

Edge shape effects and quasi-closed form expressions for the conductor loss of microstrip lines

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Abstract.

Much work has been done in the past to determine the conductor loss of microwave and millimeter-wave integrated circuits (MIMICs). However, much of this work is limited by constraints on the conductor thickness, skin depth, or the shape of the edge. In this paper, a technique will be presented that results virtually in a quasi-closed form expression for this loss in many cases. This expression will allow the calculation of the attenuation constant for a planar circuit for an arbitrarily shaped edge and any conductor thickness. The question of how different edge shapes affect the conductor loss will be addressed.

1. Introduction

Characterizing the conductor loss for planar circuits, such as for microstrips or coplanar waveguide (CPW), has been the object of much attention, especially recently [Pucel *et al.*, 1968; Wheeler, 1942; Heinrich, 1990, 1991; Paleczny *et al.*, 1990; Vakanas *et al.*, 1990; Waldow and Wolff, 1987; Goldfarb and Platzker, 1990; Finlay *et al.*, 1988; Horton *et al.*, 1971; Van Heuven, 1974; Nokilskiy and Kozlov, 1987; Lee and Itoh, 1989; Tischer, 1976; Mirshekar-Syahkal and Davies, 1979]. There are various techniques that can be used to calculate this loss. These techniques range from quasi-analytical ones like Wheeler's incremental inductance rule to full numerical approaches such as mode matching or the moment method. The moment method and mode matching procedures both have worked well, producing results that agree well with experiment. The main drawback to these two procedures is that a closed form expression for the attenuation constant (conductor loss) is not obtained; each time a structure (microstrip or CPW) is analyzed with new parameter values, a full, numerically intensive, computer run must be made.

As conductor loss is generally a small effect, it does not make sense to expend large amounts of computer time on their evaluation in design procedures. Closed-form expressions are instead desirable. Traditionally, the work of Pucel *et al.* [1968] has been used to evaluate the attenuation constant based on the Wheeler's incremental inductance rule [1942], and is of the form:

$$\alpha_c = \frac{R_s}{2\mu_0 Z_0} \sum_j \frac{\partial L}{\partial n_j}$$

where Z_0 is the characteristic impedance, $\partial L / \partial n_j$ denotes the derivative of L with respect to the incremental recession of wall j of the structure in the normal direction, and R_s is the standard Leontovich surface impedance [1948].

For a wide range of applications this expression works well. However, if t/δ_{sk} is small (where δ_{sk} is the skin depth and t is the thickness of a conductor) or even comparable to 1, then the Wheeler rule breaks down and will give very poor results, due in part to the fact that the Leontovich surface impedance is no longer valid.

Consider the microstrip shown in Figure 1. If the edges of the strip conductor are not exactly 90°, but are 60° or 70° edges (which is very possible when common processing techniques are utilized), then Pucel's results (which were obtained assuming an an-

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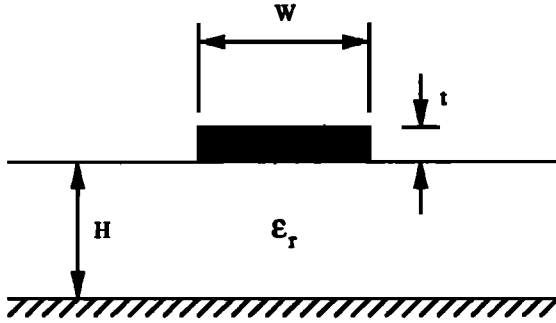


Figure 1. Microstrip geometry.

gle of 90°) are no longer valid, and full numerical techniques are the only option.

It is tempting to determine the conductor loss of this microstrip line by assuming the strip is infinitely thin and carrying out an integral over the top and bottom portion of the strip (see Figure 2) according to standard perturbation methods for computing wall loss:

$$\alpha_c \simeq \frac{R_s}{2Z_0} \int_{\text{top+bottom}} \left(\frac{J_s}{I} \right)^2 dl \quad (1)$$

where J_s is the current on the strip (assuming the strip is a perfect conductor). However, the current on an infinitely thin microstrip blows up as $1/\sqrt{r}$ at the edges, and the integration of $|J|^2$ will result in a logarithmic divergence.

A technique used to avoid this singularity was developed independently by Lewin [1984] and Vainshtein and Zhurav [1987]. This Lewin/Vainshtein procedure states that the loss can be approximated by carrying the integral (shown in equation (1)) out not to the edge, but to a point just before the edge of the strip. The position of this stopping point is a function only of the local edge geometry: the strip thickness and the shape of the edge. Then for a microstrip the conductor loss in the strip in terms of the stopping point Δ becomes

$$\alpha_c \simeq \frac{R_{sm}}{2Z_0} \frac{1}{I^2} \int_{-W/2+\Delta}^{W/2+\Delta} [J_{\text{top}}^2 + J_{\text{bottom}}^2] dx, \quad (2)$$

where the stopping point is a function of both edge shape and strip thickness. In the microstrip and other planar configurations, the density on the substrate side of the conductor may be much higher or

equivalent to the current density on the air side. The topic is covered in more detail at the end of section 5.

In the work by Lewin/Vainshtein and later Barsotti *et al.* [1991], an approximate procedure was used in order to determine this stopping point Δ . For a given edge shape, α_c is calculated by integrating the current distribution (J_s) over the contour of the edge of the actual thick strip (where J_s is assumed to be the current distribution on a perfectly conducting thick strip). This value of α_c is then compared to α_c obtained from equation (2) to determine Δ (where J_s is (2) is the current distribution of a infinite thin strip). The problem with the standard Lewin/Vainshtein stopping point is that it is only valid for a small skin depth (that is, very high frequency).

In this paper a more accurate stopping point (Δ) will be developed that will be valid for both small and large skin depths as compared to strip thickness and for any arbitrarily shaped edge. To determine this stopping point, the fields in the vicinity of a conducting strip with arbitrary edge shape is numerically determined both inside and outside the conducting region (see Figure 1). With these numerical values for the fields, the power dissipated in the strip is obtained. A philosophy similar to the Lewin/Vainshtein technique (but more precisely formulated using matched asymptotic expansions) is then employed to determine this stopping point.

In this paper it will be shown that once the stopping point Δ is determined for a given edge shape, it is possible to obtain a simple expression (often in closed form, as for microstrip) for the attenuation constant for any planar circuit accurate under most practical ranges of parameter values. It should be mentioned that when we refer to a closed form ex-

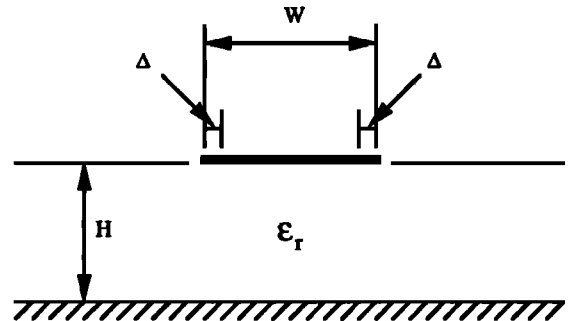


Figure 2. Infinitely thin microstrip geometry.

pression we are referring to an expression that is in terms of a parameter (the stopping point Δ) that is calculated numerically. Thus we call it a quasi-closed form expression.

2. Analysis of Fields Near an Isolated Strip Edge

The conventional approach for determining the attenuation constant of a waveguide is to use a perturbation procedure as discussed earlier, which results in an integral involving the current density over the surface of the structure. In general, this current may contain all three vector components:

$$\vec{J} = \vec{a}_x J_x + \vec{a}_y J_y + \vec{a}_z J_z$$

If the fields in the region local to the edge are expanded in an asymptotic series about a small parameter ν (where $\nu = k_d t/2$, t is strip thickness, and k_d is the wave number in the dielectric), then it can be demonstrated that up to first order in ν the fields completely decouple into two different polarizations. One polarization corresponds to J_z , while the other polarization corresponds to J_x and J_y . It can also be shown that in a region local to an edge of the structure under study, both J_x and J_y are zero to first order in ν .

The power loss can be calculated as the sum of the losses associated with each polarization. The perturbation procedure when applied to an infinitely thin structure containing only J_x and J_y behaves well, since these currents vanish at the edges of the conductor. Therefore nothing special needs to be done when applying the perturbation procedure to determine the loss.

The situation is quite different for the other polarization (J_z), as we have noted in the introduction. In this section a modification of the perturbation method for this polarization that is essentially the Lewin/Vainshtein method will be derived in a systematic manner, based on the method of matched asymptotic expansions.

The problem that needs to be solved is shown in Figure 3. In this polarization, there are only three field components that are needed: E_z , H_x and H_y . For this polarization, Maxwell's equations will reduce to

$$\begin{aligned} \vec{a}_z \times \nabla E_z &= j\omega\mu_{d,c} \vec{H} \\ \nabla \times \vec{H} &= j\omega\epsilon_{d,c} \vec{a}_z E_z \end{aligned} \quad (3)$$

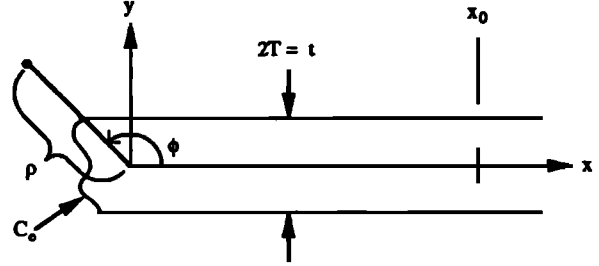


Figure 3. Isolated edge.

where d denotes the dielectric region and c denotes the conductor. The boundary conditions are as follows:

$$\begin{aligned} E_z^c|_C &= E_z^d|_C \\ \vec{H}_t^c|_C &= \vec{H}_t^d|_C \end{aligned} \quad (4)$$

In this problem it is convenient to represent the field in terms of the vector potential $\vec{A} = \vec{a}_z A_z$. For simplicity, we denote $A_z = A$.

The fields are then expressed as

$$\begin{aligned} \vec{H} &= \frac{1}{\mu} \nabla \times (\vec{a}_z A) \\ E_z &= -j\omega A \end{aligned} \quad (5)$$

From Maxwell's equations, it follows that the vector potential satisfies Helmholtz's equation

$$(\nabla^2 + \omega^2 \mu \epsilon_{d,c}) A = 0 \quad (6)$$

with the following boundary conditions:

$$\begin{aligned} A^c|_C &= A^d|_C \\ \frac{1}{\mu_c} \frac{\partial}{\partial n} A^c|_C &= \frac{1}{\mu_d} \frac{\partial}{\partial n} A^d|_C \end{aligned} \quad (7)$$

In this analysis it is necessary to separate the behavior of the potential close to the edge from that behavior far from the edge. Similar problems have been solved in acoustics by means of a matched asymptotic technique as shown by *Simons* [1980] and *Crighton and Leppington* [1973]. These papers suggest that a similar procedure can be used to find the vector potential for the isolated strip.

Before we proceed with the asymptotic expansion of the potential A , we need to determine how to handle the order of largeness of ϵ_c ; ϵ_c can be expressed as some inverse power of the thickness of the strip. Choosing

$$\epsilon_c = \frac{\epsilon_o \epsilon_{rd} G}{(k_d T)^2} \quad (8)$$

for some $G = O(1)$ results in the skin depth (δ_{sk}) being of the same order as T ($T = t/2$, where t is the total thickness of the conductor). This representation will ensure that the fields in the conductor behave as if the conductor is a "good" conductor.

2.1. Behavior of the Fields Far From the Edge: Outer Expansion

The methods of multiple scales and asymptotic expansion are used to obtain the so-called outer expansion for A , valid far from the edge. The multiple scales procedure is needed because of possible rapid variations in the y direction of the fields in the conductor region. This rapid variation is not present in the dielectric region, nor is any rapid x variation present in either region.

We define a "stretched" or "scaled" variable as

$$\tilde{y} = \frac{y}{\nu} \quad (9)$$

where ν is a small dimensionless parameter given by

$$\nu = k_d T \quad (10)$$

where T is half the total strip thickness and k_d is the wave number in the dielectric region:

$$k_d = \frac{2\pi}{\lambda_d} \quad (11)$$

It is assumed that T is small compared to all length scales in the problem, and most important, small compared to the wavelength λ_d , so that $\nu \ll 1$.

In the conductor, A is a function of x and $\tilde{y}$ and is defined as

$$A^c = U(x, \tilde{y}). \quad (12)$$

In the dielectric, A is a function of x and y . The Helmholtz equation in dielectric regions reduces to

$$\frac{\partial^2}{\partial x^2} A^d + \frac{\partial^2}{\partial y^2} A^d + k_d^2 A^d = 0 \quad (13)$$

and with the definition of ϵ_c given in equation (8), the Helmholtz equation in the conductor becomes

$$\frac{\partial^2}{\partial x^2} U + \frac{1}{\nu^2} \left[\frac{\partial^2}{\partial \tilde{y}^2} U + \omega^2 \epsilon_o \mu_c G U \right] = 0. \quad (14)$$

The first boundary condition states that the potential is continuous at the conductor surface. The interface corresponds to

$$y = \pm T \quad (15)$$

in the unstretched or unscaled y variable, or

$$\tilde{y} = \pm \frac{1}{k_d} \quad (16)$$

in $\tilde{y}$. The second boundary condition relates the normal derivatives of the potential in the two regions at the conductor interface. In the dielectric the normal derivative becomes

$$\frac{\partial}{\partial n} = \pm \frac{\partial}{\partial y} \quad (17)$$

where in the conductor region the normal derivative becomes

$$\frac{\partial}{\partial n} = \pm \frac{1}{\nu} \frac{\partial}{\partial \tilde{y}}. \quad (18)$$

With these definitions, the boundary conditions become

$$A^d(x, \pm T) = U(x, \pm \frac{1}{k_d}) \quad (19)$$

$$\frac{1}{\mu_d} \frac{\partial A^d(x, y)}{\partial y} \Big|_{y=\pm T} = \frac{1}{\mu_c \nu} \frac{\partial U(x, \tilde{y})}{\partial \tilde{y}} \Big|_{\tilde{y}=\pm \frac{1}{k_d}}$$

For the outer solution, an expansion is desired that will be valid for large distances from the origin (where the origin is just inside the conductor near to the edge). This expansion is not valid near C_e , the contour of the edge (see Figure 3).

The first step in this matched asymptotic procedure is to expand the fields in an asymptotic series about the small parameter ν :

$$\begin{aligned} A^d &\sim A^{d0}(x, y) + \nu A^{d1}(x, y) + \nu^2 A^{d2}(x, y) + O(\nu^3) \\ U &\sim U^0(x, \tilde{y}) + \nu U^1(x, \tilde{y}) + \nu^2 U^2(x, \tilde{y}) + O(\nu^3) \end{aligned} \quad (20)$$

It should be noted that the A^{d0} may contain an incident wave that is exciting this structure.

These asymptotic expansions for A^d and U are substituted into the two Helmholtz equations shown in equations (13) and (14). Next, the different powers of ν that appear in these equations can be grouped to obtain the following. In the dielectric we have

$$(\nabla^2 + k_d^2) A^{di}(x, y) = 0 \quad (21)$$

where the index i corresponds to the different orders in the asymptotic expansion. While in the conductor

$$\nu^{-2} : \frac{\partial^2 U^0}{\partial \tilde{y}^2} + k_G^2 U^0 = 0 \quad (22)$$

$$\nu^{-1} : \frac{\partial^2 U^1}{\partial \tilde{y}^2} + k_G^2 U^1 = 0 \quad (23)$$

and

$$\frac{\partial^2 U^i}{\partial y^2} + k_G^2 U^i = -\frac{\partial^2 U^{(i-2)}}{\partial x^2} \quad i = 2, 3, \dots, \quad (24)$$

where

$$k_G = \nu k_c = \nu \omega \sqrt{\mu_c \epsilon_c} \quad (25)$$

and

$$k_G^2 = \omega^2 \mu_c \epsilon_d G; \quad \text{Im}(k_G) < 0 \quad (26)$$

In order to complete the determination of our expansion, the boundary conditions given in equation (19) must be expanded in terms of the small parameter (ν). The fields in the dielectric A^d are governed by the Helmholtz equation with boundary conditions applied at $y = \pm T$. Since T is small, we extrapolate A^d to this convenient location independent of T ($y = 0^\pm$) with the aid of a Taylor series expansion of the form

$$A^d(x, \pm T) = A^{do}(x, 0^\pm) \pm T \left. \frac{\partial A^d(x, y)}{\partial y} \right|_{y=0^\pm} \pm \dots \quad (27)$$

in order to impose the boundary condition. Since for $|y| < T$, the values of A^d are extrapolated, they are not necessarily continuous at $y = 0$.

This expansion is substituted into the boundary conditions given in equation (19). Powers of ν are then combined to obtain the following:

$$A^{do}(x, 0^\pm) = U^o(x, \pm 1)$$

$$A^{d1}(x, \pm 1) = \pm \left. \frac{\partial A^{do}(x, y)}{\partial y} \right|_{y=0^\pm} + U^1(x, \pm 1) \quad (28)$$

$$\left. \frac{\partial U^o}{\partial y} \right|_{y=\pm 1/k_d} = 0 \quad (29)$$

$$\left. \frac{1}{\mu_c} \frac{\partial U^1}{\partial y} \right|_{y=\pm 1/k_d} = \left. \frac{1}{\mu_d} \frac{\partial A^{do}}{\partial y} \right|_{y=0^\pm}$$

and so on for higher orders.

The given equations for the 0th-order field in the conductor are

$$\frac{\partial^2 U^o}{\partial y^2} + k_G^2 U^o = 0 \quad (30)$$

$$\left. \frac{\partial U^o}{\partial y} \right|_{y=\pm 1/k_d} = 0$$

The sequential solution becomes

$$U^o = A_o e^{-jk_G y} + B_o e^{jk_G y} \quad (31)$$

By applying the boundary condition and since $\text{Im}(k_G) \neq 0$, it can be shown that

$$U^o \equiv 0 \quad (32)$$

Now the 0th-order field in the dielectric can be investigated. The following is the governing equation for this field:

$$\begin{aligned} (\nabla^2 + k_d^2) A^{do}(x, y) &= 0 \\ A^{do}(x, 0^\pm) &= 0 \end{aligned} \quad (33)$$

Equation (33) can be identified as that for E_z in the presence of an infinitesimal half-plane. Thus in some neighborhood not too far from the edge, the solution of A^{do} is of the form

$$\begin{aligned} A^{do} &= \sum_{m=1}^{\infty} f_m \sin\left(\frac{m\phi}{2}\right) J_{\frac{m}{2}}(k_d \rho) \\ &+ \sum_{m=1}^{\infty} g_m \sin\left(\frac{m\phi}{2}\right) H_{\frac{m}{2}}^{(2)}(k_d \rho) \end{aligned} \quad (34)$$

Normally, the edge condition would be applied at $x = y = 0$ to render the solution unique.

Since this outer expansion for A^{do} breaks down as $\rho \rightarrow 0$, A^{do} cannot be rendered unique by the edge condition. The more singular terms ($H_{m/2}^{(2)}$) may need to be present. The constants f_m are eventually determined by the source, while the constants g_m are determined by matching this outer solution to the inner solution. This will be discussed in a later section.

The first-order field in the conductor is defined by the following:

$$\begin{aligned} \frac{\partial^2 U^1}{\partial y^2} + k_G^2 U^1 &= 0 \\ \text{with} \end{aligned} \quad (35)$$

$$\left. \frac{1}{\mu_c} \frac{\partial U^1}{\partial y} \right|_{y=\pm 1/k_d} = \left. \frac{1}{\mu_d} \frac{\partial A^{do}}{\partial y} \right|_{y=0^\pm}$$

It can be shown that

$$U^1 = A_1(x) e^{-jk_G y} + B_1(x) e^{jk_G y}, \quad (36)$$

where $A_1(x)$ and $B_1(x)$ are obtained by applying the boundary condition and are found to be

$$\begin{aligned} A_1(x) &= -\frac{\mu_{cr}}{2k_G \sin(2k_G)} \left[e^{-jk_G} \left. \frac{\partial A^{do}}{\partial y} \right|_{y=0^+} - e^{jk_G} \left. \frac{\partial A^{do}}{\partial y} \right|_{y=0^-} \right] \\ B_1(x) &= -\frac{\mu_{cr}}{2k_G \sin(2k_G)} \left[e^{jk_G} \left. \frac{\partial A^{do}}{\partial y} \right|_{y=0^+} - e^{-jk_G} \left. \frac{\partial A^{do}}{\partial y} \right|_{y=0^-} \right] \end{aligned} \quad (37)$$

where $\mu_{cr} = \mu_c/\mu_d$ and $\partial A^{do}/\partial y$ is known from above.

These values for $A_1(x)$ and $B_1(x)$ can be used to obtain an expression for the first-order field in the dielectric region valid at the fictitious $y = 0$ plane. With A^{d1} defined in (28), equations (36) and (37) can be used to show that $A^{d1}(x, 0^\pm)$ reduces to

$$\begin{aligned} A^{d1}(x, 0^+) &= -\left(1 + \frac{\mu_{cr}}{k_G} \cot(2k_G)\right) \frac{\partial A^{do}}{\partial y} \Big|_{y=0^+} \\ &\quad + \frac{\mu_{cr}}{k_G \sin(2k_G)} \frac{\partial A^{do}}{\partial y} \Big|_{y=0^+} \\ A^{d1}(x, 0^-) &= \left(1 + \frac{\mu_{cr}}{k_G} \cot(2k_G)\right) \frac{\partial A^{do}}{\partial y} \Big|_{y=0^-} \\ &\quad - \frac{\mu_{cr}}{k_G \sin(2k_G)} \frac{\partial A^{do}}{\partial y} \Big|_{y=0^-} \end{aligned} \quad (38)$$

Away from $y = 0$, A^{d1} obeys $(\nabla^2 + k_d^2)A^{d1} = 0$.

Now that the 0th and first-order fields are determined, we can analyze the behavior of these fields to obtain an impedance boundary condition that is valid far from the edge. This will be the topic of the next section.

5.2. Impedance Boundary Condition Far From the Edge

With equations (5) and (20), an asymptotic expansion for E_z^d and H_z^d in the dielectric region can be expressed as the following:

$$\begin{aligned} E_z^d &\sim -j\omega (A^{do} + \nu A^{d1}) + O(\nu^2) \\ H_z^d &\sim \frac{1}{\mu} \left[\frac{\partial A^{do}}{\partial y} + \nu \frac{\partial A^{d1}}{\partial y} \right] + O(\nu^2) \end{aligned} \quad (39)$$

These fields can be extrapolated to $y = 0^\pm$:

$$\begin{aligned} E_z(x, 0^\pm) &\sim -j\omega (A^{do}(x, 0^\pm) + \nu A^{d1}(x, 0^\pm)) \\ &\quad + O(\nu^2) \\ H_z(x, 0^\pm) &\sim \frac{1}{\mu} \left[\frac{\partial A^{do}(x, 0^\pm)}{\partial y} + \nu \frac{\partial A^{d1}(x, 0^\pm)}{\partial y} \right] + O(\nu^2) \end{aligned} \quad (40)$$

From the boundary condition (33) for A^{do} at this fictitious plane, E_z^d becomes

$$E_z(x, 0^\pm) \sim -j\omega \nu A^{d1}(x, 0^\pm) + O(\nu^2). \quad (41)$$

From (38), (40), and (41), $E_z^d(x, 0^\pm)$ is also expressible in terms of $H_z(x, 0^\pm)$ as follows:

$$\begin{aligned} E_z(x, 0^\pm) &\sim \pm (j\omega \mu_d \nu + j\xi_c \cot(2k_G)) H_z(x, 0^\pm) \\ &\quad \mp j\xi_c \csc(2k_G) H_z(x, 0^\mp) \end{aligned} \quad (42)$$

The fictitious top and bottom surface currents at the conducting plane ($y = 0^\pm$) are given by

$$J_z^+ \equiv -H_z(x, 0^+) \quad J_z^- \equiv H_z(x, 0^-) \quad (43)$$

$E_z(x, 0^\pm)$ can now be written in terms of these fictitious top and bottom currents:

$$\begin{aligned} E_z(x, 0^\pm) &\sim - (j\omega \mu_d \nu + j\xi_c \cot(2k_G)) J_z^\pm \\ &\quad - j\xi_c \csc(2k_G) J_z^\mp \end{aligned} \quad (44)$$

The first term corresponds to the interaction between the E and H fields on the same side of the strip. The second term gives the interaction between the E field on one side to the H field on the opposite side. This term relays information about the fields on either side of the conducting strip and is the coupling term. The information from the two sides are contained in two surface impedances:

$$\begin{aligned} Z_s &= \xi_c \coth(j2k_G) = -j\xi_c \cot(2k_G) \\ Z_m &= \xi_c \operatorname{csch}(j2k_G) = -j\xi_c \csc(2k_G) \end{aligned} \quad (45)$$

where Z_s is a self-impedance and Z_m is a mutual impedance.

An equivalent (first order) generalized impedance boundary condition valid for the half-plane far from the edge in terms of the extrapolated E fields and the fictitious surface current $J_z^\pm$ is obtained with these impedances:

$$E_z(x, 0^\pm) = (j\omega \mu_c \nu + Z_s) J_z^\pm + Z_m J_z^\mp \quad (46)$$

Except for the term $(j\omega \mu_c \nu)$, this is equivalent to the generalized transfer impedance boundary condition given by *Horton et al.* [1971]. The extra term is a purely reactive impedance reflecting the extrapolation from points on the actual conductor surface $y = \pm T$ to the fictitious half-plane $y = 0$. Since this term is purely reactive, it plays no role in the calculation of conductor loss.

Before moving on to the inner expansion, an investigation of these self and mutual impedances is in order. More important, the question of how the E

fields behave at the two limiting situations for T/δ_{sk} needs to be answered. The two limiting cases are as $T/\delta_{sk} \rightarrow 0$ and $T/\delta_{sk} \rightarrow \infty$.

It can be shown that as t/δ_{sk} gets large, equation (46) will reduce to

$$E_x(x, 0^\pm) = \frac{(1+j)}{\sigma_c \delta_{sk}} J_z^\pm + (j\omega\mu_c k_d T) J_z^\pm. \quad (47)$$

This is essentially the Leontovich boundary condition with a shift in half the strip thickness. When t/δ_{sk} gets small, it is readily shown that

$$Z_m = Z_s \quad (48)$$

and (46) reduces to

$$E_x(x, 0^\pm) = Z_s (J_z^\pm + J_z^\mp). \quad (49)$$

This means that for small t the E fields are continuous across the strip.

2.3. Matching Principle

Much has been written about matching principles for asymptotic expansions [Simons, 1980; Crighton and Leppington, 1973; Fraenkel, 1969; Van Dyke, 1965]. Fraenkel [1969] discusses a systematic approach to the matching principle which is used by Simons [1980] and Crighton and Leppington [1973]. Van Dyke [1965] summarizes this as follows: Whenever two neighboring expansions have overlapping domains, the following holds:

$$(A^i)^\circ = (A^\circ)^i \quad (50)$$

or simply "the outer limit of (the inner expansion)" = "the inner limit of (the outer expansion)".

By applying the matching principle we are able to accomplish two things. First, insight into the behavior of both the outer and inner expansions is obtained. This information can be used to determine if the asymptotic expansions for the fields in both the inner and outer regions are valid. Second, the matching principle allows the information obtained at the outer region to be coupled into the inner solution. This information serves the same purpose as a boundary condition: It allows us to determine the unknown constants in the expansions. Even though Van Dyke [1965] shows that the matching principle can be explained in a very straightforward manner, insight and experience are often required in order to apply the matching principle properly, a reflection of

the fact that much of the matched asymptotic procedure is as yet an art.

By applying this technique to the outer expansion of the fields just determined above, it is possible not only to determine the unknown constants g_m but also to decide how the inner expansion should be represented. From equation (34) it is seen that A^{do} behaves as

$$A^{do} \rightarrow f_m \sin\left(\frac{m\phi}{2}\right) J_{\frac{m}{2}}(k_d \rho) + g_m \sin\left(\frac{m\phi}{2}\right) H_{\frac{m}{2}}^{(2)}(k_d \rho). \quad (51)$$

Even though A^{do} is not an accurate approximation to A as $\rho \rightarrow 0$, this limit is still needed to perform the matching procedure. Therefore as $\rho \rightarrow 0$, $J_{m/2}$ and $H_{m/2}^{(2)}$ behave as

$$\begin{aligned} H_{\frac{m}{2}}^{(2)}(k_d \rho) &\rightarrow \frac{i}{\pi} \Gamma\left(\frac{m}{2}\right) \left(\frac{k_d T \rho}{2}\right)^{-\frac{m}{2}} \text{ as } \rho \rightarrow 0 \\ J_{\frac{m}{2}}(k_d \rho) &\rightarrow \frac{1}{2^{\frac{m}{2}} \Gamma(1+\frac{m}{2})} (k_d \rho)^{\frac{m}{2}} \text{ as } \rho \rightarrow 0 \end{aligned} \quad (52)$$

This shows that $H_{m/2}^{(2)}$ blows up as $\rho \rightarrow 0$, but the leading term of A^{do} in the outer expansion must be matchable to an inner expansion and must remain finite. We will see that the inner expansion does not blow up as $\rho \rightarrow \infty$ (or more precisely, as some scaled $\rho \rightarrow \infty$, see next section). This leads to only one possible choice for all the g_m :

$$g_m \equiv 0 \quad (53)$$

With this, A^{do} behaves as

$$A^{do} \rightarrow f_m \sin\left(\frac{m\phi}{2}\right) J_{\frac{m}{2}}(k_d \rho), \quad (54)$$

and equation (52) shows

$$J_{\frac{m}{2}} = O(T^{\frac{m}{2}}). \quad (55)$$

Because of the form of this inner limit of the outer expansion ($\rho \rightarrow 0$), matching with the inner expansion requires that the inner expansion must have the following form:

$$\begin{aligned} A_{\text{inner}}^d &\sim A_{\text{inner}}^{do} + \nu^{1/2} A_{\text{inner}}^{d1} + \nu^1 A_{\text{inner}}^{d2} + O(\nu^{3/2}) \\ A_{\text{inner}}^c &\sim A_{\text{inner}}^{co} + \nu^{1/2} A_{\text{inner}}^{c1} + \nu^1 A_{\text{inner}}^{c2} + O(\nu^{3/2}) \end{aligned} \quad (56)$$

2.4. Behavior of Fields Local to the Edge: Inner Expansion

For the inner expansion, a solution is desired that will properly represent the behavior of the fields close to the edge. In this region the fields exhibit fast variation in not only the y direction (as in the case of the outer solution), but also in the x direction. Therefore both coordinates need to be stretched according to

$$\begin{aligned}\tilde{x} &= \frac{x}{\nu} = \frac{x}{k_d T} \\ \tilde{y} &= \frac{y}{\nu} = \frac{y}{k_d T}\end{aligned}\quad (57)$$

The fields in both the dielectric and conducting regions will be represented as the following:

$$\begin{aligned}A^d(x, y) &= V(\tilde{x}, \tilde{y}) \\ A^c(x, y) &= W(\tilde{x}, \tilde{y})\end{aligned}\quad (58)$$

With both coordinates stretched, the del operator takes the form

$$\nabla = \frac{1}{\nu} \tilde{\nabla}, \quad (59)$$

where

$$\tilde{\nabla} = \tilde{a}_x \frac{\partial}{\partial \tilde{x}} + \tilde{a}_y \frac{\partial}{\partial \tilde{y}}. \quad (60)$$

As shown above, the fields in both regions satisfy the Helmholtz equation, and in terms of the stretched coordinates they are given by

$$\begin{aligned}(\tilde{\nabla}^2 + k_d^2 \nu^2) V &= 0 \\ (\tilde{\nabla}^2 + k_g^2) U &= 0\end{aligned}\quad (61)$$

and the boundary conditions reduce to

$$\begin{aligned}V|_C &= W|_C \\ \frac{\partial W}{\partial \tilde{n}}|_C &= \mu_{cr} \frac{\partial V}{\partial \tilde{n}}|_C\end{aligned}\quad (62)$$

and, as before, $\mu_{cr} = \mu_c / \mu_d$.

As discussed in the previous subsection, the inner limit of the outer expansion behaves as a power series in $\nu^{1/2}$, so the inner expansions are expressed as

$$\begin{aligned}V &\sim V_0(\tilde{x}, \tilde{y}) + \nu^{1/2} V_1(\tilde{x}, \tilde{y}) + \nu V_2(\tilde{x}, \tilde{y}) + O(\nu^{3/2}) \\ W &\sim W_0(\tilde{x}, \tilde{y}) + \nu^{1/2} W_1(\tilde{x}, \tilde{y}) + \nu W_2(\tilde{x}, \tilde{y}) + O(\nu^{3/2})\end{aligned}\quad (63)$$

If these expansions are substituted into equation (61), powers of ν can be grouped to obtain sets of

equations governing the different orders of the fields. For the dielectric region,

$$\begin{aligned}\tilde{\nabla}^2 V_i &= 0 \quad i = 0, 1, 2, 3. \\ \tilde{\nabla}^2 V_i &= -k_d^2 V_{i-4} \quad i = 4, 5, \dots\end{aligned}\quad (64)$$

and in the conducting strip,

$$(\tilde{\nabla}^2 + k_g^2) W_i = 0 \quad \text{for } i = 0, 1, 2, 3, \dots \quad (65)$$

The boundary conditions for the fields reduce to

$$\begin{aligned}V_i|_C &= W_i|_C \\ \frac{\partial W_i}{\partial \tilde{n}}|_C &= \mu_{cr} \frac{\partial V_i}{\partial \tilde{n}}|_C\end{aligned}\quad (66)$$

These solutions will still contain some arbitrary constants. This arbitrariness is removed by applying the matching principle as explained in the previous subsection. However, when stretched coordinates are used, the matching principle is expressed slightly differently (which is detailed in *Simons* [1980] and *Crighton and Leppington* [1973]): "terms up to $O(\nu^m)$ in the outer expansion are rewritten in terms of the stretched coordinates $(\tilde{x}, \tilde{y})$ and re-expanded in powers of ν . The results up to terms of $O(\nu^g)$ must be equal to the result of taking terms in the inner expansion up to $O(\nu^g)$, rewriting in terms of (x, y) (or, for U , in terms of $(x, \tilde{y})$) and expanding up to $O(\nu^m)^n$."

For $m = 0$ and $g = 0$, we have

$$\lim_{\rho \rightarrow 0} U^o(x, \frac{y}{\nu}) = \lim_{\tilde{\rho} \rightarrow \infty} W_o(\tilde{x}, \tilde{y}) \quad (67)$$

Since $U^o \equiv 0$,

$$\lim_{\tilde{\rho} \rightarrow \infty} W_o(\tilde{x}, \tilde{y}) = 0. \quad (68)$$

In the dielectric region (for $m = 0$ and $g = 0$) it was shown that

$$\lim_{\rho \rightarrow 0} A^{do} \Rightarrow O(\sqrt{\frac{2k_d \tilde{\rho}}{\pi}}) \Rightarrow 0. \quad (69)$$

Thus

$$\lim_{\tilde{\rho} \rightarrow \infty} V_o(\tilde{x}, \tilde{y}) = 0. \quad (70)$$

Now that these boundary conditions for the 0th order fields of the inner expansion have been determined, there is enough information to formulate these fields without any arbitrariness in the solution.

The eddy current problem of these 0th fields reduces to

$$\begin{aligned}\tilde{\nabla}^2 V_o &= 0 \\ (\tilde{\nabla}^2 + k_d^2) W_o &= 0\end{aligned}\quad (71)$$

with boundary conditions

$$\begin{aligned}V_o|_C &= W_o|_C \\ \frac{\partial W_o}{\partial \mathbf{n}}|_C &= \mu_{cr} \frac{\partial V_o}{\partial \mathbf{n}}|_C \\ \lim_{\tilde{\rho} \rightarrow \infty} V_o(\tilde{x}, \tilde{y}) &= 0 \\ \lim_{\tilde{\rho} \rightarrow \infty} W_o(\tilde{x}, \tilde{y}) &= 0.\end{aligned}\quad (72)$$

With no sources present, from the uniqueness theorem for the eddy current problem we find that

$$\begin{aligned}V_o &\equiv 0 \\ W_o &\equiv 0\end{aligned}\quad (73)$$

Now the first-order solution of the inner expansion can be investigated. Recall that the small argument behavior of A^{do} is

$$A^{do} \simeq \sum_{n=1}^{\infty} f_n \sin\left(\frac{n\phi}{2}\right) \frac{(k_d \rho)^{m/2}}{2^{m/2} \Gamma(1+m/2)} \quad (74)$$

If ρ is small enough, then A^{do} can be approximated by $n=1$:

$$A^{do} \simeq f_1 \sin\left(\frac{\phi}{2}\right) \sqrt{\frac{2k_d \rho}{\pi}} \quad (75)$$

If x and y are changed to $\nu \tilde{x}$ and $\nu \tilde{y}$, respectively, then A^{do} is expressed as

$$A^{do}(x, y) = A^{do}(\tilde{x}, \tilde{y}) \sim \nu^{1/2} f_1 \sin\left(\frac{\phi}{2}\right) \sqrt{\frac{2k_d \tilde{\rho}}{\pi}} + O(\nu). \quad (76)$$

The matching principle gives

$$\begin{aligned}\lim_{\tilde{\rho} \rightarrow \infty} V_1(\tilde{x}, \tilde{y}) &= \lim_{\rho \rightarrow \infty} A^{do}(x, y) \\ &= \tilde{\rho}^{1/2} f_1 \sin\left(\frac{\phi}{2}\right) \sqrt{\frac{2k_d}{\pi}}\end{aligned}\quad (77)$$

The constant f_1 that results from whatever excitation that is present would be known from the edge behavior of the field for the infinitely thin perfect conductor.

Now look at the inner limit of U^1 (the first-order term in the outer expansion of the field in the con-

ductor's outer region). It can be shown that U^1 is $O(\nu)$, so there is nothing to match the $O(\nu^{1/2})$ term of the field in the conductor in the inner expansion to (the $\nu^{1/2}$ term in the inner limit of νU^1 is zero). This leads to

$$\lim_{\tilde{\rho} \rightarrow \infty} W_1(\tilde{x}, \tilde{y}) = 0 \quad (78)$$

As seen from equation (77), the far field of the inner expansion in the dielectric has a dominant term of the form

$$\tilde{\rho}^{1/2} \sin\left(\frac{\phi}{2}\right) f_1 \sqrt{\frac{2k_d}{\pi}}$$

This term will define the local amplitude of the fields at the tip. In order to analyze this localized behavior, a new set of fields are defined by subtracting this dominant term and multiplying by the normalizing factor:

$$\frac{1}{f_1} \sqrt{\frac{\pi}{2k_d}}$$

The new first-order field now becomes

$$\begin{aligned}Q_1(\tilde{x}, \tilde{y}) &\equiv \frac{1}{f_1} \sqrt{\frac{\pi}{2k_d}} V_1(\tilde{x}, \tilde{y}) - \tilde{\rho}^{1/2} \sin\left(\frac{\phi}{2}\right) \\ P_1(\tilde{x}, \tilde{y}) &\equiv \frac{1}{f_1} \sqrt{\frac{\pi}{2k_d}} W_1(\tilde{x}, \tilde{y})\end{aligned}\quad (79)$$

From equations (64) and (65) it follows that Q_1 and P_1 obey Laplace and Helmholtz equations, respectively:

$$\begin{aligned}\tilde{\nabla}^2 Q_1 &= 0 \\ (\tilde{\nabla}^2 + k_d^2) P_1 &= 0\end{aligned}\quad (80)$$

and from the boundary conditions (66) we have

$$\begin{aligned}P_1|_C - Q_1|_C &= \tilde{\rho}^{1/2} \sin\left(\frac{\phi}{2}\right) \\ \frac{\partial P_1}{\partial \mathbf{n}}|_C - \mu_{cr} \frac{\partial Q_1}{\partial \mathbf{n}}|_C &= \mu_{cr} \tilde{\rho}^{1/2} \sin\left(\frac{\phi}{2}\right) \tilde{\nabla} \left(\tilde{\rho}^{1/2} \sin\left(\frac{\phi}{2}\right) \right)|_C,\end{aligned}\quad (81)$$

where

$$\tilde{\rho}^{1/2} \sin\left(\frac{\phi}{2}\right) = \sqrt{\frac{\tilde{\rho} - \tilde{\epsilon}}{2}} = \frac{|\tilde{y}|}{\sqrt{2(\tilde{\rho} + \tilde{\epsilon})}} \quad (82)$$

and

$$\begin{aligned}\tilde{\nabla} \left(\tilde{\rho}^{1/2} \sin\left(\frac{\phi}{2}\right) \right) &= \frac{1}{2^{3/2} \tilde{\rho}} \left[-\frac{|\tilde{y}|}{\sqrt{2(\tilde{\rho} + \tilde{\epsilon})}} \right. \\ &\quad \left. + \text{sgn}(\tilde{y}) \sqrt{2(\tilde{\rho} + \tilde{\epsilon})} \tilde{a}_y \right].\end{aligned}\quad (83)$$

The functions shown in equations (82) and (83) all approach zero as $\tilde{x} \rightarrow \infty$ on $\tilde{y} = \pm 1$. For numerical purposes it is sufficient to place a fictitious boundary at a large distance from the origin and impose the following on it:

$$\frac{\partial P_1}{\partial \tilde{n}} = 0 \quad \text{and} \quad \frac{\partial Q_1}{\partial \tilde{n}} = 0. \quad (84)$$

3. Calculation of the Modified Lewin/Vainshtein Stopping Point

In this section the power associated with the conducting strip (local to the edge) is calculated with both the outer and inner expansions. The Lewin/Vainshtein philosophy is applied when the outer expansion is used. By equating the power loss determined in this way to that of the inner expansion, a modified Lewin/Vainshtein stopping point Δ can be determined.

3.1. Power Loss Determined From Inner Expansion

The power dissipated in the strip for the portion local to the edge is needed. Let $(x_o, 0)$ be a point in the conductor where both the outer and inner expansions are valid (i.e., where the matching procedure is performed) as shown in Figure 3. The average power dissipated into the conductor for that portion of the strip lying in $x < x_o$ is given by

$$P_L = -\frac{1}{2} \text{Re} \int_{C_{x_o}} \bar{E} \cdot (\bar{a}_n \times \bar{H}^*) dS, \quad (85)$$

where $C_{x_o} = C \cap \{x < x_o\}$ and $\bar{a}_n$ is the unit vector out of the conductor (see Figure 3).

In terms of the A , the power loss dissipated in the strip now becomes

$$P_L = -\frac{\omega}{2\mu_c} \text{Im} \int_{C_{x_o}} A^c \frac{\partial A^{c*}}{\partial n} dl \quad (86)$$

Since C_{x_o} is the contour defined from the top side at $x = x_o$ around the actual edge of the strip to the bottom side at $x = x_o$, the actual calculation must be carried out on the basis of the inner expansion. Note that the outer expansion is not valid this close to the edge and cannot be used in equation (86) to calculate the loss. The inner expansion of A^c to order $O(\nu)$ is given by

$$A^c(\tilde{x}, \tilde{y}) \sim \nu^{1/2} W_1\left(\frac{x}{\nu}, \frac{y}{\nu}\right) \quad (87)$$

and from equation (79)

$$A^c(\tilde{x}, \tilde{y}) = f_1 \sqrt{\frac{2k_d T}{\pi}} P_1\left(\frac{x}{\nu}, \frac{y}{\nu}\right). \quad (88)$$

With this definition the power loss is given by

$$\begin{aligned} P_L &= -\frac{\omega k_d T}{\mu_c \pi} |f_1|^2 \text{Im} \int_{C_{x_o}} P_1\left(\frac{x}{\nu}, \frac{y}{\nu}\right) \frac{\partial P_1^*\left(\frac{x}{\nu}, \frac{y}{\nu}\right)}{\partial n} dl \\ &= -\frac{\omega k_d T}{\mu_c \pi} |f_1|^2 \text{Im} \int_{C_{x_o}} P_1(\tilde{x}, \tilde{y}) \frac{\partial P_1^*(\tilde{x}, \tilde{y})}{\partial n} dl, \end{aligned} \quad (89)$$

where C_{x_o} is C_{x_o} defined in $(\tilde{x}, \tilde{y})$ coordinates.

3.2. Power Loss Determined From Outer Expansion: Calculation of the Stopping Point

The Lewin/Vainshtein technique proposes the use of the outer expansion in the loss calculation associated with the edge. Even though this expansion is no longer valid as we approach the edge, suppose that a wall-loss perturbation formula can be employed on an "infinitesimally" thin conductor, only up to a stopping point Δ :

$$\begin{aligned} P_L &= -\frac{1}{2} \text{Re} \int_{\Delta}^{x_o} \left[\bar{E} \cdot (\bar{a}_n \times \bar{H}^*) \Big|_{y=T} \right. \\ &\quad \left. + \bar{E} \cdot (\bar{a}_n \times \bar{H}^*) \Big|_{y=-T} \right] dx, \end{aligned} \quad (90)$$

where Δ is some point just before the edge and x_o is same point away from the edge (see Figure 4). By extrapolating these fields to the top and bottom sides of a fictitious half-plane at $y = 0^\pm$, the power loss becomes (to leading order)

$$P_L \simeq \frac{1}{2} \text{Re} \int_{\Delta}^{x_o} E_z J_{z_o}^* dx + \frac{1}{2} \text{Re} \int_{\Delta}^{x_o} E_z J_{z_{bottom}}^* dx \quad (91)$$

Let $J_z^\pm$ be given by the values they would have if the strip were infinitely thin and perfectly conducting. That is, it is found from A^{do} alone. If $E_z(x, 0^\pm)$

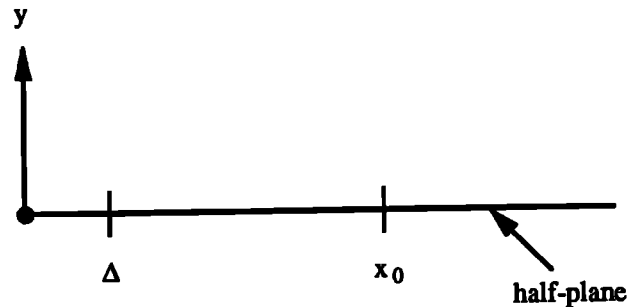


Figure 4. Power loss in a region far from the edge.

is determined approximately from equation (46), this leads to the following for the average power dissipated by both sides of the indicated segment:

$$P_L \simeq \frac{1}{2} \text{Re} \int_{\Delta}^{x_0} [E_z(x, 0^+) J_z^{+*} + E_z(x, 0^-) J_z^{-*}] dl. \quad (92)$$

With $E_z(x, 0^{\pm})$ defined in equation (46), P_L reduces to

$$P_L \simeq \frac{1}{2} \text{Re} \int_{\Delta}^{x_0} [Z_s (|J_z^+|^2 + |J_z^-|^2) + 2Z_m \text{Re} (J_z^+ J_z^{-*})] dx. \quad (93)$$

Earlier it was shown that A^{do} behaves as

$$A^{do} = \sum_{m=1}^{\infty} f_m \sin\left(\frac{m\phi}{2}\right) J_{m/2}(k_d \rho). \quad (94)$$

For small arguments, $J_{m/2}(k_d \rho)$ behaves as

$$J_{m/2} \approx \frac{1}{2^{m/2} \Gamma(1 + \nu)} (k_d \rho)^{m/2}, \quad (95)$$

so as $\rho \rightarrow 0$,

$$A^{do} \simeq f_1 \sin\left(\frac{\phi}{2}\right) \sqrt{\frac{2k_d \rho}{\pi}}. \quad (96)$$

Recalling that

$$J_z^+ \equiv -H_s(x, 0^+) \text{ and } J_z^- \equiv H_s(x, 0^-) \quad (97)$$

and using equation (40) for H_s to 0th order, we get

$$J_z^+ \simeq J_z^- \sim H_s \sim \frac{1}{\mu} \frac{\partial A^{do}}{\partial y} \sim -\frac{f_1}{\mu_d} \sqrt{\frac{k_d}{2\pi x}}. \quad (98)$$

Substituting this into equation (93) and evaluating the integral, the following is obtained for the power dissipated in the strip:

$$P_L \simeq \frac{|f_1|^2 k_d}{2\pi \mu_d^2} R_{sm} \ln\left(\frac{x_0}{\Delta}\right), \quad (99)$$

where R_{sm} is the modified Horton impedance boundary condition as discussed in section 2.2 and given by

$$R_{sm} = \omega \mu_c t \text{Im} \left(\frac{\cot(k_c t) + \csc(k_c t)}{k_c t} \right). \quad (100)$$

The first thing to note from equation (99) is that as $\Delta \rightarrow 0$, the expression for the power loss blows up. The Lewin/Vainshtein technique states that a value of Δ can be chosen such that the power loss determined from the outer expansion (equation (99)) will be equivalent to the loss determined from the inner expansion (equation (89)). Therefore

$$\frac{|f_1|^2 k_d}{2\pi \mu_d^2} R_{sm} \ln\left(\frac{x_0}{\Delta}\right) = -\frac{\omega k_d T}{\mu_c \pi} |f_1|^2 \cdot \text{Im} \int_{C_{s_0}} P_1(\tilde{x}, \tilde{y}) \frac{\partial P_1^*(\tilde{x}, \tilde{y})}{\partial n} dl$$

and after some simplification the modified Lewin/Vainshtein stopping point is determined by

$$\ln\left(\frac{x_0}{\Delta}\right) = -\frac{1}{\mu_c^2 \pi} \text{Im} \int_{C_{s_0}} P_1(\tilde{x}, \tilde{y}) \frac{\partial P_1^*(\tilde{x}, \tilde{y})}{\partial n} dl \quad (101)$$

where

$$r = \text{Im} \left(\frac{\cot(k_c t) + \csc(k_c t)}{k_c t} \right). \quad (102)$$

As $\tilde{x}_0 \rightarrow \infty$ (or x_0 is fixed as $\nu \rightarrow 0$), the value calculated for Δ should approach a constant. A choice of $x_0 \gg t$ is probably sufficient for numerical considerations. It is not as evident that the results are independent of the location of the origin (within $O(\nu)$), although from physical considerations it must be true.

4. Numerical Results for the Modified Lewin/Vainshtein Stopping Point

A finite-element program was written to solve for the fields defined in equation (80). Some unusual features present themselves in this problem, and will be treated in a separate publication (C. L. Holloway and E. F. Kuester, A variational formulation for a general class of eddy current problem, manuscript in preparation, 1994). Once these fields are determined, the value for the stopping point can be obtained from equation (101). Figure 5 shows these numerical results for t/Δ for a wide range of t/δ_{sk} . The original Lewin [1984] and Vainshtein [1987] stopping point is valid only for the very high frequency limit. For the 90° edge they obtained $t/\Delta|_{LV} = 290.8$. This value is also shown in Figure 5. The curve generated in this analysis appears to approach the asymptotic level of 290.8. However, it would seem that this value is not attained until T/δ_{sk} is of the order of 40 or above. This may seem high, but *Tikhomirov and*

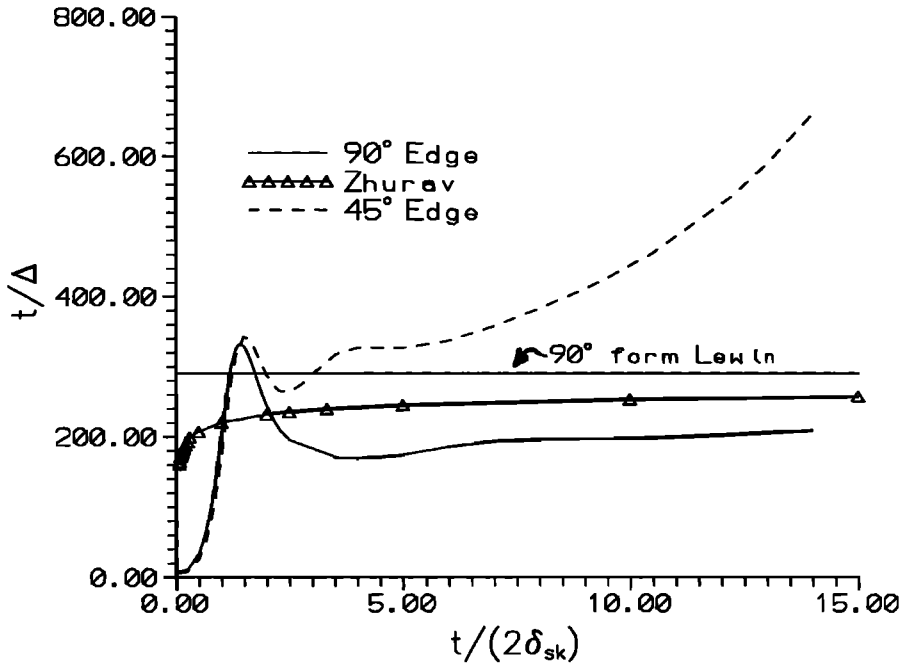


Figure 5. Numerical results for the stopping point Δ for both the 90° and 45° edges.

Manenkov [1990] have observed similar behavior in an analysis of the skin-effect in conductors of square cross-section.

Zhurav [1987] attempted to modify the original Lewin/Vainshtein stopping point to correct for smaller values of t/δ_{sk} , and the results he obtained are also plotted in Figure 5. Zhurav's results do a decent job of predicting Δ for $T/\delta_{sk} > 5$. Below $T/\delta_{sk} = 5$, Zhurav's expression does not show the resonance type behavior that is seen in the results presented here. More important, Zhurav predicts a dc value much larger than the numerical result; $t/\Delta = 160$ as compared to $t/\Delta = 9.2$.

Because of the fabrication processes, the shape of an edge in most cases is not a simple 90° profile. Various different edge profiles could be analyzed to determine how the stopping point Δ is changed. We chose a 45° edge as a sort of worst case scenario, giving an idea of the magnitudes of changes in Δ for different edge shapes. Results for the 45° edge are also shown in Figure 5.

From Figure 5, it is seen that the 45° and 90° curves are essentially identical for $t/\delta_{sk} < 3$. However, for $t/\delta_{sk} > 3$ the effects of the sharper edge start to become important and the two curves begin to diverge. This tells us that as long as $t/\delta_{sk} < 3$,

then the effects of the different edge shapes are negligible.

5. Calculation for the Attenuation Constant: Derived From the Stopping Point Δ

To determine the attenuation constant, the standard wall-loss perturbation analysis is performed on a general wave-guiding structure. Reciprocity can be used to obtain the standard equation for the change in the two propagation constants:

$$\gamma_m - \gamma_{mo} = - \frac{\oint_C (\bar{\mathcal{E}}_{-m} \times \bar{\mathcal{H}}_{mo}) \cdot \bar{a}_n dl}{\int_{S_o} (\bar{\mathcal{E}}_{mo} \times \bar{\mathcal{H}}_{-m} - \bar{\mathcal{E}}_{-m} \times \bar{\mathcal{H}}_{mo}) \cdot \bar{a}_z dS} \quad (103)$$

where $\bar{\mathcal{E}}_{mo}$, $\bar{\mathcal{E}}_{mo}$, and γ_{mo} are the fields and propagation constant of modes that correspond to the waveguide with perfectly conducting walls. $\bar{\mathcal{E}}_{-m}$ and $\bar{\mathcal{H}}_{-m}$ are the fields of a backward traveling mode of a guide with conducting (not perfectly conducting) walls; γ_m is the propagation constant of the conducting-wall guide.

The perturbation procedure involves assuming that the fields associated with the guide that has conducting walls are approximately the same as the fields

associated with perfectly conducting walls, as far as the denominator of (103) is concerned:

$$\gamma_m - \gamma_{mo} = - \frac{\oint_G \bar{\mathcal{E}}_{-m} \cdot (\bar{a}_n \times \bar{\mathcal{H}}_{mo}) dl}{2 \int_{S_o} (\bar{\mathcal{E}}_{mo} \times \bar{\mathcal{H}}_{mo}) \cdot \bar{a}_z dS} \quad (104)$$

The denominator of this equation can be defined in terms of the power carried by the mode

$$P = \frac{1}{2} \int_{S_o} (\bar{\mathcal{E}}_{mo} \times \bar{\mathcal{H}}_{mo}) \cdot \bar{a}_z dS = \frac{Z_o I^2}{2} \quad (105)$$

where Z_o is a "power/current" definition of characteristic impedance, and I is the total current. It should be noted that a quasi-TEM (transverse electromagnetic) mode is assumed in the waveguide. Thus the current on the conductor is approximated as

$$\bar{J}_s \simeq \bar{a}_z J \quad (106)$$

and as a result, Z_o is the characteristic impedance of a quasi-TEM line.

The numerator of equation (104) is calculated by means of the stopping point Δ . It is assumed that the positive conductor shown in Figure 1 is infinitely thin, and as a result the contour in the numerator will reduce to line integrals along the top and bottom side of the positive conductor only. With the definition given in equation (104), the difference in the propagation constants can be given as

$$\gamma_m - \gamma_{mo} \simeq - \frac{\int_{top} \bar{\mathcal{E}}_{-m}^{top} \cdot (\bar{a}_n \times \bar{\mathcal{H}}_{mo}^{top}) dl}{2Z_o I^2} - \frac{\int_{bottom} \bar{\mathcal{E}}_{-m}^{bottom} \cdot (\bar{a}_n \times \bar{\mathcal{H}}_{mo}^{bottom}) dl}{2Z_o I^2} \quad (107)$$

If the currents for an infinitely thin conductor are used in this equation, then the integral will become singular. The Lewin/Vainshtein philosophy says that instead of evaluating the integral at the end (where it is singular) the limits of this integral must be evaluated at some point just before the edge (the stopping point Δ). Therefore equation (107) is written as

$$\gamma_m - \gamma_{mo} \simeq - \frac{\int_{C_\Delta} \bar{\mathcal{E}}_{-m}^{top} \cdot (\bar{a}_n \times \bar{\mathcal{H}}_{mo}^{top}) dl}{2Z_o I^2} - \frac{\int_{C_\Delta} \bar{\mathcal{E}}_{-m}^{bottom} \cdot (\bar{a}_n \times \bar{\mathcal{H}}_{mo}^{bottom}) dl}{2Z_o I^2} \quad (108)$$

For example, in a microstrip, C_Δ is defined between $-W + \Delta$ and $W - \Delta$, as shown in Figure 2.

The first term in the boundary condition given in equation (46) can be neglected because it is a purely imaginary number which will contribute nothing to the power loss. Therefore, the generalized impedance boundary condition can be written as

$$\begin{aligned} \bar{E}_{tan}^{top} &= Z_s \bar{J}^{top} + Z_m \bar{J}^{bottom} \\ \bar{E}_{tan}^{bottom} &= Z_s \bar{J}^{bottom} + Z_m \bar{J}^{top} \end{aligned} \quad (109)$$

Instead of working with both the top and bottom currents, it is preferred to work with the total and difference current on the conductor:

$$\bar{J}_s^{top} = \frac{1}{2} \bar{J}_s - \delta \bar{J}_s, \quad \bar{J}_s^{bottom} = \frac{1}{2} \bar{J}_s + \delta \bar{J}_s \quad (110)$$

where $\delta \bar{J}_s$ is the difference currents. It should be noted that since it was assumed that $\bar{J}_s$ is approximated by $\bar{J}_s \simeq \bar{a}_z J$, then $\delta \bar{J}_s$ is approximated by

$$\delta \bar{J}_s \simeq \bar{a}_z \delta J$$

With the current defined as in equation (110), the Horton boundary conditions can be written as

$$\begin{aligned} \bar{E}_{tan}^{top} &= \frac{Z_s + Z_m}{2} \bar{J}_s - \frac{Z_s - Z_m}{2} \delta \bar{J}_s \\ \bar{E}_{tan}^{bottom} &= \frac{Z_s + Z_m}{2} \bar{J}_s + \frac{Z_s - Z_m}{2} \delta \bar{J}_s \end{aligned} \quad (111)$$

and once substituted into equation (108), the following is obtained:

$$\begin{aligned} \gamma_m - \gamma_{mo} \simeq & \frac{Z_s + Z_m}{4Z_o} \int_{C_\Delta} \left(\frac{J}{I} \right)^2 dl \\ & + \frac{Z_s - Z_m}{2Z_o} \int_{C_o} \left(\frac{\delta J}{I} \right)^2 dl \end{aligned} \quad (112)$$

Note that the second integral can be extended over all of C_o because this integral is not singular at the edges.

By taking the real part of this equation, an expression for the attenuation constant of a planar line with finite conductivity is obtained:

$$\alpha_c = \text{Re}(\gamma_m - \gamma_{mo}) \quad (113)$$

Now suppose conditions are such that δJ can be neglected. This is true for quasi-TEM CPW and for sufficiently narrow microstrips ($Z_o > 40\Omega$). Then the attenuation for such a line is simply

$$\alpha \simeq \frac{\text{Re}(Z_s + Z_m)}{4Z_o} \int_{C_\Delta} \left(\frac{J}{I} \right)^2 dl = \frac{R_{tm}}{4Z_o} \int_{C_\Delta} \left(\frac{J}{I} \right)^2 dl \quad (114)$$

If the current distribution of a planar circuit is known, then the attenuation constant for that structure can be determined through equation (114). This paper will demonstrate how equation (114) can be used to determine the conductor loss for a microstrip line.

6. Loss Calculation for Microstrip

Inside the strip there is a possibility of having all three vector components of current: J_x , J_y , and J_z . However, in most planar circuits the J_z component of the current is the dominant component, as shown by *Schumacher* [1979], *Kobayashi* [1985], and *Butler* [1984]. *Schumacher* [1979] shows that, up to 12 GHz, the ratio $|J_z|_{\max}/|J_x|_{\max} = 0.1$. When this ratio is squared, the result is only 1.0% of the total power loss. *Kobayashi* [1985] also shows that, even as high as 20 GHz, the ratio $|J_z|_{\max}/|J_x|_{\max} = 0.15$.

This shows that by neglecting the J_x and J_y components of the current, only a small amount of error is accumulated in the calculation of the power loss. Therefore in this analysis we will assume that only the J_z currents are present in the conducting strip.

In this section the microstrip geometry shown in Figure 1 will be investigated. There are several known current distributions that can be used in equation (114). For this analysis Maxwell's distribution will be used, as it is accurate for sufficiently narrow strips:

$$J = \frac{I}{\sqrt{(W/2)^2 - x^2}} \quad (115)$$

As stated above, C_Δ is defined between $-(W/2) + \Delta$ and $(W/2) - \Delta$, and since J is symmetric, the integral is simply

$$\begin{aligned} \int_{-W/2+\Delta}^{W/2-\Delta} \left(\frac{J}{I}\right)^2 dx &= \frac{2}{\pi^2} \int_0^{W/2-\Delta} \frac{dx}{(W/2)^2 - x^2} \\ &= \frac{1}{\pi^2(W/2)} \ln \left(\frac{W}{\Delta} - 1 \right) \end{aligned} \quad (116)$$

From (114) we get the following expression for the conductor loss in the strip:

$$\alpha \simeq \frac{R_{sm}}{Z_o} \frac{\ln \left(\frac{W}{\Delta} - 1 \right)}{2\pi^2 W} \quad (117)$$

with R_{sm} defined in equation (100).

Comparison of the predictions of (117) with other results in the literature will now be made. The

first case is a microstrip on a GaAs substrate with $H = 200 \mu\text{m}$, $\tan\delta = 3 \times 10^{-4}$ and $\epsilon_r = 12.9$, a ground plane with $\sigma = 3.333 \times 10^7 \text{ S/m}$, and a strip with $\sigma = 3.333 \times 10^7 \text{ S/m}$. Figure 6 shows the attenuation constant for a wide range of strip thicknesses (t) for $f = 10 \text{ GHz}$. Also shown in Figure 6 are the theoretical results that *Heinrich* [1990] obtained from a mode matching technique for this same geometry. The results presented here agree well with *Heinrich's* theoretical results.

Next we compare our results with published experimental results. The first case is that analyzed by *Finlay et al.* [1988]. This structure is similar to the microstrip shown in Figure 1 with the exception that there is a lossy cover layer over the whole structure. The microstrip line has a GaAs substrate ($\epsilon_r = 12.9$, $\tan\delta = 3 \times 10^{-4}$, and $H = 200 \mu\text{m}$), a dielectric cover layer ($\epsilon_r = 3.4$, $\tan\delta = 0.05$), a ground plane with $\sigma = 1.77 \times 10^7 [\Omega \cdot \text{m}]^{-1}$, and a strip with $W = 30 \mu\text{m}$, $t = 6 \mu\text{m}$ and $\sigma = 1.77 \times 10^7 [\Omega \cdot \text{m}]^{-1}$. We plot our theoretical results for this case in Figure 7, along with the experimental results from *Finlay*.

The next two geometries have no cover layer and are composed of a GaAs substrate with $H = 100 \mu\text{m}$, $\tan\delta = 3 \times 10^{-4}$ and $\epsilon_r = 12.9$, a ground plane with $\sigma = 4.1 \times 10^7 \text{ S/m}$, and a strip with $t = 3 \mu\text{m}$ and $\sigma = 4.1 \times 10^7 \text{ S/m}$. Figures 8 and 9 show attenuation constants for a wide range of frequencies for $W = 70 \mu\text{m}$ ($Z_o = 51.2\Omega$) and $W = 20 \mu\text{m}$ ($Z_o = 79.3\Omega$) respectively. Also shown in Figures 8 and 9 are experimental results obtained from *Goldfarb and Platzker* [1990] for these same geometries.

The results for narrow strips agree very well with the experimental value; however, for the 51.2Ω line (Figure 8), the agreement is not so good. This is explained by recalling that the expression in equation (114) is only for the strip loss, and the ground plane loss is not included. For narrow microstrips the ground plane loss is a small part of the total loss, whereas for wide microstrips the ground plane loss becomes more important. For $W/H \gg 1$ the ground plane loss will nearly equal the strip loss. By determining a closed form expression for the current on the ground plane of a microstrip, we have been able to study this point in detail. This work will be the topic of a separate publication (C. L. Holloway and E. F. Kuester, Closed-form expressions for the current density on the ground plane of a microstrip line, submitted to *IEEE Trans. on Microwave Theory and Tech.*, 1994).

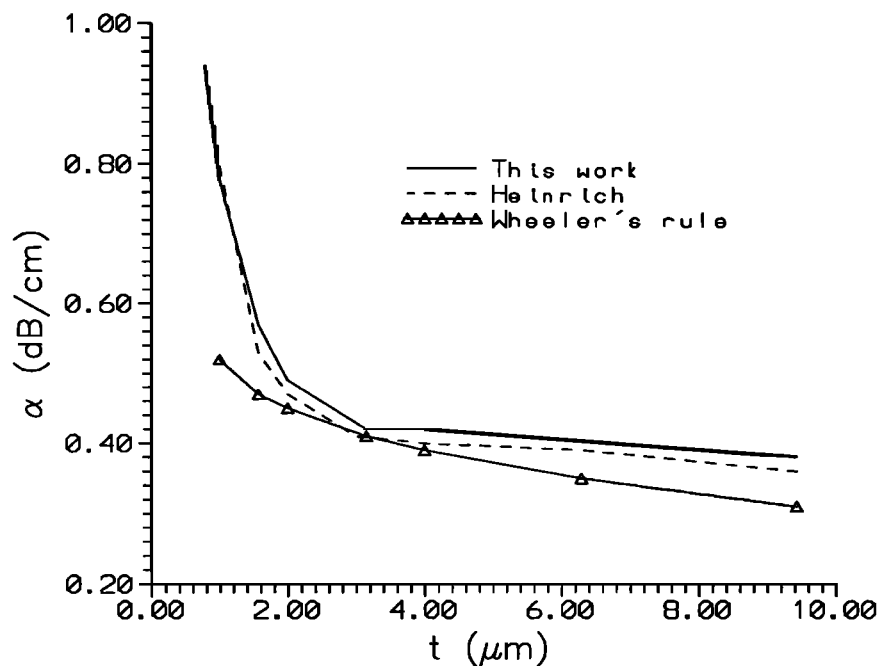


Figure 6. Strip loss of a microstrip for a range of t with $W = 30 \mu\text{m}$, $H = 200 \mu\text{m}$, $\sigma = 3.333 \times 10^7$, and $\epsilon_r = 12.9$.

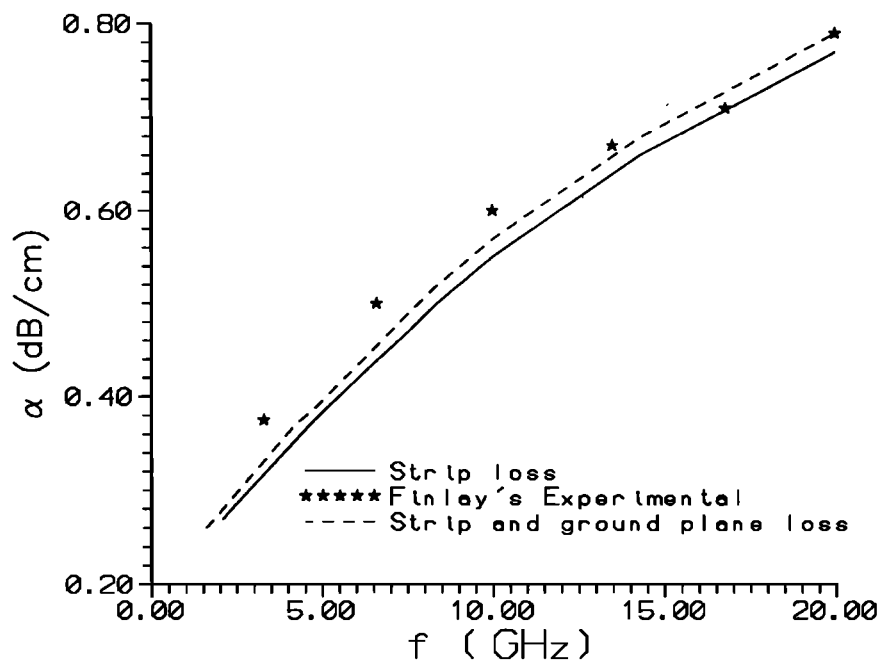


Figure 7. Strip conductor loss of a microstrip for a range of frequencies with $W = 30 \mu\text{m}$, $H = 200 \mu\text{m}$, $t = 6 \mu\text{m}$, $\sigma = 1.77 \times 10^7$, and $\epsilon_r = 12.9$.

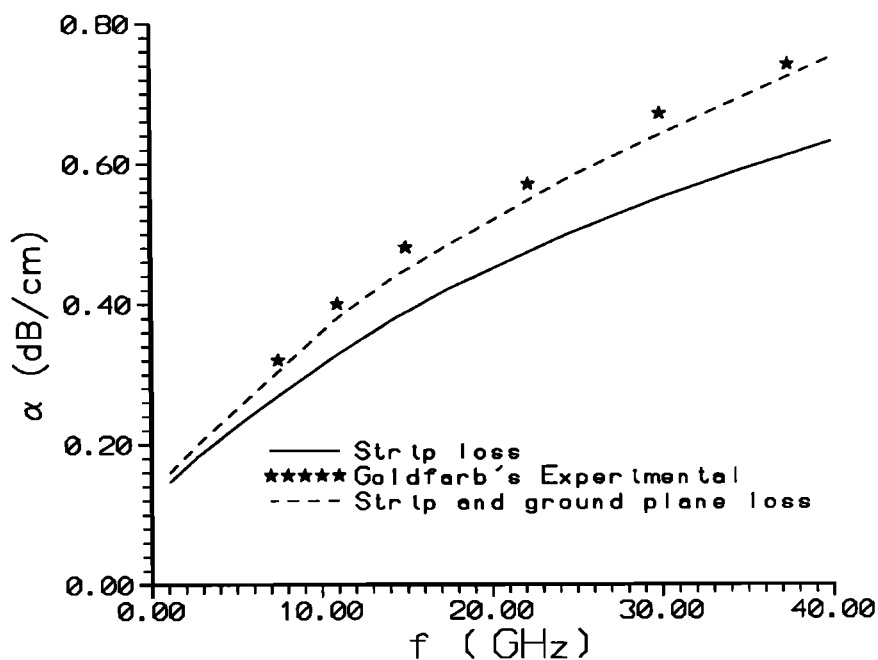


Figure 8. Strip conductor loss of a microstrip for a range of frequencies with $W = 70 \mu\text{m}$, $H = 100 \mu\text{m}$, $t = 3 \mu\text{m}$, $\sigma = 4.1 \times 10^7$, and $\epsilon_r = 12.9$.

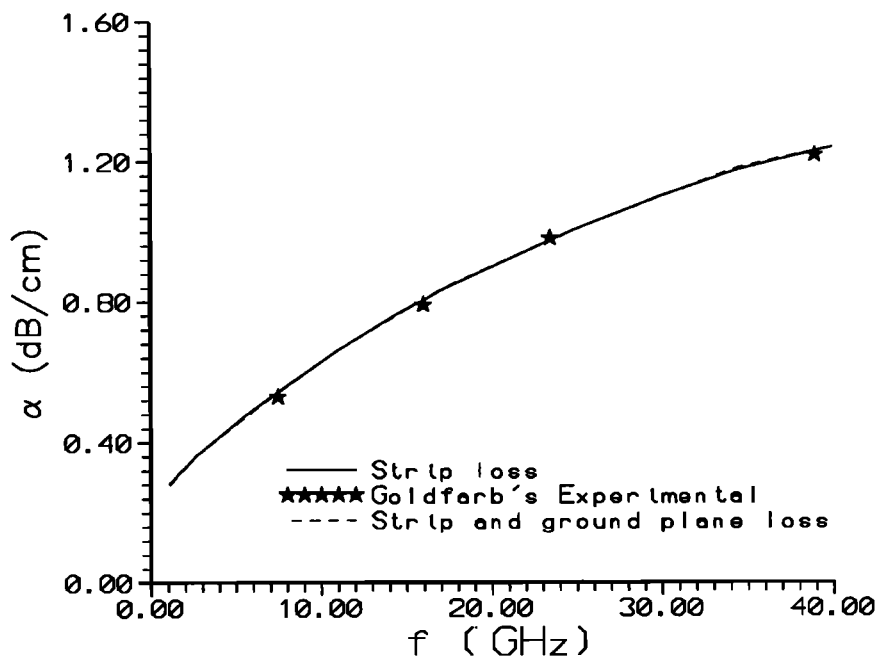


Figure 9. Strip conductor loss of a microstrip for a range of frequencies with $W = 20 \mu\text{m}$, $H = 100 \mu\text{m}$, $t = 3 \mu\text{m}$, $\sigma = 4.1 \times 10^7$, and $\epsilon_r = 12.9$.

Also shown in Figures 7-9 are results for the attenuation when both the strip and ground plane losses are included. As expected, for the narrow strips the ground plane losses do not change the results very much. (It should be noted that the results in Figure 7 do not agree as well as one would like; however, there is a highly lossy cover layer in the structure whose dielectric losses have not been taken into account in our theory.) On the other hand, for the wider strip, when the ground plane loss is included, the predicted conductor loss correlates very well with the experimental values.

Let us compare the results of our model to results obtained from Wheeler's incremental inductance rule. Figure 6 shows the comparison for a microstrip with $Z_o = 86\Omega$ for a wide range of strip thickness. Figures 10 and 11 show the comparison for a wide range of frequencies for $Z_o = 50\Omega$ and $Z_o = 80\Omega$, respectively. These figures show that Wheeler's rule predicts losses between 12% and 30% lower than those obtained from both our model and from experimental results.

Once it has been demonstrated that the technique presented here correctly predicts the loss for the 90° edges, the effects of conductor loss with sharper edges can be investigated. As stated above, by analyzing the most drastic edge shapes (say 45°), it will be pos-

sible to determine the worst case scenario. Figure 12 shows a comparison of the expected loss for both a 90° and 45° edge for the microstrip for a range of thicknesses t at $f = 10$ GHz, with $W = 30\mu\text{m}$, $H = 200\mu\text{m}$, $\sigma = 3.333 \times 10^7$, and $\epsilon_r = 12.9$. Other comparisons for these two different edges are shown in Figures 13 ($Z_o = 80\Omega$) and 14 ($Z_o = 50\Omega$) for a range of frequencies, for $t = 3\mu\text{m}$. It is seen from all these curves that for small t/δ_{sk} the 45° edge does not increase loss over the loss of the 90° by a significant amount. However, when t/δ_{sk} gets large the edge shape can be important and may need to be taken into account when calculating the conductor loss.

7. Conclusion

In this paper we have been able to develop an expression for the conductor loss of a planar line valid for any thickness and edge shape. This expression involves integrating the current profile up to some stopping point Δ . Upon applying this stopping point technique to microstrip, excellent agreement with experimental values was demonstrated. It should be noted that we have also analyzed CPW structures, and the same favorable agreement has been obtained. These results will be published separately.

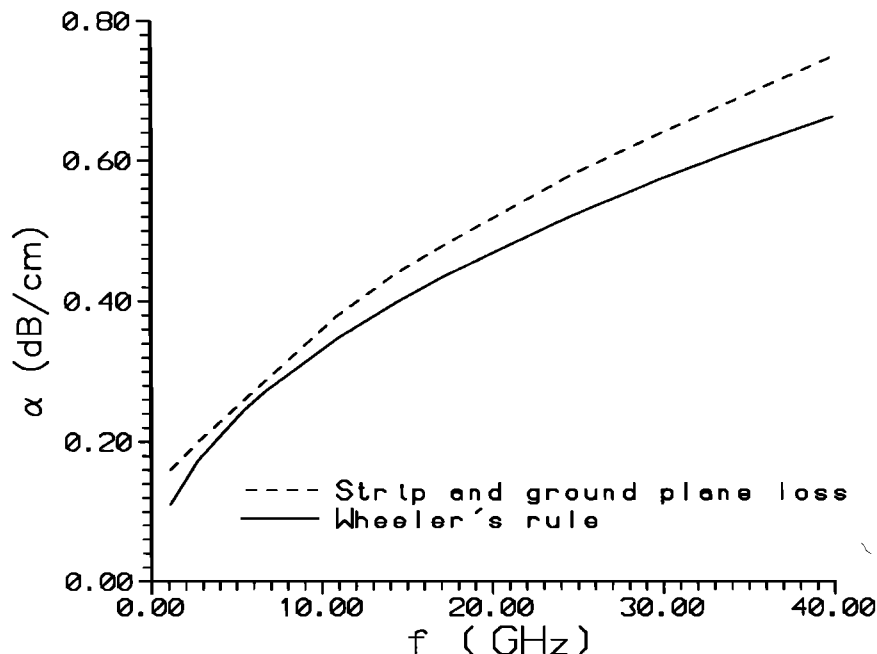


Figure 10. Comparison of our results to the results obtained from Wheeler's rule for $Z_o = 50\Omega$.

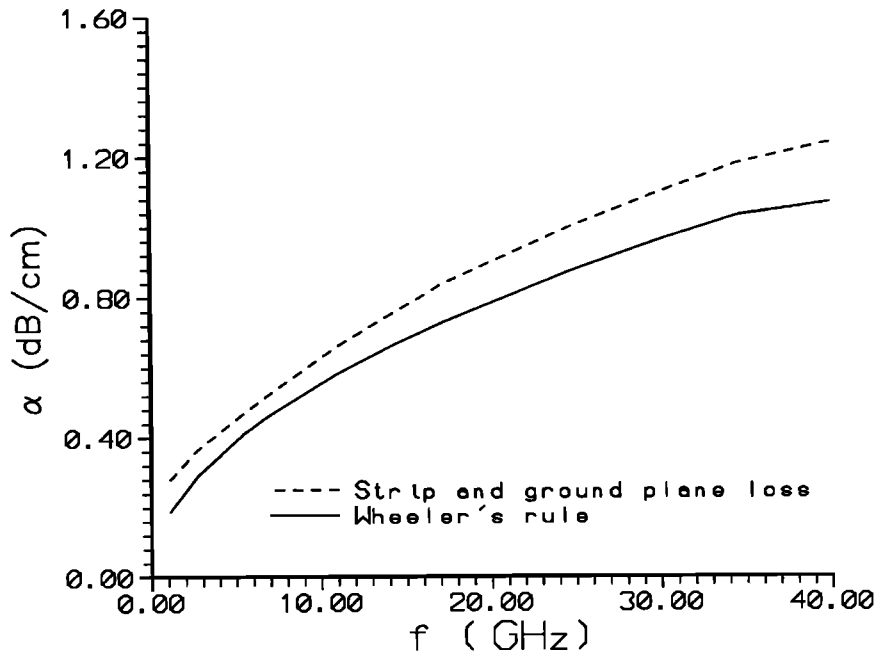


Figure 11. Comparison of our results to the results obtained from Wheeler's rule for $Z_0 = 80\Omega$.

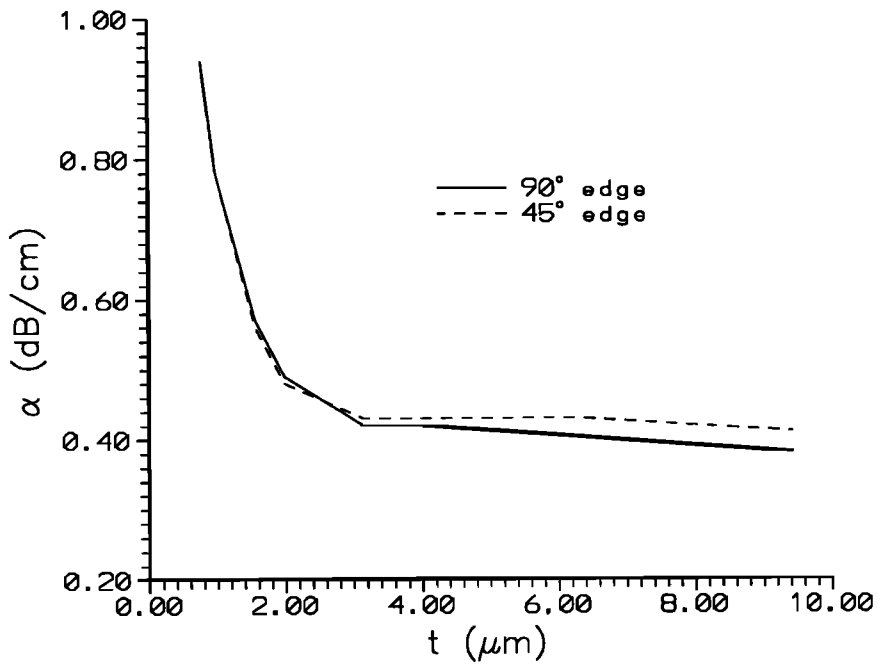


Figure 12. Comparison for conductor loss for both the 90° and 45° edge for microstrips with a range of t at $f = 10$ GHz with $W = 30\ \mu\text{m}$, $H = 200\ \mu\text{m}$, $\sigma = 3.333 \times 10^7$, and $\epsilon_r = 12.9$.

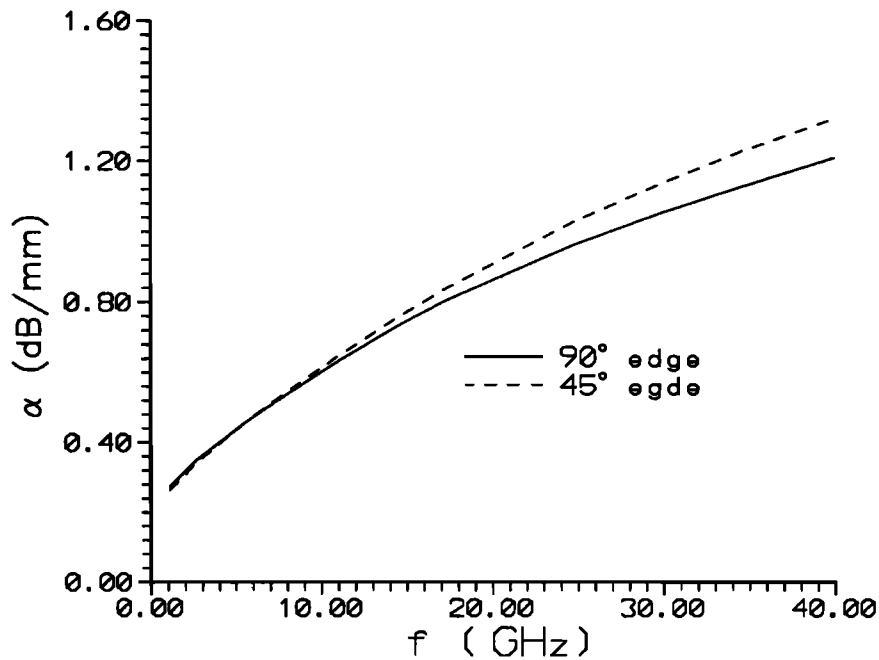


Figure 13. Comparison for conductor loss for both the 90° and 45° edge for microstrips with a range of frequencies with $t = 3 \mu\text{m}$, $W = 20 \mu\text{m}$, $H = 100 \mu\text{m}$, $\sigma = 4.1 \times 10^7$, and $\epsilon_r = 12.9$.

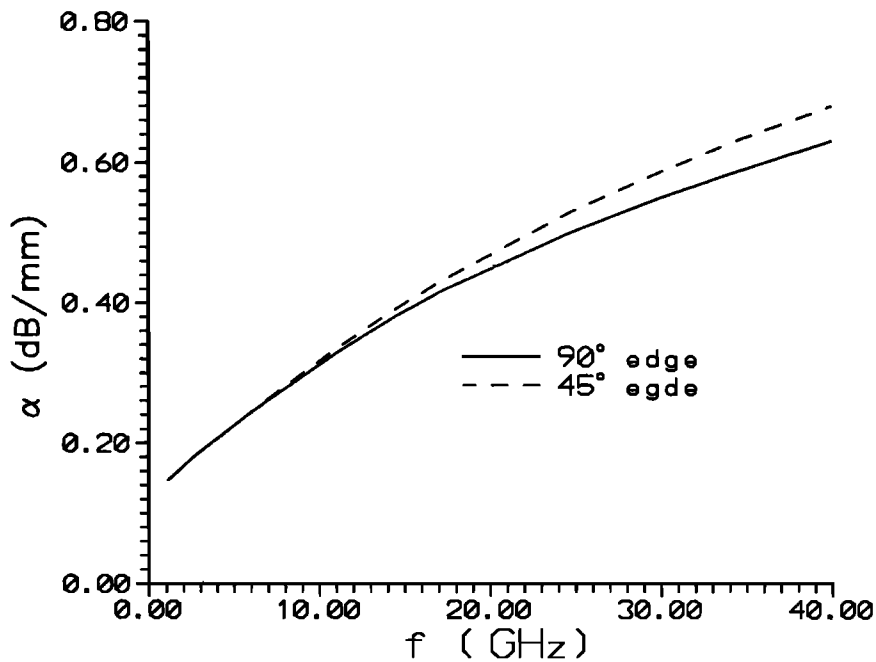


Figure 14. Comparison for conductor loss for both the 90° and 45° edge for microstrips with a range of frequencies with $t = 3 \mu\text{m}$, $W = 70 \mu\text{m}$, $h = 100 \mu\text{m}$, $\sigma = 4.1 \times 10^7$, and $\epsilon_r = 12.9$.

More important, we have been able to address the question of how different edge shapes affect power dissipation. It was demonstrated in Figures 5 and 12-14 that even with as drastic an edge as 45° , little additional power loss results when $t/\delta_{sk} \leq 3$. However, when $t/\delta_{sk} \geq 3$, the edge shape may be important. Therefore it is concluded that the edge shape can indeed alter the loss associated with the microstrip and may need to be taken into account depending on the application.

The results obtained in this study are presently being used to develop an effective surface impedance that will take into account this edge effect. This surface impedance is currently being implemented in a moment method code. In closing, it should be pointed out that the work presented here is general enough to allow the analysis of conductors with multiple-metallization layers (commonly found in microwave and millimeter-wave integrated circuits (MIMICs)). This will be the topic of a future publication.

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