J. Opl Res. Soc. Vol. 35, No. 1, pp. 69–74, 1984
Printed in Great Britain. All rights reserved
0160-5682/84 \$3.00 + 0.00

Linear Programming with Special Ordered Sets

R. D. SNYDER
Monash University, Victoria, Australia

Concave objective functions which are both piecewise linear and separable are often encountered in a wide variety of management science problems. Provided the constraints are linear, problems of this kind are normally forced into a linear programming mould and solved using the simplex method. This paper takes another look at the associated linear programs and shows that they have special structural features which are not exploited by the simplex algorithm. It suggests that their variables can be divided into special ordered sets which can then be used to guide the pivoting strategies of the simplex algorithm with a resultant reduction in basis changes.

Key words: linear programming, non-linear programming, production planning, statistics

INTRODUCTION

THIS PAPER considers mathematical programming problems involving the maximization of a concave, separable, piecewise linear objective function subject to linear constraints. They arise in a variety of situations where, for example, holding and backlog costs are associated with stock levels; costs are linked to adjustments in the level of production; hiring and firing costs are incurred with changes in the level of employment; penalty payments are made for working overtime; the absolute error criterion is used in regression analysis;[1–7] goal programming is utilized for the case of multiple objectives;[8–10] and separable programming is employed with non-linear objective functions.[11–13] They also encompass those phase I methods[14–16] for finding initial, feasible solutions to linear programmes which implicitly employ a 'big-M' approach.

Normally problems with piecewise linear objectives are converted to equivalent linear programmes and then solved with the simplex algorithm.[17] The paper suggests improvements to this procedure which exploit certain special structural features in the problems. More specifically, special ordered sets are used to control the pivoting operations and reduce the number of basis changes in the simplex algorithm.

LINEAR PROGRAMMING WITH PIECEWISE LINEAR OBJECTIVE FUNCTION

Consider a problem involving n decision variables $x_1, \dots, x_n$ which are related by the m equations

$$Ax = b, \tag{1.1}$$

where A is an $m \times n$ matrix; x is an $n \times 1$ vector of the variables $x_1, \dots, x_n$, and b is an $m \times 1$ vector of the right-hand side coefficients. The aim is to select x so as to maximize the additively separable function

$$v = \sum_{j=1}^{n} f_j(x_j), \tag{1.2}$$

where the typical component $f_j(x_j)$ is a piecewise linear function of x_j consisting of n_j linear segments with slopes $c_{j1} \dots, c_{jn_j}$ respectively. It is assumed that each component is also concave so that the segment gradients satisfy the condition

$$c_{j1} \geqslant c_{j2} \geqslant \dots \geqslant c_{jn_j}. \tag{1.3}$$

The problem will be called a concave piecewise linear programme (P.W.L.P.).

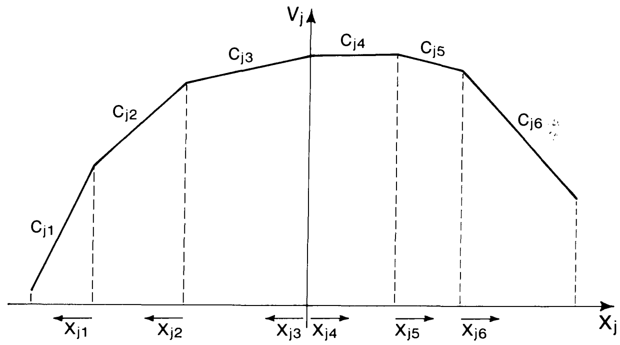

FIG. 1. *Concave piecewise linear function.*

It is possible to construct a linear programme whose optimal, feasible solution is also the optimal, feasible solution of the P.W.L.P. To this end, consider a typical graph of the component $f_j(x_j)$ shown in Figure 1. It will be convenient to adopt the following unusual convention for measuring distances along the x_j-axis. The distance between two points to the right (left) of the origin will be the positive (negative) difference between them. The distance between points on opposite sides of the origin shall remain undefined. Note that the x_j-axis can be partitioned into intervals corresponding to the segments of the function. Thus the intervals to the left of the origin have negative lengths. Now associate the variable x_{jk} with the typical interval k lying to the right (left) of the origin which measures the distance from the lower (upper) bound of the interval and which is bounded from above (below) by the length of the interval. The variable x_j can therefore be represented by the sum of the ordered set of component variables $x_{j1}, x_{j2}, \ldots, x_{jn_j}$ provided that:

(1) if a component is at its upper bound, then all predecessors equal their upper bounds;
(2) if a component is at its lower bound, then all successors equal their lower bounds.

A set of components satisfying these conditions will be called a special ordered set. It is equivalent to type-2 special ordered sets used by Beale and Forrest[18] in the context of integer programming to represent non-concave piecewise linear functions.

Consider the linear programme

$$\text{Maximize } v = \sum_{j=1}^{n} \sum_{k=1}^{n_j} c_{jk}\, x_{jk} \tag{2.1}$$

$$\text{subject to} \sum_{j=1}^{n} \sum_{k=1}^{n_j} a_{ij}\, x_{jk} = b_i \quad (i = 1, \ldots, m) \tag{2.2}$$

$$l_{jk} \leqslant x_{jk} \leqslant u_{jk} \quad (j = 1, \ldots, n;\ k = 1, \ldots, n_j), \tag{2.3}$$

where l_{jk} and u_{jk} are lower and upper bounds whose values depend on whether interval k lies to the left or right of the origin of the x_j-axis. More specifically, if the interval lies to the right of the origin, then l_{jk} equals zero and u_{jk} equals the length of the interval. In the second case the reverse is true. It is not difficult to show that the optimal, feasible solution of this linear programme must satisfy the special order properties. As a consequence it is also the optimal, feasible solution to the P.W.L.P.

To illustrate these points, consider an example where a company uses two resources, labour and materials, to make two products, PRODA and PRODB. The technology is linear with the standard production requirements shown in Table 1. When selling its products the company can discriminate on the basis of price, and this leads to a segmentation of its markets. Tables 2 and 3 show the anticipated capacities and prevailing prices for each market segment during the coming week. Currently, the company possesses 5000 kilograms of materials, with a market value of 2 dollars per kilogram. It has 15 employees, who are guaranteed a 40 hour working week but, where necessary, work overtime at the penalty rates shown in Table 4. The problem is to find a production and selling plan which maximizes *controllable* profit.

TABLE 1. STANDARD PRODUCTION REQUIREMENTS

Factors of production	Product: PRODA (kg)	PRODB (kg)
Labour (hr)	0.16	0.20
Materials (kg)	2.00	1.00

TABLE 2. MARKET FOR PRODA

Market segment	Capacity (kg)	Price (dollars/kg)
1	500	10.00
2	1000	9.00
3	1000	8.00

TABLE 3. MARKET FOR PRODB

Market segment	Capacity (kg)	Price (dollars/kg)
1	700	7.00
2	800	6.50
3	900	6.00
4	5000	5.50

TABLE 4. PAY RATES PER EMPLOYEE

Work period	Amount of time (hr)	Pay rates (dollars/hr)
Normal	40	5.00
Overtime	10	7.50
Double-time	10	10.00

The problem can be formulated as a linear programme with bounded variables. Letting

A_1, A_2, A_3	be the quantities of PRODA sold in markets 1, 2 and 3;
B_1, B_2, B_3, B_4	be the quantities of PRODB sold in markets 1, 2, 3 and 4;
L_1, L_2, L_3	be the quantities of normal, overtime, and double time labour;
M	be the quantity of materials;
Z_1, Z_2	be artificial variables.

the resulting initial tableau for the problem is given in Table 5. The problem was solved with the conventional simplex algorithm to give the results shown in Table 6. In all, it took 11 iterations to obtain the optimal, feasible solution. Of these, eight were normal basis changes, while the remainder were bound substitutions.

AN AUGMENTED SIMPLEX ALGORITHM

It is interesting to note that the initial tableau in Table 5 possesses a special structural feature. More specifically, the columns in the constraint section of the tableau are duplicated for variables belonging to the same special ordered set. The simplex method fails to exploit this structure, as can be seen by examining the tableau at the beginning of the eighth iteration in Table 7. The simplex method selects B_3 as the new driving variable because it is the variable with the most negative reduced cost not currently at its upper bound. It is increased until a variable affected by the change reaches one of its bounds. In this case L_1 is the first variable to become critical. Thus the simplex method swaps L_1 with B_3 in the basis and then effectively applies a full simplex transformation to the tableau.

Closer examination of this process suggests that it is quite inefficient. The problem at this stage of the algorithm is that there is a shortage of labour once normal labour is fully utilized. But labour can work overtime, in which case it is possible to expand the production of PRODB and hence the sales B_3 even further. In other words, it seems sensible to replace L_1 by L_2 in the basis. Note that because they both belong to the same special ordered set, their columns in the constraint section of the tableau are identical. Hence a swap with an adjacent variable from the same set does not change the basis, and so no simplex transformation has to be applied to the derived constraint coefficients.

TABLE 5. INITIAL TABLEAU FOR LINEAR PROGRAMME

Basic	A_1	A_2	A_3	B_1	B_2	B_3	B_4	L_1	L_2	L_3	M	RHS	Slack
z_1	0.16	0.16	0.16	0.20	0.20	0.20	0.20	−1.00	−1.00	−1.00	0.00	0	0
z_2	2.00	2.00	2.00	1.00	1.00	1.00	1.00	0.00	0.00	0.00	−1.00	0	0
Reduced costs	−10.00	−9.00	−8.00	−7.00	−6.50	−6.00	−5.50	0.00	7.50	10.00	2.00		
Current value	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00		
Slack	500	1000	1000	700	800	900	5000	600	150	150	5000		

TABLE 6. OPTIMAL FEASIBLE PRODUCTION AND SELLING PLAN

A_1	A_2	A_3	B_1	B_2	B_3	B_4	L_1	L_2	L_3	M
500	542	0	700	800	900	517	600	150	0	5000
$A = 1042$			$B = 2917$				$L = 750$			

TABLE 7. TABLEAU FOR EIGHTH ITERATION

Basic	A_1	A_2	A_3	B_1	B_2	B_3	B_4	L_1	L_2	L_3	M	RHS	Slack
L_1	0.00	0.00	0.00	−0.12	−0.12	−0.12	−0.12	1.00	1.00	1.00	−0.08	580.00	20.00
A_3	1.00	1.00	1.00	0.50	0.50	0.50	0.50	0.00	0.00	0.00	−0.08	250.00	750.00
Reduced cost	−2.00	−1.00	0.00	−3.00	−2.50	−2.00	−1.50	0.00	7.50	10.00	−2.00		
Current values	500	1000	250	700	800	0	0	580	0	0	5000		
Slack	0	0	750	0	0	900	5000	20	150	150	0		

Assuming that L_1 is swapped with L_2 in the basis, the cost of labour increases from 0 to 7.50 dollars. A change of this kind must affect the reduced cost of B_3. Now each extra kilogram of B_3 requires an additional 0.12 manhours. Hence the unit cost of B_3 increases by 0.9 dollars, thereby changing the reduced cost of B_3 from -2.0 to -1.1. Thus, after swapping L_1 with L_2, it pays to increase the driving variable further.

The example suggests that the conventional simplex method fails to exploit the special structural features inherent in the linear programming equivalent of the P.W.L.P. problem. More specifically, there is a duplication of constraint columns for variables from the same special ordered set. Hence, when a basic variable becomes non-basic, it is desirable to replace it with an adjacent variable from the same special ordered set to avoid changing the basis. A modified version of the simplex method which does this is described below. To avoid some details which conceal the major ideas, the following assumptions are made:

(a) The problem under consideration possesses a bounded, optimal, feasible solution.
(b) Each variable x_j lies in the region $-\infty \leqslant x_j \leqslant \infty$. If this is not the case, it is always possible to extend the domain of the piecewise linear function as follows. If $x_j \geqslant l_j$, where l_j is a lower bound, add a segment in the region $x_j \leqslant l_j$ with a slope of M, where M is a very large number. If $x_j \leqslant u_j$, where u_j is an upper bound, add a segment in the region $x_j \geqslant u_j$ with a slope of $-$M.

At each iteration, the modified simplex algorithm incorporates the following steps:

1. *Selection of the pivot non-basic variable*
   This is identical to the corresponding step in the conventional simplex method for linear programming with bounded variables.
   (a) Of all the non-basic variables at their lower bounds, select that one with the most negative reduced cost.
   (b) Of all the non-basic variables at their upper bounds, select that one with the most positive reduced cost.

   If no variables satisfying the conditions of (a) and (b) are found, then the solution is optimal and the algorithm is terminated. Otherwise, select the result of (a) and (b) with the largest absolute reduced cost as the pivot non-basic variable. It will be called the driving variable.
2. *The transformation*
   Change the driving variable in its direction of improvement until either
   (a) a basic variable reaches one of its bounds. Replace the critical basis variable with the adjacent variable from its special ordered set and revise the reduced cost of the driving variable. This can be done without changing the basis because both variables, being from the same special ordered set, possess identical constraint columns, or
   (b) the driving variable reaches a bound. In this case replace the driving variable with the adjacent variable from its special ordered set and revise the reduced cost of the driving variable.
   Repeat this process until the reduced cost changes sign. If this occurs in step (a), exchange the critical basic variable with the driving variable using the normal basis change routine.

COMPUTATIONAL EXPERIENCE

To gauge the value of these modifications, the augmented simplex algorithm was applied to the production problem. It obtained the optimal, feasible solution with only four, rather than eight, full basis changes. This is a significant computational saving in such a small problem

In addition, it was applied to a typical, two-parameter linear regression problem using the sum of absolute errors criterion.[19] Involving 30 observations, or equivalently 30 constraints, it obtained the answer with only three basis changes. In contrast, the conventional simplex algorithm took over 17 changes. It is important to note that the augmented simplex algorithm collapses to a special method developed by Davies[4] and Barrodale and Roberts[5,6] for regression analysis. This result confirms their findings that significant computational savings can be made by explicitly recognizing the piecewise linear nature of the reduced objective functions in these circumstances.

FINAL REMARKS

This paper has demonstrated that the special ordered set concept can be internalized within the simplex algorithm to find solutions to concave, piecewise linear programming problems. Extensive testing on large-scale problems remains to be done, but the small examples used in this paper indicate that the modified simplex algorithm can lead to quite large reductions in the number of basis changes. The working paper[19] gives a more comprehensive discussion of the theory underlying piecewise linear programmes. It also outlines the augmented simplex algorithm in greater detail, highlighting various strategies that further reduce computational loads.

ACKNOWLEDGEMENTS

I would like to thank Professor A. Land of the London School of Economics and Political Science and Dr S. Powell of the University of Kent for their considerable advice and support while working on the contents of this paper. Also, it should be noted that since writing the paper, other closely related work by Fourer[20] and Premoli[21] have appeared. Private communications with Dr K. McKinnon of the University of Edinburgh indicate that he has also been active in the area.

REFERENCES

1 O. J. Karst (1958) Linear curve fitting using least deviations. *J. Am. statist. Ass.* **53,** 118–132.
2 A. Charnes, W. W. Cooper and R. O. Ferguson (1955) Optimal estimation of executive compensation by linear programming. *Mgmt Sci.* **1,** 138–151.
3 H. M. Wagner (1959) Linear programming techniques for regression analysis. *J. Am. statist. Ass.* **54,** 206–212.
4 M. Davies (1967) Linear approximation using the criterion of least total deviations. *J. R. statist. Soc. (B)* **29,** 101–109.
5 I. Barrodale and F. D. K. Roberts (1973) An improved algorithm for discrete l_1 linear approximations. *SIAM J. numer. Anal.* **10,** 839–848.
6 I. Barrodale and F. D. K. Roberts (1978) An efficient algorithm for discrete l_1 linear approximation with linear constraints. *SIAM J. numer. Anal.* **15,** 603–611.
7 R. D. Snyder (1982) Robust time series analysis. *Eur. J. Opl Res.* **9,** 168–172.
8 A. Charnes and W. W. Cooper (1961) *Management Models and Industrial Applications of Linear Programming, I* & II. Wiley, New York.
9 A. Charnes, W. W. Cooper, D. Klingman and R. J. Niehaus (1975) Explicit solutions in convex goal programming. *Mgmt Sci.* **22,** 438–448.
10 A. H. Kvanli (1980) Financial planning using goal programming. *Omega* **8,** 207–218.
11 C. E. Miller (1963) The simplex method for local separable programming. In *Recent Advances in Mathematical Programming* (R. L. Graves and P. Wolfe, Eds). McGraw-Hill, New York.
12 G. Hadley (1964) *Nonlinear and Dynamic Programming.* Addison–Wesley, Reading, Massachusetts.
13 J. F. Shapiro (1979) *Mathematical Programming.* Wiley, New York.
14 P. Wolfe (1965) The composite simplex algorithm. *SIAM Rev.* **7,** 42–54.
15 A. Land, G. Morton and M. P. Robertson (1965) Spatial price equilibrium. In *Operations Research Verfahren* (R. Henn, Ed.). Anton Hain, Goettingen.
16 H. J. Greenberg (1978) Pivot selection tactics. In *Design and Implementation of Optimization Software* (H. J. Greenberg, Ed.). Sijthoff & Noordoff, Alphen aan den Rijn.
17 G. Dantzig (1963) *Linear Programming and Extensions.* Princeton University Press, Princeton, New Jersey.
18 E. M. L. Beale and J. J. H. Forrest (1976) Global optimization using special ordered sets. *Math. Prog.* **10,** 52–69.
19 R. D. Snyder (1981) Programming with piecewise linear objective functions. Working Paper No. 9/81, Department of Econometrics and Operations Research, Monash University.
20 R. Fourer (1983) Piecewise linear programming. Working Paper, Department of Industrial Engineering and Management Sciences, Northwestern University.
21 A. Premoli (1981) Piecewise linear programming: the compact (C.P.L.P.) algorithm. Gruppo Sistemi e Circuiti, Instituto Ellectrotecnico Nazionale Galileo Farraris, Torino, Italy.