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Analytic elements for multiaquifer flow

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Abstract

The objective of this paper is to present an analytic element formulation for groundwater flow in multiaquifer systems. Analytic element equations are presented for wells, line-sinks, and circular infiltration areas. Each analytic element is a solution to the governing system of differential equations, and thus simulates the leakage between aquifers exactly; the head, discharge, and leakage may be computed analytically at any point in the aquifer. Superposition of these analytic elements allows for the simulation of regional multiaquifer flow. A hypothetical example is presented of a system with three aquifers, a river network, and three pumping wells. It is demonstrated that the leakage between aquifers may vary significantly over short distances and that each aquifer has its own water divide.

Keywords: Groundwater; Multiaquifer flow; Leakage; Analytic element method

1. Introduction

Simple analytic element models, consisting of wells, line-sinks and area-sinks, are applied on a regular basis in groundwater modeling practice. These simple models may be used for the delineation of capture zones in wellhead protection studies (Strack et al., 1994; Kraemer et al., 2000), as a first step in the development of more complex models (Haitjema, 1995; Hunt et al., 1998; Bakker et al., 1999), or for education (Strack, 1989). The wells are used to simulate flow to or from pumping wells. Line-sinks are used to simulate inflow or outflow along segments of streams or rivers. Area-sinks are used to simulate infiltration from the surface or leakage to and from surface water features. The mathematical derivation of these analytic elements may be found, for flow in a single aquifer, in Strack (1989) or Haitjema (1995); the derivation of more complicated elements may be found in the same references and in, e.g., Fitts (1997) and Strack and DeLange (1999).

Area-sinks have been used to simulate the leakage between aquifers in a multiaquifer system, where the aquifers are separated by leaky layers. These area-sinks of constant strength model the leakage approximately. Area-sinks have been developed with a spatially varying strength (Strack and Janković, 1999) to simulate the distribution of the leakage more accurately. The use of area-sinks for the modeling of leakage amounts to discretizing the leakage and diminishes a major advantage of the analytic element

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method, namely that the model domain is not discretized.

The objective of this paper is to present an analytic element formulation for flow in a multiaquifer system that simulates the leakage between aquifers exactly everywhere and does not require any discretization of the model domain. The formulation allows for an arbitrary number of aquifers. The top and bottom of the aquifer system will be treated as confining layers, although the top layer may be semi-confining in places, allowing for a given infiltration to enter the top aquifer. Aquifer properties are kept homogeneous in this paper. Unconfined flow in the top aquifer may be approximated by using an average, but constant, transmissivity. Equations will be presented for the discharge potential for a well, a line-sink, and a circular area-sink with a constant infiltration rate. The expressions for a well are well known and are due to Hemker (1984). The expressions for the line-sink and circular area-sink are new (Eqs. (25), (28) and (29) in what follows). Analytic expressions for a line-sink in a two or three-aquifer system were derived previously by Heitzman (1977) and Keil (1982); their expressions could only be evaluated up to a distance of twice the leakage factor. Nienhuis (1997) implemented a numerical integration scheme for multiaquifer line-sinks. Circular area-sinks were also implemented by Nienhuis (1997), but their equations were not reported. They are obtained here through the combination of known solutions. A comparison with MODFLOW for a two-aquifer system, and an example of flow in a three-aquifer system with wells, rivers and infiltration are presented at the end of this paper.

2. Problem description

Consider a confined multiaquifer system consisting of M aquifers and $M+1$ leaky layers. Aquifers and leaky layers are numbered from the top down, so that leaky layer m is on top of aquifer m (Fig. 1). All aquifers and leaky layers are homogeneous and isotropic, and flow is confined and at steady-state everywhere. The resistance to vertical flow is neglected within an aquifer (the Dupuit–Forchheimer approximation) and flow in leaky layers is approximated as vertical. Aquifer properties are written as column vectors; component m represents layer m.

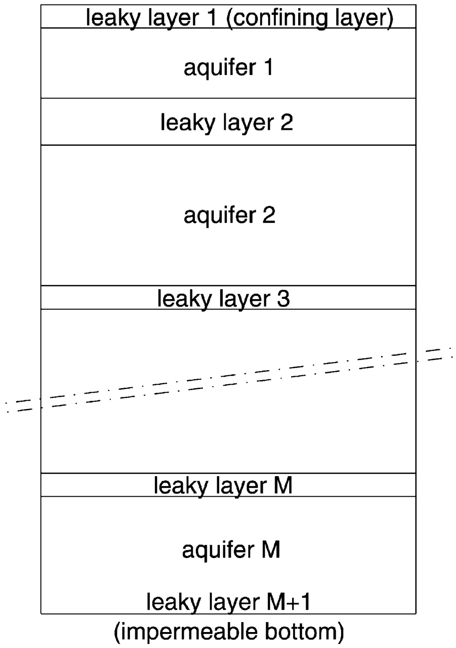

Fig. 1. Aquifer system definition.

The transmissivity of the aquifer system is $\vec{T}$ [L^2/T] and the resistance to vertical flow of the leaky layers is $\vec{c}$ [T], where $c_1 = c_{M+1} = \infty$. Flow in the aquifer system may be formulated in terms of a discharge potential $\vec{\Phi}$ [L^3/T] defined as (Strack, 1989; Bakker, 2002a)

$$\vec{\Phi} = \vec{T} \odot \vec{h} \tag{1}$$

where '$\odot$' stands for the term by term multiplication of two vectors (the Hadamard product, e.g. Magnus and Neudecker, 1999), and $\vec{h}$ [L] is a column vector of piezometric heads. Steady flow in an aquifer system with no areal infiltration is governed by the following system of differential equations (Hemker, 1984)

$$\nabla^2 \vec{\Phi} - \mathbf{A}\vec{\Phi} = 0 \tag{2}$$

where $\mathbf{A}$ is the system matrix, a tri-diagonal M by M matrix with diagonal terms

$$A_{m,m} = \frac{1}{c_m T_m} + \frac{1}{c_{m+1} T_m} \tag{3}$$

and off-diagonal terms

$$A_{m,m-1} = \frac{-1}{c_m T_{m-1}}, \qquad A_{m,m+1} = \frac{-1}{c_{m+1} T_{m+1}} \tag{4}$$

For a confined aquifer system, the system matrix has $M-1$ positive eigenvalues ω_m and corresponding real eigenvectors $\vec{v}_m$ plus one zero eigenvalue corresponding to the eigenvector $\vec{\tau} = \vec{T}/T$, where T is the comprehensive transmissivity of the aquifer system. It is common to introduce leakage factors λ_m [L] defined as

$$\lambda_m = 1/\sqrt{\omega_m} \tag{5}$$

3. Solution method

Hemker (1984) was the first to show in the English literature that for a confined aquifer system, the general solution to Eq. (2) may be written as

$$\vec{\Phi} = \Phi_L \vec{\tau} + \sum_{m=1}^{M-1} \Phi_m \vec{v}_m \tag{6}$$

where Φ_L fulfills Laplace's equation

$$\nabla^2 \Phi_L = 0 \tag{7}$$

and Φ_m fulfills the modified Helmholtz equation

$$\nabla^2 \Phi_m = \Phi_m/\lambda_m^2 \tag{8}$$

Note that Eq. (8) is the differential equation governing steady flow in a single semi-confined aquifer with a leakage factor $\lambda = \sqrt{kHc}$ (Strack, 1989). The function Φ_L will be referred to here as the harmonic potential and the functions Φ_m will be referred to as modified-Helmholtz potentials. The system of differential Equations, Eq. (2), may be solved alternatively using matrix differential calculus (Maas, 1986).

It is important to note that Φ_L is the comprehensive potential of the aquifer system, the sum of the potentials of all aquifers. This may be seen when it is realized that the sum of the components of each eigenvector $\vec{v}_m$ equals zero (Bakker, 2001). The head corresponding to $\Phi_L \vec{\tau}$ is the same in each aquifer and thus creates no leakage between aquifers. The leakage between aquifers is created by the terms $\Phi_m \vec{v}_m$; these terms redistribute flow among the aquifers but do not have a net discharge. The modified-Helmholtz potentials have unknown coefficients that are chosen such that the proper boundary condition is met; systems of linear equations will be presented to determine these coefficients.

The discharge vector $(\vec{Q}_x, \vec{Q}_y)$ [L^2/T] represents the vertically integrated flow in each aquifer. The x and y components of the discharge vector are each written as column vectors, with their components representing the flow in the different aquifers. The components of the discharge vector may be obtained from the potential by differentiation

$$\vec{Q}_x = -\frac{\partial \vec{\Phi}}{\partial x}, \qquad \vec{Q}_y = -\frac{\partial \vec{\Phi}}{\partial y} \tag{9}$$

4. Wells

The potential for a well of discharge Q [L^3/T] and screened at $(x, y) = (x_w, y_w)$ in aquifer numbered P is (Hemker, 1984)

$$\vec{\Phi} = \frac{Q}{2\pi} \ln(r)\vec{\tau} + \sum_{m=1}^{M-1} \frac{A_m}{2\pi} \mathrm{K}_0(r/\lambda_m)\vec{v}_m \tag{10}$$

where $r = \sqrt{(x-x_w)^2 + (y-y_w)^2}$ and K_0 is the modified Bessel function of the second kind and order zero. Expressions for the coefficients A_m may be obtained from a set of linear equations (Bakker, 2001)

$$\sum_{m=1}^{M-1} A_m v_{p,m} = Q\tau_p, \quad p = 1,\ldots,M; \quad p \neq P \tag{11}$$

where $v_{p,m}$ is component p of eigenvector m, and τ_p is component p of the normalized transmissivity vector $\vec{\tau}$. It is noted that Eq. (10) is exact for well screens that penetrate an aquifer fully, and approximate otherwise.

5. Line-sinks

The potential for a line-sink is obtained by integration of the expression for a well Eq. (10) along a line. Consider a line-sink with length L and a constant extraction rate σ [L^2 /T]. Use is made of a local, dimensionless X, Y coordinate system with the X axis along the line-sink and the origin at the center of the line-sink (Fig. 2); the length of the line-sink in the X, Y coordinate system is 2. The transformation from the x, y plane to the X, Y plane is given by the complex transformation (Strack, 1989)

$$Z = \frac{z - \frac{1}{2}(z_1 + z_2)}{\frac{1}{2}(z_2 - z_1)} \tag{12}$$

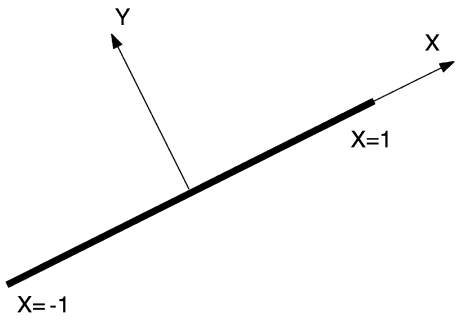

Fig. 2. The local X, Y coordinate system.

where $z = x + \mathrm{i}y$, $Z = X + \mathrm{i}Y$, and z_1 and z_2 are the complex coordinates of the beginning and end points of the line-sink. Since the transformation includes a scaling by $L/2$, the leakage factor λ in the x, y plane becomes $\Lambda = 2\lambda/L$ in the X, Y plane. The harmonic potential for a line-sink, obtained through integration of the logarithmic term in Eq. (10), may be found in Strack (1989) as

$$\Phi_L = \frac{\sigma L}{4\pi}\Re\{(Z+1)\ln(Z+1) - (Z-1)\ln(Z-1)\} \tag{13}$$

The modified-Helmholtz potential for a line-sink is derived for a generic leakage factor λ and a generic strength s (i.e. σ is replaced by s and the subscript m is dropped for convenience in Eqs. (14)–(24)) and is obtained through integration of the modified-Helmholtz potential for a well, the second part of Eq. (10), from $\Delta = -1$ to $\Delta = +1$

$$\Phi = -\frac{sL}{4\pi}\int_{-1}^{1} \mathrm{K}_0(r/\Lambda)\mathrm{d}\Delta \tag{14}$$

where $sL\mathrm{d}(\Delta)/2$ represents the discharge of an incremental section $\mathrm{d}(\Delta)$ of the line-sink and

$$r = \sqrt{(X-\Delta)^2 + Y^2} \tag{15}$$

The integration in Eq. (14) may be carried out analytically when K_0 is approximated by a polynomial of the following form

$$\mathrm{K}_0\left(\frac{r}{\Lambda}\right) = \sum_{n=0}^{N}\left[a_n \ln\left(\frac{r}{\Lambda}\right)^2 + b_n\right]\left(\frac{r}{\Lambda}\right)^{2n} \tag{16}$$

Abramowitz and Stegun (1965) give coefficients for a polynomial approximation of this form that converges for $r \leq 2\Lambda$. The approximation used here converges for $r \leq 8\Lambda$ and was obtained from Clenshaw (1962). Clenshaw presented the approximation in terms of Chebyshev polynomials, using $N = 17$ terms and providing values for the constants up to 10^{-20}. Here, only the first nine terms are used, using 16 digit precision. Coefficients of like powers in the Chebyshev approximation of Clenshaw are gathered to obtain an expression in the form Eq. (16); the coefficients are presented in Table 1. It is noted that the function $\mathrm{K}_0(r/\Lambda)$ equals zero for most practical purposes when $r > 6\Lambda$.

To facilitate integration, r^2/Λ^2 is written as

$$\left(\frac{r}{\Lambda}\right)^2 = (\delta - \zeta)(\delta - \bar{\zeta}) \tag{17}$$

where

$$\zeta = \frac{X + \mathrm{i}Y}{\Lambda}, \qquad \bar{\zeta} = \frac{X - \mathrm{i}Y}{\Lambda}, \qquad \delta = \frac{\Delta}{\Lambda} \tag{18}$$

so that

$$\mathrm{K}_0\left(\frac{r}{\Lambda}\right) = \sum_{n=0}^{N}\{a_n \ln[(\delta-\zeta)(\delta-\bar{\zeta})] + b_n\}(\delta-\zeta)^n \times (\delta-\bar{\zeta})^n \tag{19}$$

Furthermore, $(\delta - \bar{\zeta})^n$ may be written as

$$(\delta - \bar{\zeta})^n = \sum_{m=0}^{n} \gamma_{n,m}(\delta - \zeta)^m \tag{20}$$

where

$$\gamma_{n,m} = \binom{n}{m}(\zeta - \bar{\zeta})^{n-m} \tag{21}$$

Table 1
Coefficients of the K_0 approximation

n	a_n	b_n
0	$-0.500004211065677 \times 10^{0}$	$0.115956920789028 \times 10^{0}$
1	$-0.124989431448700 \times 10^{0}$	$0.278919134951974 \times 10^{0}$
2	$-0.781685695640975 \times 10^{-2}$	$0.252752008621167 \times 10^{-1}$
3	$-0.216324415010288 \times 10^{-3}$	$0.841879407543506 \times 10^{-3}$
4	$-0.344525393452639 \times 10^{-5}$	$0.152425102734818 \times 10^{-4}$
5	$-0.315133836828774 \times 10^{-7}$	$0.148292488399579 \times 10^{-6}$
6	$-0.296636186265427 \times 10^{-9}$	$0.157622547107156 \times 10^{-8}$
7	$-0.313689942474032 \times 10^{-12}$	$0.117975437124933 \times 10^{-11}$
8	$-0.112031912249579 \times 10^{-13}$	$0.655345107753534 \times 10^{-13}$

Substitution of Eq. (20) for $(\delta - \bar{\zeta})^n$ in Eq. (19) gives, after a lengthy process of gathering coefficients of like powers

$$K_0\left(\frac{r}{\Lambda}\right) = \sum_{n=0}^{2N} \{\alpha_n \ln[(\delta-\zeta)(\delta-\bar{\zeta})] + \beta_n\}(\delta-\zeta)^n \tag{22}$$

where

$$\alpha_n = \sum_{m=\max(0,n-N)}^{n/2} a_{n-m}\gamma_{n-m,m},$$
$$\beta_n = \sum_{m=\max(0,n-N)}^{n/2} b_{n-m}\gamma_{n-m,m} \tag{23}$$

Substitution of Eq. (22) for K_0 in Eq. (14), replacing $\ln[(\delta-\zeta)(\delta-\bar{\zeta})]$ by $2\Re\ln(\delta-\zeta)$, and replacing $L\mathrm{d}(\Delta)$ by $2\lambda\mathrm{d}(\delta-\zeta)$ gives

$$\Phi = -\frac{s\lambda}{2\pi}\Re\int_{\delta_1}^{\delta_2}\sum_{n=0}^{2N}[2\alpha_n\ln(\delta-\zeta)+\beta_n](\delta-\zeta)^n\mathrm{d}(\delta-\zeta) \tag{24}$$

where $\delta_1 = -1/\Lambda$ and $\delta_2 = 1/\Lambda$. This integral may be integrated by parts, which gives the modified-Helmholtz potential for a line-sink (the indices m are reintroduced here for completeness)

$$\Phi_m = -\frac{s_m\lambda_m}{2\pi} \times\Re\left\{\sum_{n=0}^{2N}\left[2\alpha_n\ln(\delta_2-\zeta) - \frac{2\alpha_n}{n+1} + \beta_n\right]\frac{(\delta_2-\zeta)^{n+1}}{n+1} - \sum_{n=0}^{2N}\left[2\alpha_n\ln(\delta_1-\zeta) - \frac{2\alpha_n}{n+1} + \beta_n\right]\frac{(\delta_1-\zeta)^{n+1}}{n+1}\right\} \tag{25}$$

The coefficients s_m in Eq. (25) are chosen such that the line-sink takes water only from the aquifer in which it is present. If the line-sink is present in aquifer numbered P, the coefficients may be obtained from the set of linear equations

$$\sum_{m=1}^{M-1} s_m\nu_{p,m} = -\sigma\tau_p, \quad p=1,\ldots,M; \quad p\neq P \tag{26}$$

It is recalled that the polynomial approximation Eq. (16) converges only within a circle of radius 8λ. Hence, the length of the line-sink must be significantly less than 8λ. The modified-Helmholtz potential due to a longer line-sink may be obtained by dividing the line-sink in smaller sections. The implementation used in this paper uses sections that are at most λ long.

Expressions for the discharge vector corresponding to the harmonic potential may be found in Strack (1989). Expressions for the discharge vector corresponding to the modified-Helmholtz potential may be obtained by differentiation of Eq. (25). Alternatively, the integration and differentiation may be reversed and the analysis Eqs. (19)–(25) repeated. Expressions for the discharge vector are derived, using the latter approach, in the Appendix A.

6. Circular infiltration area

Consider a circular infiltration area with radius R, centered at $(x, y) = (x_p, y_p)$ at the top of the aquifer system; the infiltration rate is constant and equal to N [L/T]. Infiltration into the aquifer system requires that the system of differential equations, Eq. (2), be modified as follows (Hunt, 1986)

$$\nabla^2\vec{\Phi} - \mathbf{A}\vec{\Phi} = -\vec{N} \tag{27}$$

where $\vec{N}$ is a column vector with its first component equal to the infiltration rate N and with all other components equal to zero. The potential for a circular infiltration area may be obtained through a combination of the solution for radial infiltration in a semi-confined aquifer system presented by Hunt (1986) and the solution for a pond in a single aquifer of Strack (1989), and is written in terms of the local radial coordinate $r = \sqrt{(x-x_p)^2 + (y-y_p)^2}$

$$\vec{\Phi} = -\frac{N}{4}(r^2 - R^2)\vec{\tau} + \sum_{m=1}^{M-1} B_m[\lambda_m/R - \mathrm{K}_1(R/\lambda_m) \times \mathrm{I}_0(r/\lambda_m)]\vec{\nu}_m, \qquad r \le R \tag{28}$$

$$\vec{\Phi} = -\frac{NR^2}{2}\ln(r/R)\vec{\tau} + \sum_{m=1}^{M-1} B_m \mathrm{I}_1(R/\lambda_m) \times \mathrm{K}_0(r/\lambda_m)\vec{\nu}_m, \qquad r > R \tag{29}$$

where $\mathrm{I}_0, \mathrm{I}_1, \mathrm{K}_0, \mathrm{K}_1$ are modified Bessel functions; the coefficients B_m are obtained from the $M-1$ linear equations

$$\sum_{m=1}^{M-1} \frac{B_m}{R\lambda_m}\nu_{p,m} = -N\tau_p, \qquad p = 2, \ldots, M \tag{30}$$

where $\nu_{p,m}$ is component p of eigenvector $\vec{\nu}_m$.

It may be verified that Eq. (28), with coefficients (30), fulfills Eq. (27) when it is used that a function F of the form

$$F = \lambda/R - D\mathrm{I}_0(r/\lambda) \tag{31}$$

fulfills the differential equation

$$\nabla^2 F - F/\lambda^2 = -1/(R\lambda) \tag{32}$$

for an arbitrary value of the constant D. Furthermore, continuity of potentials (28) and (29) and its radial derivative across the boundary of the infiltration area ($r = R$) may be verified by using the identity (Abramowitz and Stegun, 1965)

$$\mathrm{K}_1(a)\mathrm{I}_0(a) + \mathrm{K}_0(a)\mathrm{I}_1(a) = 1/a \tag{33}$$

which holds for any real a.

7. Application

Simple analytic element models of multiaquifer systems may be built through superposition of the potentials of a multiaquifer well Eq. (10), multiaquifer line-sink, Eqs. (13) and (25), and multiaquifer circular infiltration area Eqs. (28) and (29), plus a constant potential

$$\vec{\Phi} = C\vec{\tau} \tag{34}$$

where C is an arbitrary constant. The strengths of the line-sinks σ, representing the inflow or outflow along stream segments, are normally not known *a priori*. In practice, the strengths of the line-sinks may be computed by specifying the heads at the centers of the line-sinks, which represent the water levels in the streams (this is common practice in the analytic element method; Strack, 1989). The constant C may be obtained in a similar fashion, for example by specifying the head at one additional point in the aquifer system. The resulting system of linear equations in σ and C may be solved using a standard technique.

8. Comparison

Analytic solutions can be validated by verifying internal consistency and by checking boundary conditions. For example, accurate numerical evaluation of the Laplacian allows for the verification that the solution, as implemented, satisfies the governing system of differential equations at any point; boundary conditions can be checked more easily. These methods of validation are perhaps unfamiliar to some readers, who may be more familiar with standard numerical codes such as MODFLOW (McDonald and Harbaugh, 1988). A comparison between an analytic element solution and a MODFLOW solution is therefore presented here. Such a comparison requires the MODFLOW—grid to be sufficiently small as compared to the smallest leakage factor in the system (Bakker, 1999). The geometry of the problem is chosen such that it fits well with the numerical grid (i.e. wells, line-sinks and model boundaries are chosen along centers of cells); the model domain is surrounded by a closed boundary with specified heads.

Consider a confined two-aquifer system. The transmissivities of the top and bottom aquifers are 10 and 60 m^2/d, respectively, and the vertical resistance of the leaky layer is 1000 days (a leakance of 0.001 d^{-1}). The model domain is a square with sides of 400 m; the origin of an (x, y) coordinate system is at the lower-left hand corner. Along the boundary of the model the head is fixed at 40 m in both aquifers. In the top aquifer, a line-sink is present with endpoints (45,145) and (235,335); the total discharge of the line-sink is 195 m^3/d, equally divided over the line. In the bottom aquifer is a well at (250,150) with a discharge of 1000 m^3/d. In the analytic element model, the boundary of the domain is modeled with 16 head-specified line-sinks in each aquifer. The cell size in the MODFLOW model is chosen to be 10 m, such that the total number of cells is 1681 in each aquifer; the cell size is significantly less than the leakage factor ($\lambda = 93$ m).

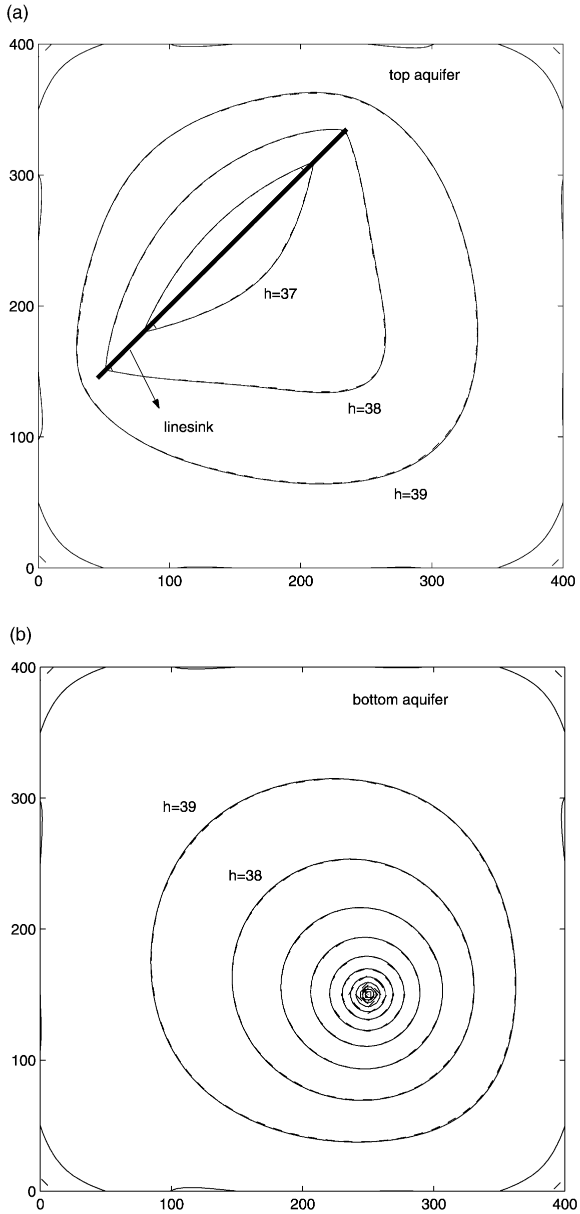

Fig. 3. Head contours of analytic element model (solid lines) and MODFLOW model (dashed lines). Contour interval is 1 m.

Contour plots of the heads in the two aquifers are very similar (Fig. 3); the solid lines are for the analytic element model, the dashed lines for the MODFLOW model. Differences occur near the well, the line-sink, and the boundary. Differences near the well and line-sink are due to the discretization in MODFLOW. Differences along the boundary of the domain are caused by the relatively large line-sinks that are used to model the boundary in the analytic element model. The solutions may be improved by using more line-sinks along the boundary in the analytic element model, and smaller cell sizes in the MODFLOW model.

9. Example

A hypothetical case of flow in a system of three aquifers, labeled 1, 2, and 3 from the top down, is presented as an example to illustrate that complex leakage patterns between aquifers may be simulated accurately with relative ease. Consider a system of three aquifers; the aquifer parameters are presented in Table 2. A river with three branches cuts through the upper aquifer. The river is modeled with 7 line-sinks of constant strength and each branch is modeled with 5 line-sinks. The heads are specified at the centers of the line-sinks and are given in Fig. 4. Two wells with a discharge each of 3000 m^3/day are screened in aquifer

Table 2
Aquifer data of example

	Bottom elevation (m)	Top elevation (m)	T (m^2/d)	c (d)
Aquifer 1	140	165	50	
Leaky layer 2	120	140		2000
Aquifer 2	80	120	240	
Leaky layer 3	60	80		20000
Aquifer 3	0	60	240	

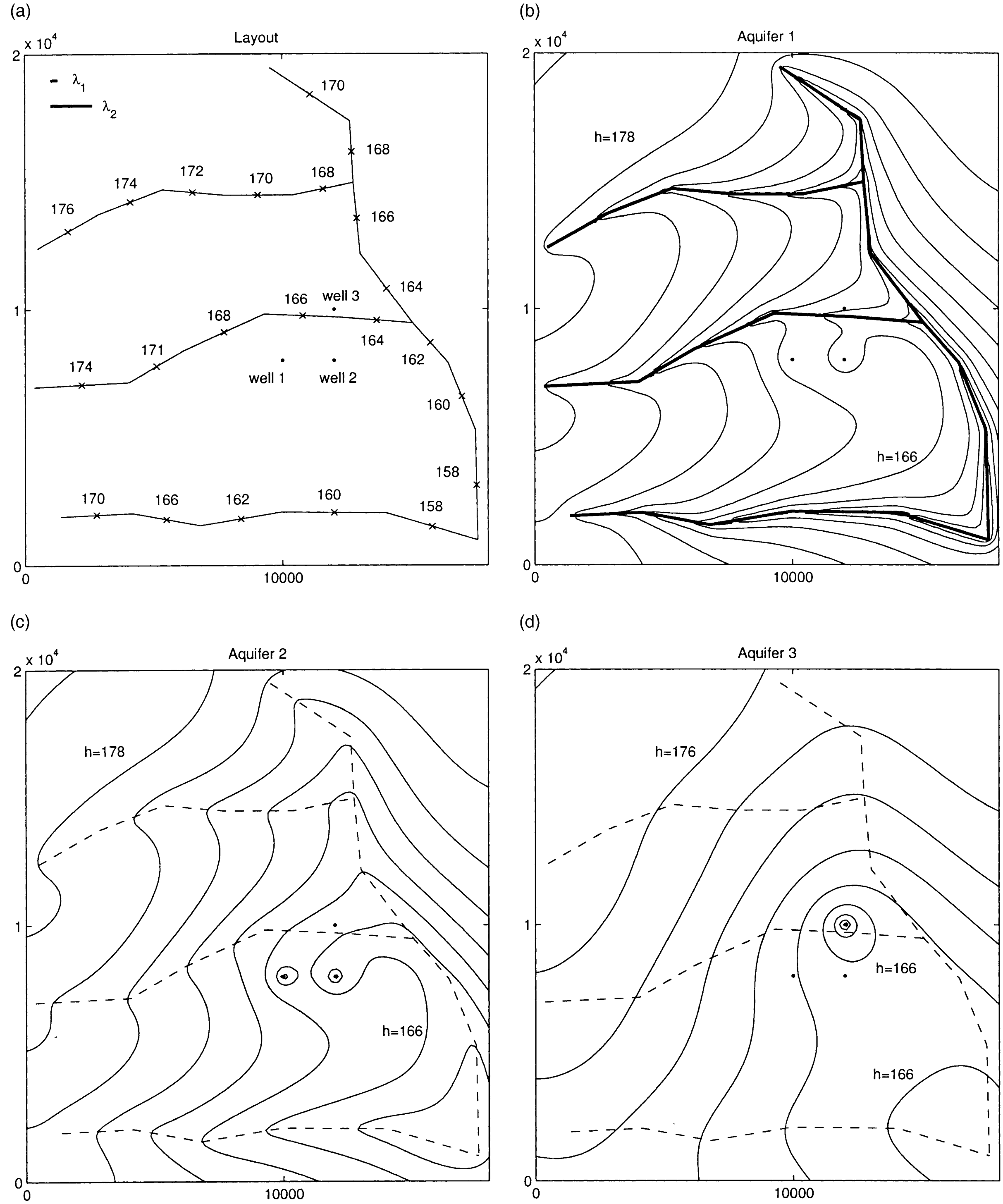

Fig. 4. Layout of analytic elements (numbers indicate heads at centers of line-sinks) and contour lines of piezometric heads in all three aquifers.

2 and one well with a discharge of 4000 m^3/d is screened in aquifer 3 (labeled well 3 in the Fig. 4). A constant recharge through the upper boundary of aquifer 1 is simulated by one large circular infiltration area that covers the entire model area; the recharge rate is $N = 0.2$ mm/d. A head of 175 m is specified in layer 1 at the upper righthand corner of the model domain. A layout of all analytic elements, except the boundary of the infiltration area, is shown in Fig. 4. The leakage factors λ_1 and λ_2 Eq. (5), which have the dimension of length, are depicted graphically in the upper lefthand corner.

Contour plots of the heads in the three aquifers are also shown in Fig. 4; the contour interval is 2 m. The leakage γ between aquifers m and $m-1$ may be computed as

$$\gamma = (h_m - h_{m-1})/c_m \qquad (35)$$

Contour plots of the leakage are shown in Fig. 5. The contour interval is 0.5 mm/d for the leakage between aquifers 2 and 1 and 0.05 mm/d for the leakage between aquifers 3 and 2. The shaded areas indicate upward leakage.

It may be seen from Fig. 5 that, in general, leakage is upward under the rivers and downward between them. This distribution of leakage is altered near pumping centers where an increase in leakage is seen into the screened aquifer. This general distribution of leakage is of course well known. The advantage of the approach presented in this paper is the accuracy and relative ease with which the leakage (and also the head and flow) can be computed. It is also emphasized that models are built with elements that directly represent hydrogeologic boundaries. The potential, thus obtained, is valid for an infinite domain, but the model is realistic only in the area where enough analytic elements are specified to control the flow. Note that it would be difficult to model this flow field if the computer model requires finite boundaries. Curves that are no-flow boundaries in the upper aquifer are not necessarily no-flow boundaries in the deeper aquifers. Furthermore, along the rivers the heads vary differently in all three aquifers. As

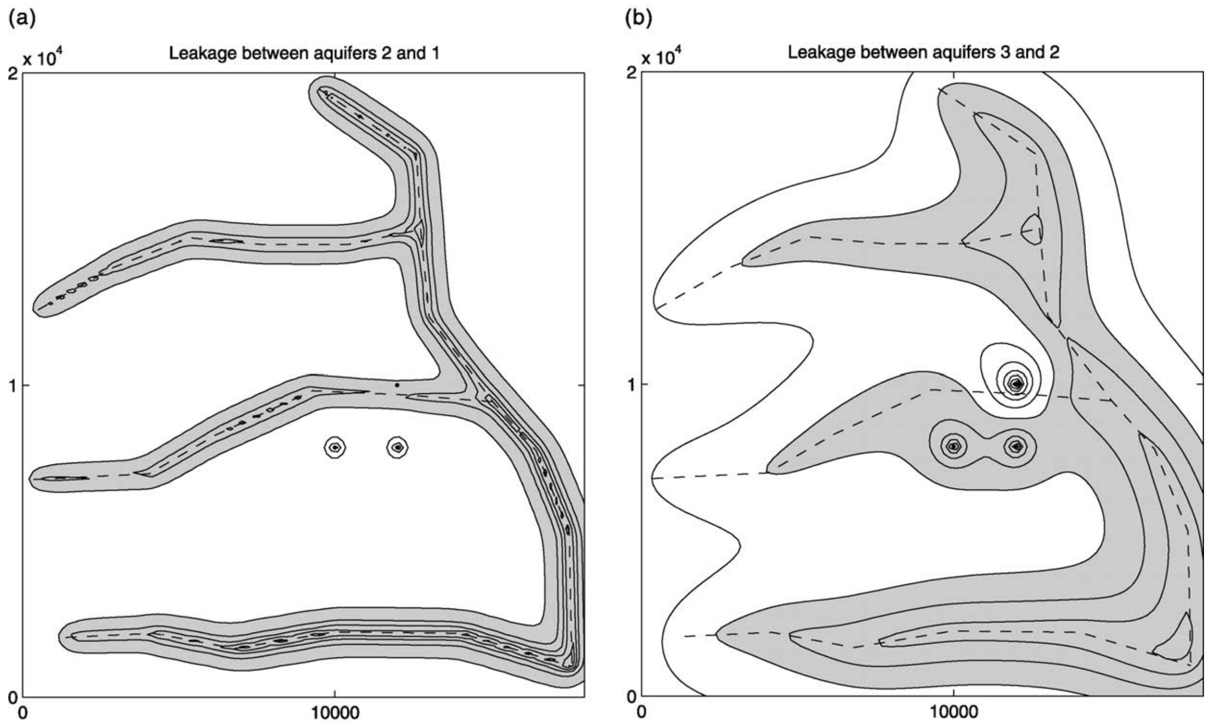

Fig. 5. Contour lines of leakage; shaded areas indicate upward leakage.

such, it would be difficult to delineate boundaries in the deeper aquifers along which boundary conditions are known a priori.

10. Conclusion

The presented analytic element formulation for multiaquifer flow has the following two main advantages

1. Head, flow, and leakage may be computed analytically at any point in the aquifer system, and the model domain does not have to be discretized.
2. Boundary conditions are specified only along hydrogeologic boundaries; each analytic element represents a hydrogeologic feature in the aquifer.

These advantages are the same as for the analytic element method for single-aquifer flow. The usefulness of the presented formulation will be enhanced further with the implementation of bounded areas with different aquifer properties called inhomogeneities. A formulation for cylindrical inhomogeneities for multiaquifer systems has already been developed by Bakker et al., 2002b; formulations for inhomogeneities bounded by ellipses or polygons are under development.

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Appendix A

The discharge vector corresponding to the modified-Helmholtz potential for a line-sink Eq. (25) is written for convenience as the complex discharge $W = Q_x - \mathrm{i}Q_y$ and is obtained by application of the chain rule

$$W = (Q_X - \mathrm{i}Q_Y)\frac{\partial Z}{\partial z} \tag{A1}$$

where Z is given by Eq. (12). Q_X is obtained by differentiation of Eq. (14)

$$Q_X = -\frac{\partial \Phi}{\partial X} = \frac{sL}{4\pi}\frac{\partial}{\partial X}\int_{-1}^{1} \mathrm{K}_0(r/\Lambda)\mathrm{d}\Delta \tag{A2}$$

Integration and differentiation may be reversed and the integration variable changed to $X - \Delta$ to obtain

$$Q_X = -\frac{sL}{4\pi}\int_{X-1}^{X+1} \frac{\partial \mathrm{K}_0(r/\Lambda)}{\partial(X-\Delta)}\mathrm{d}(X-\Delta) \tag{A3}$$

so that

$$Q_X = -\frac{sL}{4\pi}[\mathrm{K}_0(r_2/\Lambda) - \mathrm{K}_0(r_1/\Lambda)] \tag{A4}$$

where r_1 and r_2 stand for evaluation of r Eq. (15) at $\Delta = -1$ and $\Delta = 1$, respectively.

Derivation of an expression for Q_Y requires the solution of the following integral

$$Q_Y = \frac{sL}{4\pi}\int \frac{\partial \mathrm{K}_0}{\partial Y}\mathrm{d}\Delta \tag{A5}$$

The derivative of K_0 in the Y direction is, using (16) and after gathering terms

$$\frac{\partial \mathrm{K}_0}{\partial Y} = \sum_{n=0}^{N}\frac{2Ya_n}{\Lambda^2}\left(\frac{r}{\Lambda}\right)^{2n-2} + \sum_{n=1}^{N}\left[a_n \ln\left(\frac{r}{\Lambda}\right)^2 + b_n\right] \times \frac{2nY}{\Lambda^2}\left(\frac{r}{\Lambda}\right)^{2n-2} \tag{A6}$$

Introduction of the new constants

$$\tilde{a}_{n-1} = \frac{2nY}{\Lambda^2}a_n, \qquad \tilde{b}_{n-1} = \frac{2Y(a_n + nb_n)}{\Lambda^2}, \qquad n = 1,\ldots,N \tag{A7}$$

gives

$$\frac{\partial \mathrm{K}_0}{\partial Y} = \frac{2Ya_0}{r^2} + \sum_{n=0}^{N-1}\left[\tilde{a}_n \ln\left(\frac{r}{\Lambda}\right)^2 + \tilde{b}_n\right]\left(\frac{r}{\Lambda}\right)^{2n} \tag{A8}$$

Integration of the first term may be dealt with as follows, using Eqs. (17) and (18)

$$\frac{sL}{4\pi}\int \frac{2Ya_0}{r^2}\mathrm{d}\Delta = \frac{sLY}{2\pi\Lambda}\int \frac{a_0}{(\delta-\zeta)(\delta-\bar{\zeta})}\mathrm{d}\delta \tag{A9}$$

Using partial fractions this gives

$$\frac{sL}{4\pi}\int\frac{2Ya_0}{r^2}\mathrm{d}\Delta = \frac{sLYa_0}{2\pi\Lambda(\zeta-\bar{\zeta})}\ln\frac{\delta-\zeta}{\delta-\bar{\zeta}} \tag{A10}$$

evaluated between δ_1 and δ_2. The integral of the summation in Eq. (A8) may be carried out as for the potential (resulting in Eqs. (24) and (25)) and is not repeated here. The final expression for Q_Y is

$$Q_Y = \frac{sLa_0}{4\pi i}\left[\ln\frac{\delta_2-\zeta}{\delta_2-\bar{\zeta}} - \ln\frac{\delta_1-\zeta}{\delta_1-\bar{\zeta}}\right] + \frac{sL}{4\pi}\int_{-1}^{1}\sum_{n=0}^{N-1}\left[\tilde{a}_n\ln\left(\frac{r}{\Lambda}\right)^2+\tilde{b}_n\right]\left(\frac{r}{\Lambda}\right)^{2n}\mathrm{d}\Delta \tag{A11}$$

where it is used that $\zeta-\bar{\zeta}=2iY/\Lambda$.

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