Solve Nonlinear Least-Squares Problem with the Gauss-Newton Methods.

What is linear/nonlinear?

- Linear: A polynomial of degree 1.
- Nonlinear: A function cannot be expressed as a polynomial of degree 1.

What is Least-Squares Problem?

Given a residual function r(x), where x is the parameter vector, the least-squares problem aims to find the optimal parameters that minimize the sum of squared residuals.

The objective function F can be defined as:

$$F = \sum_{i=0}^{n} (r^T \Sigma^{-1} r) \tag{1}$$

Here, Σ is the covariance matrix for the measurement. and the inverse of Σ is often referred to as the information matrix. In simpler cases, it can be the identity matrix.

What is Gauss-Newton Methods?

The Gauss-Newton methods are used to solve nonlinear least-squares problems. Unlike Newton's method, Gauss-Newton methods do not require the calculation of second derivatives of the residual function, which can be difficult in some cases.

Taylor expansion around the initial guess x.

$$\begin{aligned} F &= \sum_{i=0}^{n} (r(x+\Delta x)^T \Sigma^{-1} r(x+\Delta x)) \\ &= \underbrace{\sum (r^T \Sigma^{-1} r)}_{\text{c}} + 2\underbrace{\sum (r^T \Sigma^{-1} J)}_{\text{g}} \Delta x + \Delta x^T \underbrace{\sum (J^T \Sigma^{-1} J)}_{\text{H}} \Delta x \end{aligned}$$

Here, c represents a constant, g is the gradient vector, and H is the Hessian matrix of F.

$$g = \sum g_i = \sum J_i^T \Sigma^{-1} r_i \tag{2}$$

$$H = \sum H_i \approx \sum J_i^T \Sigma^{-1} J_i \tag{3}$$

- r is the residual vector.
- J is the jacobian matrix of r.

To find the minimum value of F, we differentiate the right side of the equation and set it equal to 0.

$$\begin{aligned} \dot{F} &= \partial(c + 2g\Delta x + \Delta x^T H \Delta x)/\partial \Delta x \\ &= 2g + 2H\Delta x = 0 \end{aligned}$$

Therefore, when dx is equal to (4), the value of F is a minimum.

$$\Delta x = -H^{-1} g \tag{4}$$

Since F may be nonlinear function, we can approximate the optimal x using iterative methods.

The problem of 3D points matching

If we define the increment of SO3/SE3 as:

$$T(x_0 \boxplus \delta) \triangleq T(x_0) \exp(\delta) \tag{5}$$

The $\delta \in \mathfrak{so}(3)$ or $\delta \in \mathfrak{se}(3)$

We use a first-order Taylor expansion to approximate the original equation:

$$T(x_0 \boxplus \delta) = T_0 \exp(\delta) \cong T_0 + T_0 \hat{\delta} \tag{6}$$

The the residual function of 3D points matching problem can be defined as:

$$r(x) = T(x)a - b \tag{7}$$

a is the target point:
b is the reference point.

We can use gauss-newton method to solve this problem.
According to gauss-newton method, we need to find the Jacobian matrix of r

$$
\begin{aligned}
\dot{r} &= \frac{T_0 \exp(\delta)\, a - T_0 a}{\delta} \\
&\cong \frac{T_0 a + T_0 \hat{\delta} a - T_0 a}{\delta} \\
&= \frac{T_0 \hat{\delta} a}{\delta} \\
&= -\frac{T_0 \delta \widehat{a}}{\delta} \\
&= -T_0 \widehat{a}
\end{aligned} \tag{8}
$$

When $\delta \in \mathfrak{so}(3)$

T_0 is a 3d rotation matrix(R_0),
and $\widehat{a}$ is defined as a skew symmetric matrix for vector a

$$
\dot{r} = -R_0 [a]_\times \tag{9}
$$

When $\delta \in \mathfrak{se}(3)$

$$
\delta = [\, v, \omega] \tag{10}
$$

ω: the parameters of rotation.

v: the parameters of translation.

$$
\hat{\delta} = \begin{bmatrix} [\omega]_\times & v \\ 0 & 0 \end{bmatrix} \tag{11}
$$

$$
\begin{aligned}
\dot{r} &= \frac{R_0 \hat{\delta} a}{\delta} \\
&= \frac{T_0 \begin{bmatrix} [\omega]_\times & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ 1 \end{bmatrix}}{[v, \omega]} \\
&= \frac{T_0 \begin{bmatrix} I & [-a]_\times \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix}}{[v, \omega]} \\
&= T_0 \begin{bmatrix} I & [-a]_\times \\ 0 & 0 \end{bmatrix}
\end{aligned} \tag{12}
$$