Nonparametric Testability of Slutsky Symmetry

Florian Gunsilius
Department of Economics
Emory University
fgunsil@emory.edu

Lonjezo Sithole
Department of Economics
University of Michigan
lsithole@umich.edu

February 11, 2026

Abstract

Economic theory implies strong limitations on what types of consumption behavior are considered rational. Rationality implies that the Slutsky matrix, which captures the substitution effects of compensated price changes on demand for different goods, is symmetric and negative semi-definite. While empirically informed versions of negative semi-definiteness have been shown to be nonparametrically testable, the analogous question for Slutsky symmetry has remained open. Recently, it has even been shown that the symmetry condition is not testable via the average Slutsky matrix, prompting conjectures about its non-testability. We settle this question by deriving nonparametric conditional quantile restrictions on observable data that constitute a testable implication of Slutsky symmetry in an empirical setting with individual heterogeneity and endogeneity. The theoretical contribution is a multivariate generalization of identification results for partial effects in nonseparable models without monotonicity, which is of independent interest. This result has implications for different areas in econometric theory, including nonparametric welfare analysis with individual heterogeneity for which, in the case of more than two goods, the symmetry condition introduces nonlinear correction factors.

JEL Numbers: C01, C12, C14, C18, D12

Keywords: Demand; Heterogeneity; Identification; Rationality; Slutsky symmetry; Testability; Welfare analysis

*A previous version was titled "A nonparametric test of Slutsky Symmetry". The authors thank Shomu Banerjee, Stefan Hoderlein, and audiences at Emory, LMU Munich, NYU, the University of Michigan, the 4th Georgia Econometrics workshop, the Econometrics mini conference at the University of Iowa, the 2025 IAAE Annual Conference in Turin, Italy, the 2025 World Congress of the Econometric Society and the 2025 Canadian Econometric Study Group Meeting for helpful feedback. All errors are the authors'.

1 Introduction

Rationality is the foundational assumption of consumer demand theory. A smooth demand function is rationalizable, that is, consistent with utility maximization, if and only if the associated Slutsky matrix of compensated price effects is symmetric and negative semidefinite (Samuelson, 1948; Houthakker, 1950; Hurwicz and Uzawa, 1971). Whether these conditions hold in data is therefore a question of first-order importance for applied welfare analysis.

Testing the Slutsky conditions nonparametrically, i.e., without committing to a functional form for demand, has proven difficult. For negative semidefiniteness, Dette et al. (2016) provide a nonparametric test in an empirical model with individual heterogeneity and endogeneity. For symmetry, no analogous result has been available to date. Parametric tests exist (Barten, 1967; Laitinen, 1978; Meisner, 1979; Bera et al., 1981; Bewley, 1983; Taylor and Shonkwiler, 1985; Silver and Ali, 1989), but conflate violations of symmetry with misspecification. Nonparametric approaches such as Lewbel (1995), Haag et al. (2009), and Hoderlein (2011) rely on additional restrictions that are either unverifiable or impose substantial structure on the demand system. Recently, Kono (2025) shows that symmetry cannot be tested from the average Slutsky matrix alone, and Maes and Malhotra (2024) conjecture that in the general setting of Dette et al. (2016), nonparametric testability of Slutsky symmetry is impossible.

We show that this conjecture is false in the general setting. We derive a nonparametric restriction on observable conditional and marginal quantiles that is necessary for Slutsky symmetry, providing a testable characterization of the symmetry condition in the empirical setting of Dette et al. (2016). The key theoretical contribution is a bivariate extension of the identification result of Hoderlein and Mammen (2007) for average partial effects in nonseparable models without monotonicity. This extension, which is of independent interest, reveals that in the multivariate setting the identification result acquires correction terms that are absent in the univariate case. These terms have concrete economic content: they capture how the composition of preference types at a given demand level shifts with prices, and they govern the gap between conditional quantile demands and true Hicksian demands in multi-good welfare analysis.

Our approach is based on continuous demands in a nonseparable model with unrestricted heterogeneity. Hausman and Newey (2016) provide the foundational framework for nonparametric welfare analysis in this setting, but their analysis is restricted to two goods where symmetry holds by construction. We generalize their approach to multiple goods, where symmetry has empirical content and must be verified.

2 A nonparametric Slutsky symmetry condition

This section contains the main identification result for the Slutsky symmetry condition, which leads to a testable nonparametric restriction. To state it, we start with the required assumptions on the data and an extension of the ideas in Hoderlein and Mammen (2007) to conditional quantiles.

2.1 Setup and assumptions

The key idea is based on the connection between quantile functions of the data-generating process and properties of the Slutsky matrix in an empirical model with individual heterogeneity and endogeneity. Dette et al. (2016) posit an IV model for a nonparametric demand system with a potentially endogenous covariate (for example, income or total expenditures). The endogeneity of total expenditure is, in part, because categories of individual expenditures are often mismeasured, and hence the aggregate expenditure measure is mismeasured. Moreover, the authors model expenditure while allowing for unobserved individual heterogeneity. In the sequel, we carry over the notation and model from Dette et al. (2016) to offer a unified treatment:

$$\begin{aligned} Y &= \psi(P, X, U) = \psi(P, X, \nu(Q, A)) = \phi(P, X, Q, A) \\ X &= \mu(P, Q, S, V). \end{aligned} \tag{1}$$

Here Y, a vector of quantities of goods demanded, takes values in $\mathbb{R}^{L-1}$; and X is an endogenous variable, for instance income. We assume that there exists exactly one exogenous shifter S for the endogenous variable. This is done for ease of exposition, analogous to the result in Dette et al. (2016), and can be generalized to higher dimensions (e.g. Gunsilius, 2023). V captures the unobservable variation in this first stage regression. V can be solved for and used as residuals in a control function approach. For standard identification arguments (e.g. Imbens and Newey, 2009), therefore, we assume μ is invertible in its last argument. $P \in \mathbb{R}^{L-1}$ is the vector of prices for the goods; and U is an unrestricted unobservable describing the preferences. Making U as general as possible is crucial to account for any form of preference function. By writing $U = \nu(Q, A)$, we assume that the preference function is a function of observable characteristics Q, like age for instance, and unobservable characteristics A. A can therefore be interpreted as residual unobserved heterogeneity after controlling for the observable part of individual heterogeneity. Putting everything together, we have the same assumption on the model as Dette et al. (2016), to which we refer for further detail.

Assumption 2.1 (Model). *Let $(\Omega, \mathcal{F}, P)$ be a complete probability space on which we define the random variables $A : \Omega \to \mathcal{A} \subseteq \mathbb{R}^\infty$, and $(Y, P, X, Q, S, V) : \Omega \to \mathcal{Y} \times \mathcal{P} \times \mathcal{X} \times \mathcal{Q} \times \mathcal{S} \times \mathcal{V}, \mathcal{Y} \subseteq \mathbb{R}^{L-1}, \mathcal{P} \subseteq \mathbb{R}^{L-1}, \mathcal{X} \subseteq \mathbb{R}, \mathcal{Q} \subseteq \mathbb{R}^K, \mathcal{S} \subseteq \mathbb{R}, \mathcal{V} \subseteq \mathbb{R}$, with L and K finite integers, such that*

$$
\begin{aligned}
Y &= \phi(P, X, Q, A) \\
X &= \mu(P, Q, S, V),
\end{aligned}
$$

where $\phi : \mathcal{P} \times \mathcal{X} \times \mathcal{Q} \times \mathcal{A} \to \mathcal{Y}$ and $\mu : \mathcal{P} \times \mathcal{Q} \times \mathcal{S} \times \mathcal{V} \to \mathcal{X}$ are Borel functions, and realizations of (Y, P, X, Q, S) are observable, whereas (A, V) are latent variables. The function μ is invertible in its last argument, for every (p, q, s).

The Slutsky matrix $\mathcal{S}(p, x, u)$ takes the form

$$
\mathcal{S} = D_p \psi(p, x, u) + \partial_x \psi(p, x, u) \psi(p, x, u),
$$

where D_p is the Jacobian of Marshallian demands with respect to prices and ∂_x is a vector of partial derivatives of Marshallian demands with respect to income.
Following Dette et al. (2016), we denote $W = (P, X, Q, V)$ and introduce $k(\alpha, b \mid w)$ as the α-quantile defined as

$$
P\left(b'Y \leq k(\alpha, b \mid w) \mid W = w\right) = \alpha
$$

for any nonzero $b \in \mathbb{R}^{L-1}$. To derive a condition on the observable quantiles that captures the symmetry condition of the Slutsky matrix, only considering univariate marginal quantiles is not enough, as one has to compare off-diagonal terms. However, we can focus on only two elements at a time to capture the symmetry condition. Still, only considering bivariate joint quantiles does not provide identification of the relevant conditions, because the result in Hoderlein and Mammen (2007) cannot be generalized to the multivariate setting directly (Hoderlein and Mammen, 2009).

We show that symmetry becomes testable in this setting by considering a combination of *conditional- and marginal quantiles*, for which we provide a generalization of the result in Hoderlein and Mammen (2007) in section 2.2. For symmetry, we only need to consider the case where b is a unit vector, that is, we only need to consider elements of the form $e_i'Y$ for $e_i \in \mathbb{R}^{L-1}$ the i-th unit vector in $\mathbb{R}^{L-1}$. In this case the above defined α-quantile reduces to the α-quantile of the marginal distribution of $Y_i = e_i'Y$. For notational purposes, we define the *marginal α-quantile* as

$$P\left(Y_i \leq k_{\alpha,i}(w) \mid W=w\right)=\alpha$$

In the same vein, we define the *conditional γ-quantile* of the element $Y_i \equiv e_i'Y$ given that the element $Y_j \equiv e_j'Y$ lies at the α marginal level by

$$P\left(Y_i \leq k_{\gamma,i}\left(w, k_{\alpha,j}(w)\right) \mid W=w, Y_j=k_{\alpha,j}(w)\right)=\gamma.$$

This conditional quantile depends not only on W but also on the marginal quantile $k_{\alpha,j}(w)$, which in turn also depends on W. In fact, it will be the interplay of the conditional and the marginal quantile which will enable us to derive conditions on the observable data that captures the Slutsky symmetry. We also need to make use of the *joint bivariate α-quantile* $K_{ij}(\alpha \mid w)$ for elements $(Y_i, Y_j) \equiv \left(e_i'Y, e_j'Y\right)$ of the vector $Y \in \mathbb{R}^{L-1}$, which is defined as

$$P\left((Y_i, Y_j) \leq K_{ij}(\alpha \mid w) \mid W=w\right)=\alpha.$$

Note that K_{ij} is not unique without further assumption: it is not a point but an isoquant in $\mathbb{R}^2$, which is the main complication for testing symmetry nonparametrically.

Since we generalize the identification result in Hoderlein and Mammen (2007) to conditional and marginal quantiles, we aim to keep the same notation and assumptions. These assumptions are also stated in the same notation as the ones in Dette et al. (2016) to keep the two results unified. We set $Z := (Q, V)$.

Assumption 2.2 (Independence)**.** *$A \perp\!\!\!\perp (P, X) \mid Z$.*

The independence assumption requires that after controlling for observable preference characteristics and unobservable first-stage, the residual unobserved heterogeneity represented by A is independent of prices and the endogenous variable. This conditional independence assumption is key to identification; it maps the space of nonseparable demand functions, which is inherently unobservable, to something we can observe: regression quantiles. It is this feature that allows us to derive a testable restriction on the quantiles of the data generating process, develop a test based on the restriction and infer Slutsky symmetry for the unobserved nonseparable demand functions.

In the following, we denote by $\partial_{w_{1s}} k_{\alpha,j}(w^*)$ the partial derivative of the quantile $k_{\alpha,j}$ with respect to the price of good s (the s^{th} element in the price vector, which in turn is the first element of w) at w^*. For the conditional quantile $k_{\gamma,i}\left(w^*, k_{\alpha,j}\left(w^*\right)\right)$, we define by $\partial_{1,w_{1s}} k_{\gamma,i}\left(w^*, k_{\alpha,j}\left(w^*\right)\right)$ the partial derivative with respect to price of good s (the s^{th} element of the price vector, which in turn is the first argument of w) at w^* and by $\partial_2 k_{\gamma,i}\left(w^*, k_{\alpha,j}\left(w^*\right)\right)$ the partial derivative with respect to the second argument, i.e. $k_{\alpha,j}\left(w^*\right)$.

Furthermore, we write the conditional density $f_{e_i'Y|W,e_j'Y}(y_i)$ at a point $(w^*, k_{\alpha,j}(w^*))$ as $f_{e_i'Y|W=w^*,e_j'Y=k_{\alpha,j}(w^*)}(y_i)$. Similarly, for the conditional distribution function $F_{Y_i|W,Y_j=y_j}(y_i)$, we write $\nabla_{p,1}F_{Y_i|W=w^*,Y_j=y_j}(y_i)$ for the gradient with respect to the price components of W holding the conditioning value y_j fixed, $\partial_{x,1}F_{Y_i|W=w^*,Y_j=y_j}(y_i)$ for the partial derivative with respect to income holding y_j fixed, and $\partial_2 F_{Y_i|W=w^*,Y_j=y_j}(y_i)$ for the partial derivative with respect to the conditioning value y_j. We impose regularity conditions on the conditional distribution function, the conditional density, the conditional quantile and the demand function, which we have relegated to the appendix.

2.2 A generalization of Hoderlein and Mammen (2007)

We can now state a multivariate extension of the identification result in Hoderlein and Mammen (2007):

Lemma 2.1. *Under Assumption 2.1 and Assumption 2.2 along with the regularity conditions in Assumption A.1, fix a point (y_i^*, y_j^*) in the interior of the support of (Y_i, Y_j) given $W = w^*$ and define*

$$\alpha_j := F_{Y_j|W=w^*}(y_j^*), \qquad \gamma_{i|j} := F_{Y_i|W=w^*,Y_j=y_j^*}(y_i^*),$$

so that $k_{\alpha_j,j}(w^*) = y_j^*$ and $k_{\gamma_{i|j},i}(w^*, k_{\alpha_j,j}(w^*)) = y_i^*$. Then for $L \geq 3$, the following holds

$$\begin{aligned}
&E\Big[\partial_{w_{1s}}\phi \,\Big|\, W = w^*,\, e_i'Y = k_{\gamma_{i|j},i}(w^*, k_{\alpha_j,j}(w^*)),\, e_j'Y = k_{\alpha_j,j}(w^*)\Big] \\
&= \partial_{1,w_{1s}} k_{\gamma_{i|j},i}(w^*, k_{\alpha_j,j}(w^*)) \\
&\quad + \partial_{w_{1s}} k_{\alpha_j,j}(w^*) \left[\partial_2 k_{\gamma_{i|j},i}(w^*, k_{\alpha_j,j}(w^*)) + \frac{\partial_2 F_{Y_i|W=w^*,Y_j=k_{\alpha_j,j}(w^*)}(k_{\gamma_{i|j},i}(w^*, k_{\alpha_j,j}(w^*)))}{f_{Y_i|W=w^*,Y_j=k_{\alpha_j,j}(w^*)}(k_{\gamma_{i|j},i}(w^*, k_{\alpha_j,j}(w^*)))}\right] \\
&\quad + \frac{\partial_{w_{1s}} F_{Y_i|W=w^*,Y_j=k_{\alpha_j,j}(w^*)}(k_{\gamma_{i|j},i}(w^*, k_{\alpha_j,j}(w^*)))}{f_{Y_i|W=w^*,Y_j=k_{\alpha_j,j}(w^*)}(k_{\gamma_{i|j},i}(w^*, k_{\alpha_j,j}(w^*)))}
\end{aligned}$$

for $s = 1, \ldots, L$.

This result is analogous to the univariate version of Hoderlein and Mammen (2007). The difference is that the conditional quantile is affected through several paths when changing w, instead of just the marginal $k_{\alpha,j}(w)$. This results in the additional complex correction terms in the expression. The underlying idea of the proof lies in the fact that the inherently 2-dimensional quantile function $K_{ij}(\alpha, w)$ can be represented in two different ways by the two respective marginal- and the corresponding conditional quantiles. This representation is unique for regular quantiles.

Lemma 2.1 can be straightforwardly extended to more than two dimensions by repeated conditioning. In this case, the expression becomes even more complicated with additional terms; this stems from the fact that for d-dimensions, one needs to iteratively condition on $d-1$ quantiles. Since we only care about the Slutsky symmetry case, which is bivariate, we omit the d-dimensional version. Note that this result does not hinge on the generality of Assumption 2.1: in applications without endogeneity, this result still applies with a "reduced form" nonparametric demand model (the second stage model in Assumption 2.1).

2.3 A Slutsky symmetry condition on conditional and marginal quantiles

Lemma 2.1 now enables us to derive the condition on the quantile functions which captures the symmetry condition of the Slutsky matrix. Based on Lemma 2.1, the intuition is as follows: the joint quantile $K_{ij}(\alpha|w)$ can be represented uniquely in two ways, corresponding to two different combinations of marginal- and conditional quantiles, which captures the "off-diagonal" terms in the Slutsky matrix via these marginal and corresponding conditional quantile functions. This leads to the following representation:

Theorem 2.1 (Slutsky symmetry)**.** *Let Assumptions 2.1–A.1 hold. Fix a point* (y_i^*, y_j^*) *in the interior of the support of* (Y_i, Y_j) *given* $W = w^*$*, and define the following quantile indices:*

$$\alpha_i := F_{Y_i|W=w^*}(y_i^*), \qquad \alpha_j := F_{Y_j|W=w^*}(y_j^*),$$
$$\gamma_{i|j} := F_{Y_i|W=w^*, Y_j=y_j^*}(y_i^*), \qquad \gamma_{j|i} := F_{Y_j|W=w^*, Y_i=y_i^*}(y_j^*),$$

so that $k_{\alpha_i,i}(w^*) = y_i^*$*,* $k_{\alpha_j,j}(w^*) = y_j^*$*,* $k_{\gamma_{i|j},i}(w^*, k_{\alpha_j,j}(w^*)) = y_i^*$*, and* $k_{\gamma_{j|i},j}(w^*, k_{\alpha_i,i}(w^*)) = y_j^*$*.*

Define the correction terms

$$C_{ij}(w^*) := \partial_2 k_{\gamma_{i|j},i}(w^*, k_{\alpha_j,j}(w^*)) + \frac{\partial_2 F_{Y_i|W=w^*,Y_j=k_{\alpha_j,j}(w^*)}(k_{\gamma_{i|j},i}(w^*, k_{\alpha_j,j}(w^*)))}{f_{Y_i|W=w^*,Y_j=k_{\alpha_j,j}(w^*)}(k_{\gamma_{i|j},i}(w^*, k_{\alpha_j,j}(w^*)))},$$

$$C_{ji}(w^*) := \partial_2 k_{\gamma_{j|i},j}(w^*, k_{\alpha_i,i}(w^*)) + \frac{\partial_2 F_{Y_j|W=w^*,Y_i=k_{\alpha_i,i}(w^*)}(k_{\gamma_{j|i},j}(w^*, k_{\alpha_i,i}(w^*)))}{f_{Y_j|W=w^*,Y_i=k_{\alpha_i,i}(w^*)}(k_{\gamma_{j|i},j}(w^*, k_{\alpha_i,i}(w^*)))},$$

$$D_{ij}(w^*) := \frac{\nabla_{p,1} F_{Y_i|W=w^*,Y_j=k_{\alpha_j,j}(w^*)}(k_{\gamma_{i|j},i}(w^*, k_{\alpha_j,j}(w^*)))}{f_{Y_i|W=w^*,Y_j=k_{\alpha_j,j}(w^*)}(k_{\gamma_{i|j},i}(w^*, k_{\alpha_j,j}(w^*)))},$$

$$D_{ij}^{(x)}(w^*) := \frac{\partial_{x,1} F_{Y_i|W=w^*,Y_j=k_{\alpha_j,j}(w^*)}(k_{\gamma_{i|j},i}(w^*, k_{\alpha_j,j}(w^*)))}{f_{Y_i|W=w^*,Y_j=k_{\alpha_j,j}(w^*)}(k_{\gamma_{i|j},i}(w^*, k_{\alpha_j,j}(w^*)))},$$

$$D_{ji}(w^*) := \frac{\nabla_{p,1} F_{Y_j|W=w^*,Y_i=k_{\alpha_i,i}(w^*)}(k_{\gamma_{j|i},j}(w^*, k_{\alpha_i,i}(w^*)))}{f_{Y_j|W=w^*,Y_i=k_{\alpha_i,i}(w^*)}(k_{\gamma_{j|i},j}(w^*, k_{\alpha_i,i}(w^*)))},$$

$$D_{ji}^{(x)}(w^*) := \frac{\partial_{x,1} F_{Y_j|W=w^*,Y_i=k_{\alpha_i,i}(w^*)}(k_{\gamma_{j|i},j}(w^*, k_{\alpha_i,i}(w^*)))}{f_{Y_j|W=w^*,Y_i=k_{\alpha_i,i}(w^*)}(k_{\gamma_{j|i},j}(w^*, k_{\alpha_i,i}(w^*)))}.$$

If $\mathcal{S}$ is symmetric, then the following equality holds for all $i, j = 1, \ldots, L-1$ and all (y_i^*, y_j^*) in the interior of the support:

$$\begin{aligned}
&\nabla_{p,1} k_{\gamma_{i|j},i}(w^*, k_{\alpha_j,j}(w^*))\, e_j + \partial_{x,1} k_{\gamma_{i|j},i}(w^*, k_{\alpha_j,j}(w^*))\, k_{\alpha_j,j}(w^*) \\
&\quad + C_{ij}(w^*) \cdot \Big[\nabla_p k_{\alpha_j,j}(w^*)\, e_j + \partial_x k_{\alpha_j,j}(w^*)\, k_{\alpha_j,j}(w^*)\Big] \\
&\quad + D_{ij}(w^*) \cdot e_j + D_{ij}^{(x)}(w^*)\, k_{\alpha_j,j}(w^*) \\
&= \nabla_{p,1} k_{\gamma_{j|i},j}(w^*, k_{\alpha_i,i}(w^*))\, e_i + \partial_{x,1} k_{\gamma_{j|i},j}(w^*, k_{\alpha_i,i}(w^*))\, k_{\alpha_i,i}(w^*) \\
&\quad + C_{ji}(w^*) \cdot \Big[\nabla_p k_{\alpha_i,i}(w^*)\, e_i + \partial_x k_{\alpha_i,i}(w^*)\, k_{\alpha_i,i}(w^*)\Big] \\
&\quad + D_{ji}(w^*) \cdot e_i + D_{ji}^{(x)}(w^*)\, k_{\alpha_i,i}(w^*). \qquad (2)
\end{aligned}$$

Equation (2) gives a necessary condition for Slutsky Symmetry in the empirical setting described by Model (1) and Assumptions A.1 and 2.2. If this condition is not satisfied for any i, j pair of goods under the model assumptions, then the Slutsky symmetry is not satisfied for that set of goods in this model. The complication arises from the correction terms C_{ij}, C_{ji}, D_{ij}, D_{ji}, $D_{ij}^{(x)}$, and $D_{ji}^{(x)}$, which do not appear in the negative semidefinite case.

3 Slutsky Symmetry and Nonparametric Welfare Analysis

Our results yield new insights for nonparametric welfare analysis with individual heterogeneity in multi-good settings. Hausman and Newey (2016) show that in a two-good setting with a numeraire, quantile demands coincide with true demands, using the univariate identification result of Hoderlein and Mammen (2007) and the negative semidefiniteness condition of Dette et al. (2016). In that setting, Slutsky symmetry holds by construction. With more than two goods, symmetry has empirical content, and the univariate identification argument no longer suffices. Existing approaches either assume symmetry outright or impose restrictive conditions such as monotonicity in scalar unobserved heterogeneity.

Our bivariate extension of Hoderlein and Mammen (2007) in Lemma 2.1, together with the symmetry condition in (2), restores the quantile-based approach to welfare analysis under general heterogeneity. Theorem 2.1 shows that with vector-valued demands, true demands are identified by conditional and marginal quantiles satisfying (2). Unlike in Hausman and Newey (2016), where marginal quantile demands coincide with true demands directly, conditional quantile demands for each good coincide with true demands only up to the correction factors $C_{ij}(w^*)$, $D_{ij}(w^*)$, and $D_{ij}^{(x)}(w^*)$ induced by conditioning on the demand level of the other good.

Empirical content of the condition The symmetry condition in Theorem 2.1 involves two types of correction terms beyond the leading conditional and marginal quantile derivatives: the terms $C_{ij}(w^*)$ and $D_{ij}(w^*)$ (along with their income counterparts $D_{ij}^{(x)}(w^*)$). These terms have distinct economic content that is worth unpacking, as they govern the gap between conditional quantile demands and true Hicksian demands in multi-good settings.

The C_{ij} terms. The term $C_{ij}(w^*)$ captures how the conditional quantile of Y_i responds to shifts in the level of Y_j at which we condition. It combines a direct effect ($\partial_2 k_{\gamma_{i|j},i}$, the mechanical response of the conditional quantile to a change in the conditioning value) with an indirect effect through the conditional distribution. Economically, $C_{ij}(w^*)$ is small when the demand for good i at a given quantile is approximately insensitive to the demand level of good j. This is plausible when the two goods are weakly related in preferences (for instance, under approximate weak separability) or when each good individually constitutes a small fraction of total expenditure, so that the income reallocation channel linking their demands is muted.

The D_{ij} terms. The terms $D_{ij}(w^*)$ and $D_{ij}^{(x)}(w^*)$ capture a fundamentally different phenomenon: how the conditional distribution of Y_i given $Y_j = y_j^*$ shifts when prices or income change, *holding the conditioning level y_j^* fixed.* This effect is absent in the univariate setting of Hoderlein and Mammen (2007), where the independence assumption $A \perp\!\!\!\perp (P, X) \mid Z$ suffices to eliminate any such dependence. In the bivariate setting, the conditional distribution $F_{Y_i|W,Y_j=y_j^*}$ conditions on the level set $\{a : e_j'\phi(p, x, z^*, a) = y_j^*\}$, which shifts with (p, x) through the demand function ϕ, even though the marginal distribution of the preference heterogeneity A given Z does not depend on prices or income.

Economically, $D_{ij}(w^*) = 0$ when the *composition of preference types* demanding a fixed quantity y_j^* of good j is stable with respect to price changes. This is a natural condition in settings where the mapping from preferences to demand for good j is approximately one-to-one (so that the level set $\{a : e_j'\phi(\cdot, a) = y_j^*\}$ shifts rigidly rather than changing its composition), or more generally when the heterogeneity in demand for good i among consumers who demand the same quantity of good j is not systematically related to the price sensitivity of demand for good j. A sufficient condition for $D_{ij}(w^*) = 0$ for all w^* is that ϕ is monotone in the component of A that drives Y_j, reducing the bivariate problem to the univariate case of Hoderlein and Mammen (2007). The results of this paper, when integrated with existing negative-semidefiniteness tests, now allow researchers to comprehensively verify Slutsky conditions prior to welfare analysis with unobserved heterogeneity.

4 Conclusion

We provide a fully nonparametric analysis of the testability of the symmetry of the Slutsky matrix via empirical quantiles, complementing existing work that provides a test for the negative semidefiniteness. This work, therefore, closes the gap on testing an empirically informed version of rationality, providing a testable symmetry condition relevant for nonparametric welfare analysis with individual heterogeneity. Incidentally, the results show that the symmetry condition is testable using *conditional* quantile restrictions. This shows that a recent conjecture (Maes and Malhotra, 2024) is false and opens the door to testing a empirically informed version of rationality with heterogeneity. The symmetry condition provides empirical restrictions in empirical welfare analysis. Since it involves only conditional and marginal quantiles and their derivatives, all of its components are estimable from data via conditional quantile inference methods. In the multi-good case, the condition provides nonlinear correction factors that govern the gap between conditional quantile demands and true Hicksian demands, which must be accounted for in any nonparametric welfare analysis with individual heterogeneity beyond the two-good setting.

References

Barten, Anton P (1967). "Evidence on the Slutsky conditions for demand equations". In: *The Review of Economic and Statistics*, pp. 77–84.

Bera, A.K. et al. (1981). "Further evidence on asymptotic tests for homogeneity and symmetry in large demand systems". In: *Economics Letters* 8.2, pp. 101–105. ISSN: 0165-1765. DOI: https://doi.org/10.1016/0165-1765(81)90001-X. URL: https://www.sciencedirect.com/science/article/pii/016517658190001X.

Bewley, Ronald A (1983). "Tests of restrictions in large demand systems". In: *European Economic Review* 20.1-3, pp. 257–269.

Chang, Joseph T and David Pollard (1997). "Conditioning as disintegration". In: *Statistica Neerlandica* 51.3, pp. 287–317.

Dette, Holger et al. (2016). "Testing multivariate economic restrictions using quantiles: The example of Slutsky negative semidefiniteness". In: *Journal of Econometrics* 191.1, pp. 129–144. ISSN: 0304-4076. DOI: https://doi.org/10.1016/j.jeconom.2015.07.004. URL: https://www.sciencedirect.com/science/article/pii/S030440761500250X.

Gunsilius, Florian F (2023). "A condition for the identification of multivariate models with binary instruments". In: *Journal of Econometrics* 235.1, pp. 220–238.

Haag, Berthold R. et al. (2009). "Testing and imposing Slutsky symmetry in nonparametric demand systems". In: *Journal of Econometrics* 153.1, pp. 33–50. ISSN: 0304-4076. DOI: https://doi.org/10.1016/j.jeconom.2009.04.003. URL: https://www.sciencedirect.com/science/article/pii/S0304407609001092.

Hausman, Jerry A. and Whitney K. Newey (2016). "Individual Heterogeneity and Average Welfare". In: *Econometrica* 84.3, pp. 1225–1248.

Hoderlein, Stefan (2011). "How many consumers are rational?" In: *Journal of Econometrics* 164.2, pp. 294–309.

Hoderlein, Stefan and Enno Mammen (2007). "Identification of Marginal Effects in Nonseparable Models without Monotonicity". In: *Econometrica* 75.5, pp. 1513–1518. ISSN: 00129682, 14680262. URL: http://www.jstor.org/stable/4502038 (visited on 10/19/2024).

— (2009). "Identification and estimation of local average derivatives in non-separable models without monotonicity". In: *The Econometrics Journal* 12.1, pp. 1–25. DOI: https://doi.

org/10.1111/j.1368-423X.2008.00273.x. eprint: https://onlinelibrary.wiley.com/doi/pdf/10.1111/j.1368-423X.2008.00273.x. URL: https://onlinelibrary.wiley.com/doi/abs/10.1111/j.1368-423X.2008.00273.x.

Houthakker, Hendrik S. (1950). "Revealed Preference and the Utility Function". In: *Economica* 17.66, pp. 159–174.

Hurwicz, Leonid and Hirofumi Uzawa (1971). "On the integrability of demand functions". In: *Preferences, utility and demand*.

Imbens, Guido W and Whitney K Newey (2009). "Identification and estimation of triangular simultaneous equations models without additivity". In: *Econometrica* 77.5, pp. 1481–1512.

Kono, Haruki (2025). "Untestability of Average Slutsky Symmetry". In: *arXiv preprint arXiv:2501.18923*.

Laitinen, Kenneth (1978). "Why is demand homogeneity so often rejected?" In: *Economics Letters* 1.3, pp. 187–191.

Lewbel, Arthur (1995). "Consistent nonparametric hypothesis tests with an application to Slutsky symmetry". In: *Journal of Econometrics* 67.2, pp. 379–401.

Maes, Sebastiaan and Raghav Malhotra (2024). *Beyond the Mean: Testing Consumer Rationality through Higher Moments of Demand*. Centre for Research in Economic Theory and its Applications.

Meisner, James F. (1979). "The sad fate of the asymptotic Slutsky symmetry test for large systems". In: *Economics Letters* 2.3, pp. 231–233. ISSN: 0165-1765. DOI: https://doi.org/10.1016/0165-1765(79)90027-2. URL: https://www.sciencedirect.com/science/article/pii/0165176579900272.

Samuelson, Paul A. (1948). "Consumption Theory in Terms of Revealed Preference". In: *Economica* 15.60, pp. 243–253.

Silver, J Lew and Mukhtar M Ali (1989). "Testing Slutsky symmetry in systems of linear demand equations". In: *Journal of Econometrics* 41.2, pp. 251–266.

Taylor, Timothy G. and J.S. Shonkwiler (1985). "A size-corrected Wald test for Slutsky symmetry in systems of demand equations". In: *Economics Letters* 19.4, pp. 327–330. ISSN: 0165-1765. DOI: https://doi.org/10.1016/0165-1765(85)90229-0. URL: https://www.sciencedirect.com/science/article/pii/0165176585902290.

A Proofs

A.1 Regularity Assumptions

Assumption A.1 (Regularity)**.** *For fixed* $w^* := (p^*, x^*, z^*) \in \mathcal{P} \times \mathcal{X} \times \mathcal{Z}$ *all* $i, j = 1, 2, \ldots, L-1$*, and* $\gamma, \alpha \in [0, 1]$ *the following conditions hold:*

1. *The conditional distribution function* $F_{Y_i|W,Y_j}$ *of* $e_i'Y$ *given* $\left(W, e_j'Y\right)$ *is absolutely continuous with respect to Lebesgue measure for* $\left(p, x, e_j'y\right)$ *in a neighborhood of* $(w^*, k_{\alpha,j}(w^*))$ *and for* $z = z^*$.

2. *The conditional density* $f_{e_i'Y|W,e_j'Y}(y_i)$ *of* $e_i'Y$ *given* $\left(W, e_j'Y\right)$ *is continuous in* $(e_i'y)$ *and differentiable in* $\left(P, X, e_j'Y\right)$ *in a neighborhood of the point* $(k_{\gamma,i}(w^*, k_{\alpha,j}(w^*)), k_{\alpha,j}(w^*))$. *Furthermore,* $f_{e_i'Y|W,e_j'Y}(y_i)$ *is bounded above by an integrable function* $g(y_i)$ *on* $\mathbb{R}$ *and bounded below by some constant* $C > 0$ *in a neighborhood of* $(w^*, k_{\alpha,j}(w^*))$.

3. *The conditional quantile* $k_{\gamma,e_i}(w^*, k_{\alpha,j}(w^*))$ *is partially differentiable with respect to both arguments at* $(w^*, k(\alpha, e_j \mid w^*))$. *Similarly, the marginal quantile* $k_{\alpha,j}(w^*)$ *is partially differentiable with respect to any component of* (p, x) *at* w^*.

4. *The function* $\phi(p, x, u)$ *is approximately differentiable in all dimensions of* $w_1 := (p, x) \in \mathbb{R}^L$ *in the sense that there exist measurable functions* $\Delta_s, s = 1, \ldots, L$ *satisfying*

$$P\left[\left|\phi\left(w_{1s}^* + \delta, w_{-1s}^*, z^*, A\right) - \phi\left(w_{1s}^*, w_{-1s}^*, z^*, A\right) - \delta\Delta_s(A)\right| \geq \delta\varepsilon \mid W = w^*, e_j'Y = k_{\alpha,j}(w^*)\right] = o(\delta).$$

for $\delta \to 0$ *and fixed* $\varepsilon > 0$. *Analogously to Dette, Hoderlein δ Neumeyer (2016), we write* $\partial_{w_{1s}}\phi(w_1^*, z^*, a) := \Delta_s(a)$ *and* $\partial_{w_{1s}}\phi := \Delta_s(A)$ *for all* $s = 1, \ldots, L$.

5. *The conditional distribution of* $(e_i'Y, \partial_{w_{1s}}\phi)$ *given* $\left(P, X, Z, e_j'Y\right)$ *is absolutely continuous with respect to Lebesgue measure for* $\left(W, e_j'Y\right) = (w^*, k_{\alpha,j}(w^*))$ *and all* $i, j \in \{1, \ldots, L\}$. *The conditional density* $f_{Y_i,\partial_{w_{1s}}\phi|W,e_j'Y}$ *satisfies*

$$f_{Y_i,\partial_{w_{1s}}\phi|W=w^*,e_j'Y=k_{\alpha,j}(w^*)}(y, y') \leq Cg(y')$$

for some constant $0 < C < +\infty$ *and a positive density function* $g(y')$ *with finite first moment on* $\mathbb{R}$.

A.2 Proof of Lemma 2.1

Proof. The proof is similar to the proof of Theorem 2.1 in Hoderlein and Mammen (2007), except for several additional terms and computations. To reduce notational clutter, we assume that W is univariate in analogy to Hoderlein and Mammen (2007) (this is without loss of generality since the argument applies to each component of w separately, and the multivariate case follows by repeating the derivation for each partial derivative $\partial_{w_{1s}}$, $s = 1, \ldots, L$). By definition of the conditional quantile we have

$$
\begin{aligned}
0 &= P\Big(e_i'Y \leq k_{\gamma_{i|j},i}(w^* + \delta, k_{\alpha_j,j}(w^* + \delta)) \;\Big|\; W = w^* + \delta,\, e_j'Y = k_{\alpha_j,j}(w^* + \delta)\Big) \\
&\quad - P\Big(e_i'Y \leq k_{\gamma_{i|j},i}(w^*, k_{\alpha_j,j}(w^*)) \;\Big|\; W = w^*,\, e_j'Y = k_{\alpha_j,j}(w^*)\Big) \\
&= A_1 + A_2 + A_3,
\end{aligned}
$$

where

$$
\begin{aligned}
A_1 &= P\Big(e_i'\phi(w^* + \delta, U) \leq k_{\gamma_{i|j},i}(w^* + \delta, k_{\alpha_j,j}(w^* + \delta)) \;\Big|\; W = w^* + \delta,\, e_j'Y = k_{\alpha_j,j}(w^* + \delta)\Big) \\
&\quad - P\Big(e_i'\phi(w^* + \delta, U) \leq k_{\gamma_{i|j},i}(w^*, k_{\alpha_j,j}(w^*)) \;\Big|\; W = w^* + \delta,\, e_j'Y = k_{\alpha_j,j}(w^* + \delta)\Big), \\
A_2 &= P\Big(e_i'\phi(w^* + \delta, U) \leq k_{\gamma_{i|j},i}(w^*, k_{\alpha_j,j}(w^*)) \;\Big|\; W = w^* + \delta,\, e_j'Y = k_{\alpha_j,j}(w^* + \delta)\Big) \\
&\quad - P\Big(e_i'\phi(w^* + \delta, U) \leq k_{\gamma_{i|j},i}(w^*, k_{\alpha_j,j}(w^*)) \;\Big|\; W = w^*,\, e_j'Y = k_{\alpha_j,j}(w^*)\Big), \\
A_3 &= P\Big(e_i'\phi(w^* + \delta, U) \leq k_{\gamma_{i|j},i}(w^*, k_{\alpha_j,j}(w^*)) \;\Big|\; W = w^*,\, e_j'Y = k_{\alpha_j,j}(w^*)\Big) \\
&\quad - P\Big(e_i'\phi(w^*, U) \leq k_{\gamma_{i|j},i}(w^*, k_{\alpha_j,j}(w^*)) \;\Big|\; W = w^*,\, e_j'Y = k_{\alpha_j,j}(w^*)\Big).
\end{aligned}
$$

Consider each term step by step. We have

$$
\begin{aligned}
A_1 &= \int_{k_{\gamma_{i|j},i}(w^*, k_{\alpha_j,j}(w^*))}^{k_{\gamma_{i|j},i}(w^* + \delta, k_{\alpha_j,j}(w^* + \delta))} f_{Y_i|W=w^*+\delta, Y_j = k_{\alpha_j,j}(w^*+\delta)}(y_i)\, dy_i \\
&= \delta\Big[\partial_1 k_{\gamma_{i|j},i}(w^*, k_{\alpha_j,j}(w^*)) + \partial_2 k_{\gamma_{i|j},i}(w^*, k_{\alpha_j,j}(w^*))\, \partial_w k_{\alpha_j,j}(w^*)\Big] \\
&\qquad \cdot f_{Y_i|W=w^*, Y_j = k_{\alpha_j,j}(w^*)}(k_{\gamma_{i|j},i}(w^*, k_{\alpha_j,j}(w^*))) + o(\delta),
\end{aligned}
$$

which follows from parts 1 and 3 of Assumption A.1 and the multivariate chain rule for $\delta \to 0$.

For A_2:

$$
\begin{aligned}
A_2 &= \int_{-\infty}^{k_{\gamma_{i|j},i}(w^*,k_{\alpha_j,j}(w^*))} \Big[f_{Y_i|W=w^*+\delta,Y_j=k_{\alpha_j,j}(w^*+\delta)}(y_i) - f_{Y_i|W=w^*,Y_j=k_{\alpha_j,j}(w^*)}(y_i) \Big]\, dy_i \\
&= \delta \int_{-\infty}^{k_{\gamma_{i|j},i}(w^*,k_{\alpha_j,j}(w^*))} \Big[\partial_1 f_{Y_i|W=w^*,Y_j=k_{\alpha_j,j}(w^*)}(y_i) \\
&\qquad\qquad + \partial_2 f_{Y_i|W=w^*,Y_j=k_{\alpha_j,j}(w^*)}(y_i)\, \partial_w k_{\alpha_j,j}(w^*) \Big]\, dy_i + o(\delta) \\
&= \delta \Big[\partial_{w_{1s}} F_{Y_i|W=w^*,Y_j=k_{\alpha_j,j}(w^*)}(k_{\gamma_{i|j},i}(w^*,k_{\alpha_j,j}(w^*))) \\
&\qquad + \partial_w k_{\alpha_j,j}(w^*)\, \partial_2 F_{Y_i|W=w^*,Y_j=k_{\alpha_j,j}(w^*)}(k_{\gamma_{i|j},i}(w^*,k_{\alpha_j,j}(w^*))) \Big] + o(\delta),
\end{aligned}
$$

where the second line follows from parts 1, 2, and 3 of Assumption A.1 in combination with the dominated convergence theorem and the multivariate chain rule for $\delta \to 0$. Note that the first term $\partial_{w_{1s}} F_{Y_i|W=w^*,Y_j=k_{\alpha_j,j}(w^*)}$ does *not* vanish: while Assumption 2.2 ensures $A \perp\!\!\!\perp (P, X) \mid Z$, the conditional distribution $F_{Y_i|W,Y_j=y_j}$ conditions on the level set $\{a : e_j' \phi(w_1, z^*, a) = y_j\}$, which depends on w_1 through ϕ. This distinguishes the bivariate setting from the univariate case in Hoderlein and Mammen (2007), where no such conditioning is present and the analogous term is zero.

By the same argument as in Hoderlein and Mammen (2007), under parts 1, 4, and 5 of Assumption A.1 we obtain

$$
\begin{aligned}
A_3 = -\delta\, E\Big[\partial_w \phi \;\Big|\; W = w^*,\, e_i' Y = k_{\gamma_{i|j},i}(w^*, k_{\alpha_j,j}(w^*)),\, e_j' Y = k_{\alpha_j,j}(w^*) \Big] \\
\cdot f_{Y_i|W=w^*,Y_j=k_{\alpha_j,j}(w^*)}(k_{\gamma_{i|j},i}(w^*, k_{\alpha_j,j}(w^*))) + o(\delta).
\end{aligned}
$$

Rewriting and simplifying the equation $A_1 + A_2 + A_3 = 0$ gives the claim. □

A.3 Proof of Theorem 2.1

Proof. $\mathcal{S}$ is symmetric if and only if $e_i' \mathcal{S}\, e_j = e_j' \mathcal{S}\, e_i$ for all unit vectors e_i, e_j. Fix a point (y_i^*, y_j^*) in the interior of the support of (Y_i, Y_j) given $W = w^*$. Define $\alpha_i, \alpha_j, \gamma_{i|j}, \gamma_{j|i}$ as in the statement of the theorem. This point lies on the isoquant $K_{ij}(\beta \mid w^*)$ for $\beta := F_{Y_i,Y_j|W=w^*}(y_i^*, y_j^*)$.

If $\mathcal{S}$ is symmetric, then

$$
E\big[e_i' \mathcal{S}\, e_j \bigm| W = w^*, (Y_i, Y_j) = (y_i^*, y_j^*) \big] = E\big[e_j' \mathcal{S}\, e_i \bigm| W = w^*, (Y_i, Y_j) = (y_i^*, y_j^*) \big]. \tag{3}
$$

Although $K_{ij}(\beta \mid w^*)$ is set-valued, the disintegration theorem (the conditions of Theorem 1 in Chang and Pollard (1997) are satisfied since our measure is Radon on $\mathbb{R}$)

provides two unique decompositions of the conditioning at the point (y_i^*, y_j^*):

(i) Condition first on $Y_j = y_j^* = k_{\alpha_j,j}(w^*)$, then on $Y_i = y_i^* = k_{\gamma_{i|j},i}(w^*, k_{\alpha_j,j}(w^*))$;

(ii) Condition first on $Y_i = y_i^* = k_{\alpha_i,i}(w^*)$, then on $Y_j = y_j^* = k_{\gamma_{j|i},j}(w^*, k_{\alpha_i,i}(w^*))$.

By uniqueness of the disintegration, the σ-algebras generated by these two decompositions coincide at the point (y_i^*, y_j^*), so that

$$
\begin{aligned}
&E\Big[e_i'\mathcal{S}\, e_j \;\Big|\; W = w^*, Y_i = k_{\gamma_{i|j},i}(w^*, k_{\alpha_j,j}(w^*)),\, Y_j = k_{\alpha_j,j}(w^*)\Big] \\
&\quad = E\Big[e_j'\mathcal{S}\, e_i \;\Big|\; W = w^*, Y_j = k_{\gamma_{j|i},j}(w^*, k_{\alpha_i,i}(w^*)),\, Y_i = k_{\alpha_i,i}(w^*)\Big].
\end{aligned}
$$

The proof now proceeds similarly to the one in Dette et al. (2016), using Lemma 2.1. Consider the left-hand side only, as the analysis for the right-hand side is analogous (with the roles of i, j and the indices $\gamma_{j|i}, \alpha_i$ exchanged). By definition of the Slutsky matrix it holds

$$
\begin{aligned}
&E\Big[\mathcal{S} \;\Big|\; P = p^*, X = x^*, Z = z^*, e_i'Y = k_{\gamma_{i|j},i}(w^*, k_{\alpha_j,j}(w^*)),\, e_j'Y = k_{\alpha_j,j}(w^*)\Big] \\
&= E\Big[D_p\phi \;\Big|\; W = w^*, e_i'Y = k_{\gamma_{i|j},i}(w^*, k_{\alpha_j,j}(w^*)),\, e_j'Y = k_{\alpha_j,j}(w^*)\Big] \\
&\quad + E\Big[\partial_x\phi\,\phi' \;\Big|\; W = w^*, e_i'Y = k_{\gamma_{i|j},i}(w^*, k_{\alpha_j,j}(w^*)),\, e_j'Y = k_{\alpha_j,j}(w^*)\Big] \\
&= B_1 + B_2.
\end{aligned}
$$

Consider each term separately. Start with

$$
e_i'B_1\, e_j = e_i'E\Big[D_p\phi \;\Big|\; W = w^*, e_i'Y = k_{\gamma_{i|j},i}(w^*, k_{\alpha_j,j}(w^*)),\, e_j'Y = k_{\alpha_j,j}(w^*)\Big]\, e_j.
$$

Then by the same reasoning as Dette et al. (2016) one can write

$$
\begin{aligned}
e_i'B_1\, e_j &= E\Big[e_i'D_p\phi \;\Big|\; W = w^*, e_i'Y = k_{\gamma_{i|j},i}(w^*, k_{\alpha_j,j}(w^*)),\, e_j'Y = k_{\alpha_j,j}(w^*)\Big]\, e_j \\
&= E\Big[\nabla_p e_i'\phi \;\Big|\; W = w^*, e_i'Y = k_{\gamma_{i|j},i}(w^*, k_{\alpha_j,j}(w^*)),\, e_j'Y = k_{\alpha_j,j}(w^*)\Big]\, e_j \\
&= \nabla_{p,1} k_{\gamma_{i|j},i}(w^*, k_{\alpha_j,j}(w^*))\, e_j \\
&\quad + \left[\partial_2 k_{\gamma_{i|j},i}(w^*, k_{\alpha_j,j}(w^*)) + \frac{\partial_2 F_{Y_i|W=w^*,Y_j=k_{\alpha_j,j}(w^*)}(k_{\gamma_{i|j},i}(w^*, k_{\alpha_j,j}(w^*)))}{f_{Y_i|W=w^*,Y_j=k_{\alpha_j,j}(w^*)}(k_{\gamma_{i|j},i}(w^*, k_{\alpha_j,j}(w^*)))}\right] \nabla_p k_{\alpha_j,j}(w^*)\, e_j \\
&\quad + \frac{\nabla_{p,1} F_{Y_i|W=w^*,Y_j=k_{\alpha_j,j}(w^*)}(k_{\gamma_{i|j},i}(w^*, k_{\alpha_j,j}(w^*)))}{f_{Y_i|W=w^*,Y_j=k_{\alpha_j,j}(w^*)}(k_{\gamma_{i|j},i}(w^*, k_{\alpha_j,j}(w^*)))} \cdot e_j,
\end{aligned}
$$

where the last equality follows from Lemma 2.1.

For the second term it holds

$$
\begin{aligned}
e_i' B_2 \, e_j &= E\Big[e_i' \partial_x \phi \, \phi' \, e_j \;\Big|\; P = p^*, X = x^*, Z = z^*, e_i' Y = k_{\gamma_{i|j},i}(w^*, k_{\alpha_j,j}(w^*)),\; e_j' Y = k_{\alpha_j,j}(w^*) \Big] \\
&= E\Big[\partial_x e_i' \phi \cdot \phi' \, e_j \;\Big|\; W = w^*, e_i' Y = k_{\gamma_{i|j},i}(w^*, k_{\alpha_j,j}(w^*)),\; e_j' Y = k_{\alpha_j,j}(w^*) \Big] \\
&= E\Big[\partial_x e_i' \phi \;\Big|\; W = w^*, e_i' Y = k_{\gamma_{i|j},i}(w^*, k_{\alpha_j,j}(w^*)),\; e_j' Y = k_{\alpha_j,j}(w^*) \Big] \, k_{\alpha_j,j}(w^*) \\
&= \partial_{x,1} k_{\gamma_{i|j},i}(w^*, k_{\alpha_j,j}(w^*)) \, k_{\alpha_j,j}(w^*) \\
&\quad + \left[\partial_2 k_{\gamma_{i|j},i}(w^*, k_{\alpha_j,j}(w^*)) + \frac{\partial_2 F_{Y_i|W=w^*, Y_j = k_{\alpha_j,j}(w^*)}(k_{\gamma_{i|j},i}(w^*, k_{\alpha_j,j}(w^*)))}{f_{Y_i|W=w^*, Y_j = k_{\alpha_j,j}(w^*)}(k_{\gamma_{i|j},i}(w^*, k_{\alpha_j,j}(w^*)))} \right] \partial_x k_{\alpha_j,j}(w^*) \, k_{\alpha_j,j}(w^*) \\
&\quad + \frac{\partial_{x,1} F_{Y_i|W=w^*, Y_j = k_{\alpha_j,j}(w^*)}(k_{\gamma_{i|j},i}(w^*, k_{\alpha_j,j}(w^*)))}{f_{Y_i|W=w^*, Y_j = k_{\alpha_j,j}(w^*)}(k_{\gamma_{i|j},i}(w^*, k_{\alpha_j,j}(w^*)))} \, k_{\alpha_j,j}(w^*),
\end{aligned}
$$

where the third line follows from the conditioning ($e_j' Y = \phi' \, e_j = k_{\alpha_j,j}(w^*)$ is in the conditioning set) and the last equality follows each time from Lemma 2.1. The same result holds when switching the roles of i and j (using decomposition (ii) with indices $\gamma_{j|i}, \alpha_i$), and subtraction and rearranging gives the claim. $\square$