Graph Optimization

What is graph?

A graph is a pair $G = (V, E)$,
where V is a set of vertices, each of which contains some parameters to be optimized. E is a set of connected information, whose elements are denotes the constraint relationship between two vertices. Many robotics and computer vision problems can be represented by a graph problem.

How to solve graph problem?

A graph problem can be defined as a nonlinear least squares problems. Here, r_k and Σ_k represent the residual vector and the covariance matrix of edge k, respectively.

$$\arg\min_x F(x) = \sum_{e_k \in E} r_k^T \Sigma_k^{-1} r_k \tag{1}$$

We need to find an optimal set of vertices (i.e. V) to minimize the overall cost. As mentioned in guass_newton_method.md, once we can compute the Hessian matrix H and gradient g, we can solve this problem.

The hessian matrix H

Assuming the number of vertices in the graph is n and the number of edges is m, the block sizes of J, r, H, and g are m x n, m x 1, n x n, and n x 1, respectively. We notice that the size of H and g is independent of m.

The hessian matrix can be calculated as:

$$H = J^T \Sigma^{-1} J = \begin{bmatrix} \ddots & \vdots & \vdots \\ \vdots & \sum_{e_k \in E} {J_i^k}^T \Sigma_k^{-1} J_j^k & \vdots \\ \vdots & \vdots & \ddots \end{bmatrix} \tag{2}$$

The gradient g

The gradient vector can be calculated as:

$$g = J^T \Sigma^{-1} r = \begin{bmatrix} \vdots \\ \sum_{e_k \in E} {J_i^k}^T \Sigma_k^{-1} r_k \\ \vdots \end{bmatrix} \tag{3}$$

Here, i and j are the index of vertex, and they also indicate the row and column numbers within the Hessian matrix. k is the index of edge. J_i^k represents the partial derivative matrix of r_k with respect to x_i.

Derivative of edge between two lie groups

Suppose φ is an smooth mapping between two lie groups,
we can define the derivative of φ as J:

$$\exp(\widehat{J\delta}) = \varphi(x)^{-1} \varphi(x \oplus \delta) \tag{4}$$

x is a the parameter of φ, and δ is a small increment to x.

The the transfrom error of two lie groups can define as:

$$\varphi(A, B) = Z^{-1} A^{-1} B \tag{5}$$

Where A and B are the two lie groups, which represent the poses of two vertices. The Z represents the relative pose of A nad B, which usually measured by odometry or loop-closing.

If A and B are SO3

$$\begin{aligned} \exp(\widehat{J_A \delta}) &= (Z^{-1} A^{-1} B)^{-1} (Z^{-1} (A \exp(\hat{\delta}))^{-1} B) \\ &= B^{-1} A Z Z^{-1} \exp(-\hat{\delta}) A^{-1} B \\ &= B^{-1} A \exp(-\hat{\delta}) A^{-1} B \\ &= -\exp(B^{-1} A \hat{\delta} A^{-1} B) \\ &= -\exp(\widehat{B^{-1} A \delta}) \end{aligned} \tag{6}$$

Hence:

$$J_A = -B^{-1}A \tag{7}$$

$$\begin{aligned} \exp(\widehat{J_B\delta}) &= (Z^{-1}A^{-1}B)^{-1}(Z^{-1}A^{-1}B\exp(\hat{\delta})) \\ &= B^{-1}AZZ^{-1}A^{-1}B\exp(\hat{\delta}) \\ &= \exp(\hat{\delta}) \end{aligned} \tag{8}$$

Hence:

$$J_B = I \tag{9}$$

If A and B are SE2

The small incremental matrix of SE2 can be shown as follow:

$$\hat{\delta} = \begin{bmatrix} [\omega]_+ & v \\ 0 & 0 \end{bmatrix} \tag{10}$$

Where $\delta = \begin{bmatrix} v \\ w \end{bmatrix} \in \mathfrak{se}(2)$

ω: the parameter of rotation (is a scalar). $[w]_+ = \begin{bmatrix} 0 & -w \\ w & 0 \end{bmatrix}$

v: the parameters of translation (is a 2d vector).

We rewrite the $B^{-1}A$ as T_{BA}.

$$T_{BA} = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \tag{11}$$

We substitute (10) and (11) into (6), we get:

$$
\begin{aligned}
\exp(\widehat{J_A\delta}) &= -\exp(T_{BA}\hat{\delta}T_{BA}^{-1}) \\
&= -\exp(T_{BA}\begin{bmatrix}[\omega]_+ & v \\ 0 & 0\end{bmatrix}T_{BA}^{-1}) \\
&= -\exp(\begin{bmatrix}R & t \\ 0 & 1\end{bmatrix}\begin{bmatrix}[\omega]_+ & v \\ 0 & 0\end{bmatrix}\begin{bmatrix}R^T & -R^Tt \\ 0 & 1\end{bmatrix}) \\
&= -\exp(\begin{bmatrix}R[\omega]_+ & Rv \\ 0 & 0\end{bmatrix}\begin{bmatrix}R^T & -R^Tt \\ 0 & 1\end{bmatrix}) \\
&= -\exp(\begin{bmatrix}R[\omega]_+R^T & R[\omega]_+(-R^Tt)+Rv \\ 0 & 0\end{bmatrix}) \\
&= -\exp(\begin{bmatrix}[\omega]_+ & -[\omega]_+t+Rv \\ 0 & 0\end{bmatrix}) \\
&= -\exp(\begin{bmatrix}[\omega]_+ & -[\omega]_+t+Rv \\ 0 & 0\end{bmatrix})
\end{aligned} \tag{12}
$$

According to (10), we can rewrite (12) as:

$$
\begin{aligned}
\exp(\widehat{J_A\delta}) &= -\exp(\widehat{\begin{bmatrix}-[\omega]_+t+Rv \\ w\end{bmatrix}}) \\
&= -\exp(\widehat{\begin{bmatrix}-\omega t^\perp+Rv \\ w\end{bmatrix}}) \\
&= -\exp(\widehat{\begin{bmatrix}R & -t^\perp \\ 0 & 1\end{bmatrix}\begin{bmatrix}v \\ w\end{bmatrix}})
\end{aligned}
$$

Where $t^\perp = [1]_+ t = \begin{bmatrix}-t_2 \\ t_1\end{bmatrix}$

Hence:

$$
J_A = -\begin{bmatrix}R & -t^\perp \\ 0 & 1\end{bmatrix} = -\begin{bmatrix}R_{BA} & -t_{BA}^\perp \\ 0 & 1\end{bmatrix} \tag{13}
$$

similer with (9):

$$
J_B = I \tag{14}
$$

If A and B are SE3

The small incremental matrix of SE3 can be shown as follow:

$$
\hat{\delta} = \begin{bmatrix}[\omega]_\times & v \\ 0 & 0\end{bmatrix} \tag{15}
$$

Where $\delta = \begin{bmatrix} v \\ w \end{bmatrix} \in \mathfrak{se}(3)$

ω: the parameters of rotation (is a 3d vector). $[w]_\times$ is the skew symmetric matrix of w.

v: the parameters of translation (is a 3d vector).

Similar to (12), we get:

$$\begin{aligned}
\exp(\widehat{J_A \delta}) &= -\exp(T_{BA}\hat{\delta}T_{BA}^{-1}) \\
&= -\exp(T_{BA}\begin{bmatrix} [\omega]_\times & v \\ 0 & 0 \end{bmatrix} T_{BA}^{-1}) \\
&= -\exp(\begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} [\omega]_\times & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R^T & -R^T t \\ 0 & 1 \end{bmatrix}) \\
&= -\exp(\begin{bmatrix} R[\omega]_\times & Rv \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R^T & -R^T t \\ 0 & 1 \end{bmatrix}) \\
&= -\exp(\begin{bmatrix} R[\omega]_\times R^T & -R[\omega]_\times R^T t + Rv \\ 0 & 0 \end{bmatrix}) \\
&= -\exp(\begin{bmatrix} [R\omega]_\times & -[R\omega]_\times t + Rv \\ 0 & 0 \end{bmatrix})
\end{aligned} \tag{16}$$

According to (10), we can rewrite (16) as:

$$\begin{aligned}
\exp(\widehat{J_A \delta}) &= -\exp(\overline{\begin{bmatrix} -[R\omega]_\times t + Rv \\ Rw \end{bmatrix}}) \\
&= -\exp(\overline{\begin{bmatrix} [t]_\times R\omega + Rv \\ Rw \end{bmatrix}}) \\
&= -\exp(\overline{\begin{bmatrix} R & [t]_\times R \\ 0 & R \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix}})
\end{aligned} \tag{17}$$

Hence:

$$J_A = -\begin{bmatrix} R_{BA} & [t_{BA}]_\times R_{BA} \\ 0 & R_{BA} \end{bmatrix} \tag{18}$$

similer with (9):

$$J_B = I \tag{19}$$