Shallow water model SAM

Reference Manual

Version 1.0

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1 Model equations

The shallow water model SAM solves the shallow water equations on a spherical planet with shallow orography. The model equations and the numerical solution procedure will be described in the following subsections.

1.1 Continuity equation

In a shallow water model it is assumed that the density ρ is constant. Consequently, the continuity equation reduces to the condition that the threedimensional flow is nondivergent. In spherical coordinates the nondivergence condition reads

$$\left(\frac{1}{r\cos\varphi}\frac{\partial u}{\partial\lambda}+\frac{1}{r\cos\varphi}\frac{\partial}{\partial\varphi}(\cos\varphi v)+\frac{\partial w}{\partial r}+\frac{2w}{r}\right)=0, \tag{1.1}$$

In this equation λ, φ and r are the spherical coordinates longitude, latitude and radius, respectively and u, v and w denote the zonal, meridional and vertical velocity components, respectively. The last term on the left hand side of this equation can be neglected since it is assumed that the shallow water system is contained within a thin spherical shell. Furthermore, the radius r can be approximately replaced be the mean planet radius a where it appears in a factor. Therefore, the continuity equation will be simplified to give

$$\left(\frac{1}{a\cos\varphi}\frac{\partial u}{\partial\lambda}+\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(\cos\varphi v)+\frac{\partial w}{\partial z}\right)=0, \tag{1.2}$$

where $z = r - a$ denotes the height above the spherical surface having the radius $r = a$.

In a shallow water model it is assumed that the horizontal velocity components are vertically independent. Therefore, the continuity equation can be vertically integrated over the whole shallow water fluid. This gives

$$\frac{Dh}{Dt}-\frac{Dh_s}{Dt}+(h-h_s)\nabla_h\cdot\mathbf{v}_h=0\ , \tag{1.3}$$

where $\nabla_h\cdot\mathbf{v}_h$ denotes the divergence of the horizontal flow, h the height of the shallow water surface, h_s the height of the shallow water bottom and

$$\frac{D}{Dt}=\frac{\partial}{\partial t}+\frac{u}{a\cos\varphi}\frac{\partial}{\partial\lambda}+\frac{v}{a}\frac{\partial}{\partial\varphi}$$

is the individual time derivative operator.

1.2 Momentum equation

The fluid of the shallow water model is treated as a nondivergent ideal fluid. Therefore, the momentum equation is given by EULERs' equation in a rotating coordinate frame:

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla \left(\frac{1}{2} \mathbf{v}^2 \right) + (\nabla \times \mathbf{v} + 2\mathbf{\Omega}) \times \mathbf{v} = -\frac{1}{\rho} \nabla p - \nabla \phi, \tag{1.4}$$

where p is the pressure, ϕ the geopotential and $\mathbf{\Omega}$ the angular velocity vector of the planet rotation which is directed perpendicular to the equatorial plane ($\varphi = 0$).

The shallow water model describes flow phenomena having a vertical scale that is much smaller than its horizontal scale. In this case the hydrostatic balance is satisfied to a high degree. Therefore, it is assumed that the following hydrostatic balance equation holds

$$\frac{\partial p}{\partial z} = -\rho \frac{\partial \phi}{\partial z} \tag{1.5}$$

In this equation p denotes the pressure and ϕ the geopotential. Since density is constant, this equation can easily be integrated in vertical direction over the whole fluid depth. This gives

$$p_s - p|_{z=h} = \rho(\phi|_{z=h} - \phi_s) \ , \tag{1.6}$$

where the index s means that the variable is evaluated at the bottom of the shallow water. For shallow water systems it is usally assumed that the pressure at the free upper surface vanishes. Furtheremore, for earth-like planets the gravitational attraction force in addition with the centrifugal force of the planet rotation give rise to geopotential isosurfaces that are nearly spherical. Therefore, it is assumed that the geopotential only depends upon height z and is, furthemore, proportional to height since the corresponding acceleration is nearly constant with in a thin spherical shell. Consequently, surface pressure is related to the shallow water elevation and surface orography by

$$p_s = g\rho(h - h_s) \ , \tag{1.7}$$

where g denotes the gravitational acceleration.

Due to this equation the continuity equation (1.3) becomes a tendency equation for surface pressure:

$$\frac{Dp_s}{Dt} + p_s \nabla_h \cdot \mathbf{v}_h = \frac{\partial p_s}{\partial t} + \nabla_h \cdot (\mathbf{v}_h p_s) = 0 \ , \tag{1.8}$$

The horizontal components of EULERs equation are retained with little approximation. We obtain for the horizontal components of $(\nabla \times \mathbf{v} + 2\mathbf{\Omega}) \times \mathbf{v}$:

$$((\nabla \times \mathbf{v} + 2\mathbf{\Omega}) \times \mathbf{v})_h \tag{1.9}$$

$$= \Omega(\zeta + 2\sin\varphi)\mathbf{e}_z \times \mathbf{v}_h - \nabla_h \left(\frac{1}{2}w^2\right) + \frac{\mathbf{v}_h w}{r} + 2\Omega\cos\varphi w\mathbf{e}_\lambda \ ,$$

where ζ is the nondimensional vertical component of the vorticity vector $\nabla \times \mathbf{v}$ (relative vorticity) and the index h denotes that the vector has only horizontal components. The last two terms on the right hand side can be neglected due to the small magnitude of vertical velocity in the shallow water model.

Therefore, the EULER-equation for the horizontal velocity vector becomes:

$$\frac{\partial \mathbf{v}_h}{\partial t} + \nabla_h \left(\frac{1}{2}\mathbf{v}_h^2\right) + \Omega(\zeta + 2\sin(\varphi))\mathbf{e}_z \times \mathbf{v}_h = -\frac{1}{\rho}\nabla_h p_s - \nabla\phi_s \ . \tag{1.10}$$

1.3 Nondimensional model equations

Applying the operators $\Omega^{-2}\mathbf{e}_z\nabla_h\times$ and $\Omega^{-2}\nabla_h\cdot$ gives the nondimensional vorticity and divergence equations, respectively

$$\frac{\partial \eta}{\partial \tau} = -\frac{1}{1-\mu^2}\frac{\partial}{\partial \lambda}(\eta U) - \frac{\partial}{\partial \mu}(\eta V) \ , \tag{1.11}$$

$$\frac{\partial D}{\partial \tau} = \frac{1}{1-\mu^2}\frac{\partial}{\partial \lambda}(\eta V) - \frac{\partial}{\partial \mu}(\eta U) - \Delta_h\left(\frac{U^2+V^2}{2(1-\mu^2)} + \Phi_s + \mathrm{Bu}P\right) \ , \tag{1.12}$$

Here, we have used the following notations

$$\begin{aligned}
\mu &= \sin\varphi \quad \text{sine of latitude} \\
\tau &= \Omega t \quad \text{nondimensional time} \\
\eta &= \zeta + 2\mu \quad \text{nondimensional absolute vorticity} \\
D &= \Omega^{-1}\nabla_h \cdot \mathbf{v}_h \quad \text{nondimensional divergence} \\
(U,V) &= \cos\varphi(\Omega a)^{-1}(u,v) \quad \text{nondimensional velocity components multiplied by } \cos\varphi \\
P &= \frac{p_s - p_0}{p_0} \quad \text{nondimensional surface pressure anomaly} \\
\Phi_s &= \frac{\phi_s}{\Omega^2 a^2} \quad \text{nondimensional surface geopotential} \\
\mathrm{Bu} &= \frac{p_0}{\Omega^2 a^2 \rho} \quad \text{BURGER number} \\
\Delta_h &= \frac{1}{1-\mu^2}\frac{\partial^2}{\partial \lambda^2} + \frac{\partial}{\partial \mu}(1-\mu^2)\frac{\partial}{\partial \mu} \quad \text{nondimensional LAPLACE} - \text{operator}
\end{aligned}$$

Similarly, we obtain for the nondimensional surface pressure equation:

$$\frac{\partial P}{\partial \tau} = -\frac{1}{1-\mu^2}\frac{\partial}{\partial \lambda}(UP) - \frac{\partial}{\partial \mu}(VP) - D \ , \tag{1.13}$$

It remains to compute the horizontal velocity components from vorticity and divergence fields. This can be done with the HELMHOLTZ decomposition of the velocity vector field:

$$\mathbf{v}_h = \Omega a^2 \left(\nabla \chi + \nabla \times \mathbf{A}\right) \ . \tag{1.14}$$

where χ is a nondimensional velocity potential and $\mathbf{A}$ a nondimensional vector stream potential. Because only the horizontal velocity field is related to vorticity and divergence, it is sufficient to restrict the vector potential to a single vertical component, the so called streamfunction ψ. Therefore:

$$U = -(1-\mu^2)\frac{\partial \psi}{\partial \mu} + \frac{\partial \chi}{\partial \lambda} \ , \quad V = \frac{\partial \psi}{\partial \lambda} + (1-\mu^2)\frac{\partial \chi}{\partial \mu} \ . \tag{1.15}$$

Streamfunction and velocity potential are related to vorticity and divergence via POISSON-equations:

$$\eta = 2\mu + \Delta_h \psi \ , \quad D = \Delta_h \chi \ . \tag{1.16}$$

Equation (1.11)-(1.13), (1.15) and (1.16) form the governing equations of the shallow water model. They are solved with a semispectral method which is described in some detail in the next subsections.

1.4 Spectral representation of the model equations

All fields are projected onto a new base, namely, the so-called spherical harmonics $Y_n^m(\lambda, \mu)$. This gives for a field function $Q(\lambda, \mu, \tau)$:

$$Q(\lambda, \mu, \tau) = \sum_{m=-\infty}^{\infty} \sum_{n=|m|}^{\infty} Q_n^m(\tau) Y_n^m(\lambda, \mu) \ . \tag{1.17}$$

The spherical harmonics are defined by

$$Y_n^m(\lambda, \mu) = \sqrt{\frac{2n+1}{2}\frac{(n-m)!}{(n+m)!}}\frac{(1-\mu^2)^{\frac{m}{2}}}{2^n n!}\frac{\partial^{m+n}}{\partial \mu^{m+n}}\left[(\mu^2-1)^n\right] e^{im\lambda} \ . \tag{1.18}$$

Since Q must be a real function we have the constraint $Q_n^m = Q_n^{-m*}$. Therefore, it is sufficient to calculate only the coefficients with $m \geq 0$ and the following representation is equivalent to (1.19):

$$Q(\lambda, \mu, \tau) = \sum_{n=1}^{\infty} Q_n^0(\tau) Y_n^0(\mu) + 2 \sum_{m=1}^{\infty} \sum_{n=m}^{\infty} \mathrm{Re}\left(Q_n^m(\tau) Y_n^m(\lambda, \mu)\right) \ . \tag{1.19}$$

The spherical harmonics form an orthogonal base with respect to the scalar product

$$\langle Q_1 | Q_2 \rangle := \frac{1}{2\pi} \int_{-1}^{1} \int_{0}^{2\pi} Q_1^*(\lambda, \mu) Q_2(\lambda, \mu) d\lambda d\mu \tag{1.20}$$

Therefore:

$$\left\langle Y_n^m | Y_{n'}^{m'} \right\rangle = \delta_{m,m'} \delta_{n,n'} \tag{1.21}$$

Furthermore, the spherical harmonics are eigenfunctions of the nondimensional LAPLACE-operator Δ_h:

$$\Delta_h Y_n^m = -n(n+1) Y_n^m \ . \tag{1.22}$$

Further useful relations are:

$$\frac{\partial Y_n^m}{\partial \lambda} = im Y_n^m \ , \tag{1.23}$$

$$(1-\mu^2) \frac{\partial}{\partial \mu} Y_n^m = n d_{m,n+1} Y_{n+1}^m - (n+1) d_{m,n} Y_{n-1}^m \ , \tag{1.24}$$

where

$$d_{m,n} = \sqrt{\frac{n^2 - m^2}{4n^2 - 1}} \ .$$

With (1.22) - (1.24) the governing model equations can be converted into component form after applying the scalar product $< Y_n^m | .. >$:

$$\frac{d\eta_n^m}{d\tau} = -\left\langle Y_n^m \mid \frac{im\eta U}{1-\mu^2} \right\rangle - \left\langle Y_n^m \mid \frac{\partial(\eta V)}{\partial \mu} \right\rangle \ , \tag{1.25}$$

$$\frac{dD_n^m}{d\tau} = \left\langle Y_n^m \mid \frac{im\eta V}{1-\mu^2} \right\rangle - \left\langle Y_n^m \mid \frac{\partial(\eta U)}{\partial\mu} \right\rangle + n(n+1)\left\langle Y_n^m \mid \frac{1}{2}\frac{U^2+V^2}{1-\mu^2} \right\rangle + n(n+1)\left[\mathrm{Bu}P_n^m + \Phi_{s_n}{}^m\right] , \tag{1.26}$$

$$\frac{dP_n^m}{d\tau} = -\left\langle Y_n^m \mid \frac{imPU}{1-\mu^2} \right\rangle - \left\langle Y_n^m \mid \frac{\partial(PV)}{\partial\mu} \right\rangle - D_n^m , \tag{1.27}$$

$$U_n^m = (n-1)d_{m,n}\psi_n^m - (n+2)d_{m,n+1}\psi_{n+1}^m + im\ \chi_n^m , \tag{1.28}$$

$$V_n^m = im\ \psi_n^m - (q-1)d_{m,n}\chi_{n-1}^m + (n+2)d_{m,n+1}\chi_{n+1}^m , \tag{1.29}$$

$$\psi_n^m = -\frac{1}{n(n+1)}\left(\eta_n^m - \sqrt{\frac{8}{3}}\ \delta_{m,0}\delta_{n,1}\right) , \tag{1.30}$$

$$\chi_n^m = -\frac{1}{n(n+1)}D_n^m . \tag{1.31}$$

1.4 Numerical solution technique

An approximative solution of the model is determined by time integration of the truncated spectral equations. In the truncated model the first summation operator in (1.17) runs only from $-M$ to M. This gives a triangular truncation TM.

The terms in brackets of Eqs.(1.25)-(1.27) are nonlinear. The computation of these scalar products is very complicated and consumes a lot of computer time. Therefore, these terms are evaluated in the numerical model with the so-called spectral transform method [Orszag (1970) and Eliasen et al. (1970)]. This method uses an auxiliary grid in the physical space where point values of the dependent variables are computed. The auxiliary grid in the physical space (GAUSSian grid) is defined by M_g equally spaced longitudes and J_g GAUSSian latitudes. For the transformation from gridpoint space into spectral space the spectral coefficients are obtained by GAUSSian quadrature of the FOURIER coefficients. With this method bilinear products are calculated without error when $M_g \geq 3M+1$ and $J_g \geq (3M+1)/2$. Since only bilinear products appear as nonlinearities in the equations the spectral transform method is equivalent to the analytical determination of the nonlinear products using interaction coefficients.

Time integration is done with the semi-implicit time stepping procedure. This has the advantage that for a stable integration the time-step $\Delta\tau$ can be larger than the period of high-frequency inertial gravity modes. The time derivatives are approximated by a centered difference so that

$$\frac{dQ}{d\tau} \approx \delta_t Q := \frac{Q(\tau+\Delta\tau) - Q(\tau+\Delta\tau)}{2\Delta\tau} . \tag{1.32}$$

The tendency is splitted into an explicit and an implicit part. The explicit part is related to the sum of all nonlinear terms and is evaluated at the present time τ. The linear terms form the implicit part. They only occur in the divergence and pressure equation and enter the equations as a time mean of time step $\tau + \Delta\tau$ and $\tau - \Delta\tau$. Altogether, the finite-difference form of the prognostic model equations reads:

$$\delta_t \eta_n^m = \mathcal{N}_\eta(\tau) \ , \tag{1.33}$$

$$\delta_t D_n^m = \mathcal{N}_D(\tau) + \mathrm{Bu}\frac{n(n+1)}{2}[P_n^m(\tau - \Delta\tau) + P_n^m(\tau + \Delta\tau) + 2\Phi_{sn}{}^m] \tag{1.34}$$

$$\delta_t P_n^m = \mathcal{N}_P(\tau) - \frac{1}{2}[D_n^m(\tau - \Delta\tau) + D_n^m(\tau + \Delta\tau)] \ , \tag{1.35}$$

where $\mathcal{N}_\eta$, $\mathcal{N}_D$ and $\mathcal{N}_P$ denote the respective nonlinear tendency terms. Eliminating $P_n^m(\tau + \Delta\tau)$ in (1.35) gives

$$\left[1 + \Delta\tau^2 \mathrm{Bu}\ n(n+1)\right] \overline{D_n^m}^\tau \tag{1.36}$$

$$= D_n^m(\tau - \Delta\tau) + \Delta\tau \left\{\mathcal{N}_D(\tau) + \mathrm{Bu}\ n(n+1)\left[P_n^m(\tau - \Delta\tau) + \Phi_{sn}{}^m + \Delta\tau\mathcal{N}_P(\tau)\right]\right\} \ ,$$

where $\overline{D_n^m}^\tau = [D_n^m(\tau + \Delta\tau) + D_n^m(\tau - \Delta\tau)]/2$.

The time averaged value $\overline{D_n^m}^\tau$ can be easily calculated from equation (1.36). With this value it is straightforward to determine the future values $D_n^m(\tau+\Delta\tau)$ and $P_n^m(\tau+\Delta\tau)$.

For noise reduction a Robert/Asselin-Filter [Haltiner and Williams (1982)] is applied at every timestep to a spectral coefficient Q_n^m as follows:

$$Q_n^m(\tau) = (1 - 2\gamma)Q_n^m(\tau) + \gamma[Q_n^m(\tau + \Delta\tau) + Q_n^m(\tau - \Delta\tau)] \ . \tag{1.37}$$

In the model SAM γ takes the value 0.02 by default.

1.5 Hyperdiffusion

In a turbulent flow regime a cascade from large scales to the dissipative range of the wavenumber spectrum takes place. The model cannot resolve such small scales. Therefore, kinetic energy dissipation must occur at resolvable scales in the model without affecting the large scale too much. This is achieved by introducing hyperdiffusion by adding the terms

$$\mathcal{H}_{\eta\ n}^m = K\left[-n(n+1)\right]^{n_h}\left(\eta_n^m - \sqrt{\frac{8}{3}}\ \delta_{m,0}\delta_{n,1}\right) \tag{1.38}$$

and

$$\mathcal{H}^m_{D\ n} = K\left[-n(n+1)\right]^{n_h} D^m_n \tag{1.39}$$

to the vorticity and divergence equations, respectively. n_h denotes the order of hyper-diffusion. Without additional terms the corresponding vorticity or divergence component is damped with the dimensional e-folding timescale

$$\tau_H = \left\{\Omega K[n(n+1)]^{n_h}\right\}^{-1} \ . \tag{1.40}$$

In the model τ_H at the wavenumber where the expansion is truncated will be prescribed. Therefore, the coefficient K becomes:

$$K = \left\{\Omega \tau_H[M(M+1)]^{n_h}\right\}^{-1} \ . \tag{1.41}$$

1.6 Parameterization of baroclinic and dissipative processes

To perform simulations with baroclinic forcing and dissipative damping additional terms must be added to the spectral vorticity and divergence equations. These are given by

$$\begin{aligned}\left.\frac{d\eta^m_n}{d\tau}\right|^P &= \frac{\eta^m_{ne} - \eta^m_n}{\tau^m_{nR}} + S^m_n(\tau) - \frac{\eta^m_n - \sqrt{8/3}\delta_{m,0}\delta_{n,1}}{\tau_F} \\ &\quad -A_h\left(n^2+n-2\right)\left(\eta^m_n - \sqrt{8/3}\delta_{m,0}\delta_{n,1}\right) \ ,\end{aligned} \tag{1.42}$$

$$\left.\frac{dD^m_n}{d\tau}\right|^P = -\frac{D^m_n}{\tau_F} - A_h n(n+1) D^m_n \ , \tag{1.43}$$

where η^m_{ne} is the equilibrium state of the climatological baroclinic forcing, τ^m_{nR} the timescale of the climatological forcing (depends upon wavenumbers), $S^m_n(\tau)$ the time dependent baroclinic forcing (stochastic), τ_F the timescale of Rayleigh friction and A_h the horizontal momentum exchange coefficient.

2 User guide

SAM can like PUMA and PlaSim be compiled and started with the model starter program most.x. SAM also supports the Graphical User Interface (GUI) developed for PUMA and PlaSim. In the code several initial conditions are specified with the integer parameter iexp (1-13). The various experiments are documented in the initialization subroutine initfd. Further initial conditions can be added at discretion. The model resolution has to be specified with the namelist *resolution_namelist*. It contains the parameter NLAT that denotes the number of latitudes. NLAT should be represented

by 2^n with n being an integer to ensure optimal parallelization and consistency with the fast Fourier transform.

The namelist for SAM *somnamelist* contains the following parameters

Parameter	Default	Meaning
KICK	1	Initial noise (not implemented in the current version)
NAFTER	12	Output after intervals of NAFTER timesteps
NCOEFF	0	Number of spectral modes to be print in wrspam
NDEL	6	Order of hyperdiffusion
NDIAG	12	Parameter to determine that ASCII diagnostics are written after intervals of NDIAG timesteps
NKITS	3	Number of initial timesteps
NRUN	0	Number of timesteps to run - 0: use nyears and
NWSPINI	1	1: Write initial sp(:) to file puma_sp_ini
NOUTPUT	1	1: Write model output to file sam_output
NSTEP	0	Current timestep
NSTOP	0	Stop step - 0: compute from nyears and nmonths
NTSPD	1	Number of timesteps per day
NCU	0	Check unit (for debug output only)
NGUI	0	1: Run with GUI
NGUIDBG	0	1: Switch on GUI debug output
NYEARS	0	Simulation time in years
DISS	0.25	Hyper diffusion time scale [days]
NRUIDO	0	1: Add noise on every time step
DISP	0.0	Noise amplitude for nruido = 1
AH	0.0	Horizontal momentum exchange coefficient [m^2/s]
RESTIM	0.0	Timescale for climatological forcing [days] (0.0→ no forcing)
NDL	0	1: Print spectral vorticity and divergence modes
ROTSPD	0.	Rotations per day (should be set as in PlaSim)
TFRC	0.	Rayleigh friction timescale in days (0.0 → no friction)
LLID	.false.	Switch for introducing a rigid lid
LEQUIV	.false.	Switch for equivalent barotropic model
LBAL	.false.	Switch for balanced initial state
IEXP	4	Number of experiment (1-13, see initfd)
MMAX	0	Maximum zonal waven. for climat. forcing
NMAX	42	Maximum total waven. for climat. forcing

Other parameters can be set in the module pumamod of the code. These are:

Parameter	Default	Meaning
GA	9.81	Gravity acceleration [m/s^2]
PLARAD	6371000.0	Radius of the sphere [m]
WW	0.00007292	Angular velocity of the rotating sphere [1/s]
RHO0	1.	Density of the shallow water [kg/m^3]
HS	12000.	Height of the shallow water [m]
F0	1.e-4	Mean Coriolis parameter [1/s] (only LEQUIV=.true.)

SAM writes output into the file *sam_output*. It contains spectral coefficients of i) surface geopotential (code 129), ii) logarithm of surface pressue (code 152), iii) diver-

gence (code 155) and iv) relative vorticity (code 138). The pumaburner can process the file *sam_output* for further diagnostics.