec{u} \perp \vec{v}$

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