Filter

Bayesian Filter

A Bayesian filter estimates the uncertain state of a system. It operates sequentially by predicting the next state from the past state distribution and control input, and then correcting that prediction using new observation data.

Prediction Step
In this step, the “past state” and “control input” are used to predict the “next state” probabilistically:

$$\bar{p}(x_t) = \int p(x_t \mid u, x_{t-1}) p(x_{t-1})\, dx_{t-1} \tag{1}$$

- **Motion model** $p(x_t \mid u, x_{t-1})$: The probability of reaching state x_t from x_{t-1} with control input u .
- **Past state distribution** $p(x_{t-1})$: The probability distribution of the previous state, derived from the last update.

This step uses the Markov property, assuming x_t depends only on x_{t-1} and u.

Correction Step
This step updates the “predicted state distribution” using new observation data to improve the current state estimate:

$$p(x_t \mid z_t) = \eta\, p(z_t \mid x_t)\, \bar{p}(x_t) \tag{2}$$

- **Predicted distribution** $\bar{p}(x_t)$: The prior distribution from the prediction step.
- **Observation model** $p(z_t \mid x_t)$: The probability of observations z_t given state x_t, incorporating sensor data.
- **Normalization constant** η: Ensures the total probability is 1.

Using Bayes' theorem, this step refines the prior with new observation z_t, yielding the posterior $p(x_t \mid z_t)$. This updated state passes forward as the prior in the next cycle, continuously refining the system's state estimate in real time.

Features of Each Filtering Technique

In a Bayesian filter, distributions $p(x)$, $p(x_t \mid x_{t-1}, u)$, and $p(z_t \mid x_t)$ are all probabilities. You must model them accurately for proper computation.

Differences in how these distributions are modeled lead to two main filtering approaches:

- **Extended Kalman Filter (EKF)**:
  - Assumes state uncertainty follows a Gaussian (normal) distribution.
  - Cannot handle cases where the Gaussian assumption significantly fails.
- **Particle Filter (PF)**:
  - Does not assume a Gaussian form; instead, uses many sampled points (“particles”) to represent the uncertainty.
  - Flexible and can handle nonlinear or complex distributions but is computationally more expensive since processing increases with the number of samples.

Extended Kalman Filter (EKF)

In many systems, using a Gaussian to represent uncertainty works well.

Prediction Step

Assume at time $t-1$ the state’s probability distribution is Gaussian:

$$p(x_{t-1}) = \mathcal{N}(\bar{x}_{t-1}, P_{t-1}) \tag{3}$$

Here $\bar{x}_{t-1}$ and P_{t-1} are the mean and covariance matrix. The overbar on x ($\bar{x}$) represents the mean or predicted value.

Covariance P_{t-1} indicates how uncertain you are about the state. Formally:

$$P_{t-1} = \mathbb{E}\big((x_{t-1} - \bar{x}_{t-1})(x_{t-1} - \bar{x}_{t-1})^T\big) = \mathbb{E}(\Delta x_{t-1}, \Delta x_{t-1}^T) \tag{4}$$

where $\Delta x_{t-1} = x_{t-1} - \bar{x}_{t-1}$.

The robot’s motion is described by the motion equation f, which computes the next state x_t from x_{t-1} and a control u:

$$x_t = f(x_{t-1}, u) \tag{5}$$

Control u also has uncertainty, often modeled by a Gaussian:

$$p(u) = \mathcal{N}(\bar{u}, Q) \tag{6}$$

Here Q is the covariance of the control input u:

$$Q = \mathbb{E}((u - \bar{u})(u - \bar{u})^T) = \mathbb{E}(\Delta u, \Delta u^T) \tag{7}$$

Ignoring error, the predicted mean is:

$$\bar{x}_t = f(\bar{x}_{t-1}, \bar{u}) \tag{8}$$

We want to quantify this spread. From equation (5), x_{t-1} and u are uncertain, so the resulting distribution of next state ($\bar{x}_t$) might not strictly be Gaussian. But EKF assumes it remains Gaussian, with covariance $\bar{P}_t$. The steps to compute $\bar{P}_t$ follow.

Using (4):

$$\bar{P}_t = \mathbb{E}\big((x_t - \bar{x}_t)(x_t - \bar{x}_t)^T\big) \tag{9}$$

Substituting (5) into (9):

$$\bar{P}_t = \mathbb{E}\big(\big(f(x_{t-1}, u) - f(\bar{x}_{t-1}, \bar{u})\big), \big(f(x_{t-1}, u) - f(\bar{x}_{t-1}, \bar{u})\big)^T\big) \tag{10}$$

Letting $\Delta x = x_{t-1} - \bar{x}_{t-1}$ and $\Delta u = u - \bar{u}$, and assuming these are small, we linearize f with a first-order Taylor expansion:

$$f(x_{t-1}, u) \approx f(\bar{x}_{t-1}, \bar{u}) + F_x \Delta x + F_u \Delta u \tag{11}$$

where

$$F_x = \left.\frac{\partial f}{\partial x}\right|_{x=\bar{x}_{t-1}, u=\bar{u}}, \quad F_u = \left.\frac{\partial f}{\partial u}\right|_{x=\bar{x}_{t-1}, u=\bar{u}} \tag{12}$$

Substituting this into equation (10) yields:

$$\bar{P}_t = F_x \mathrm{E}(\Delta \mathbf{x} \Delta \mathbf{x}^T) F_x^T + 2 F_x \mathrm{E}(\Delta \mathbf{x} \Delta \mathbf{u}^T) F_u^T + F_u \mathrm{E}(\Delta \mathbf{u} \Delta \mathbf{u}^T) F_u^T$$

Since x and u are independent, $\mathrm{E}(\Delta \mathbf{x} \Delta \mathbf{u}^T) = 0$. Furthermore, substituting equations (4) and (7) yields:

$$\bar{P}_t = F_x P_{t-1} F_x^T + F_u Q F_u^T \tag{14}$$

Correction Step

In the correction step, the predicted state $\bar{x}_t$ is adjusted using the observation data to estimate a more accurate state $\hat{x}_t$. Here, the hat symbol ($\hat{}$) denotes the corrected value.

To obtain the corrected state $\hat{x}_t$, the update Δx_t is first defined based on the predicted state $\bar{x}_t$ from the prediction step:

$$\hat{x}_t = \bar{x}_t + \Delta x_t \tag{15}$$

The correction step calculation requires two main components:

1. The formula for the update Δx_t to compute $\hat{x}_t$.
2. The formula for the corrected covariance matrix $\hat{P}_t$ (which represents the uncertainty in $\hat{x}_t$).

In this section, we will derive these two equations from theory.

Firstly, let's define the sensor observation model. The sensor observation is based on the true state x_t , and the observation equation h gives the observation z. This observation includes an error v, represented by a covariance R.

$$p(z \mid x_t) = \mathcal{N}(h(x_t), R) \tag{16}$$

$$z = h(x_t) + v \tag{17}$$

From the predicted state $\bar{x}_t$, the corresponding observation can also be predicted. The difference between the actual observation z and the predicted observation $h(\bar{x}_t)$ is defined as the observation residual y.

$$y \equiv z - h(\bar{x}_t) \tag{18}$$

The covariance matrix of the observation residual y is defined as S and can be derived as follows:

$$\begin{aligned} S &= E(yy^T) \\ &= E((H\Delta x_t + v)(H\Delta x_t + v)^T) \\ &= E(H\Delta x_t \Delta x_t^T H^T) + E(vv^T) \\ &= H\bar{P}_t H^T + R \end{aligned} \tag{19}$$

The above calculation uses the linearization of h (i.e., $h(x_t) \approx h(\bar{x}_t) + H\Delta x_t$), where H is the Jacobian matrix of the observation model.

The ideal Δx_t is obtained by minimizing the discrepancy between the observation residual y and the prediction error Δx_t. Therefore, an objective function can be formulated to minimize the total error with respect to Δx_t, as shown in Equation (20):

$$
\begin{aligned}
\underset{\text{argmin}} \quad F(\bar{x} + \Delta x_t) &= \|y\|_S^2 + \|\Delta x_t\|_P^2 \\
&= \|z - h(\bar{x}_t)\|_S^2 + \|x - \bar{x}_t\|_P^2 \\
&= (z - h(\bar{x}_t))S^{-1}(z - h(\bar{x}_t))^T + (x - \bar{x}_t)P^{-1}(x - \bar{x}_t)^T
\end{aligned} \tag{20}
$$

By differentiating the objective function with respect to Δx_t and setting it to zero, the optimal Δx_t can be found:

$$
\begin{aligned}
0 &= 2H^T S^{-1}(z - h(\bar{x}_t)) + 2P^{-1}\Delta x_t \\
\Delta x_t &= PH^T S^{-1}(z - h(\bar{x}_t)) \\
\Delta x_t &= PH^T S^{-1} y
\end{aligned} \tag{21}
$$

The term $PH^T S^{-1}$ is defined as the **Kalman Gain** K:

$$
K \equiv PH^T S^{-1} \tag{22}
$$

By substituting Equations (21) and (22) into Equation (15), the corrected state $\hat{x}_t$ can be calculated as follows:

$$
\hat{x}_t = \bar{x}_t + Ky \tag{23}
$$

Next, we evaluate the corrected covariance matrix $\hat{P}_t$. The corrected covariance matrix is computed based on the mean square error between the true state x_t and the corrected state $\hat{x}_t$:

$$
\hat{P}_t = E((x_t - \hat{x}_t)(x_t - \hat{x}_t)^T) \tag{24}
$$

Expanding this equation:

$$
\begin{aligned}
\hat{P}_t &= E((x_t - \bar{x}_t - Ky)(x_t - \bar{x}_t - Ky)^T) \\
&= E\left[(x_t - \bar{x}_t)(x_t - \bar{x}_t)^T - (x_t - \bar{x}_t)(Ky)^T - (Ky)(x_t - \bar{x}_t)^T + (Ky)(Ky)^T\right]
\end{aligned} \tag{25}
$$

- First term: $E[(x_t - \bar{x}_t)(x_t - \bar{x}_t)^T] = \bar{P}_t$
- Second and third terms:

$$
\begin{aligned}
& E[(x_t - \bar{x}_t)(h(x_t) + v - h(\bar{x}_t))^T K^T] \\
=& E[(x_t - \bar{x}_t)(h(\bar{x}_t) + H(x_t - \bar{x}_t) + v - h(\bar{x}_t))^T K^T] \\
=& E[(x_t - \bar{x}_t)(H(x_t - \bar{x}_t) + v)^T K^T] \\
=& E[(x_t - \bar{x}_t)(x_t - \bar{x}_t)^T H^T K^T] \\
=& \bar{P}_t H^T K^T \\
=& KH\bar{P}_t
\end{aligned} \tag{26}
$$

- Fourth term:

$$E((Ky)(Ky)^T) = KE(yy^T)K^T = KSK^T \tag{27}$$

Thus, the corrected covariance matrix is calculated as:

$$\begin{aligned} \hat{P}_t &= \bar{P}_t - 2\bar{P}_t H^T K^T + KSK^T \\ &= \bar{P}_t - 2\bar{P}_t H^T K^T + PH^T S^{-1} SK^T \\ &= \bar{P}_t - \bar{P}_t H^T K^T \\ &= (I - KH)\bar{P}_t \end{aligned} \tag{28}$$

Summary of EKF

The calculation of the Extended Kalman Filter (EKF) is now fully derived. In summary, the EKF calculation involves using the following six equations.

Prediction Step

- Equation (8): Predicting the state: $\bar{x}_t = f(\bar{x}_{t-1}, \bar{u})$
- Equation (14): Predicting the covariance: $\bar{P}_t = F_x P_{t-1} F_x^T + F_u Q F_u^T$

Correction Step

- Equation (18): Observation residual: $y = z - h(\bar{x}_t)$
- Equation (22): Kalman Gain: $K = PH^T(H\bar{P}_t H^T + R)^{-1}$
- Equation (23): Correcting the state: $\hat{x}_t = \bar{x}_t + Ky$
- Equation (28): Correcting the covariance: $\hat{P}_t = (I - KH)\bar{P}_t$