Filling N-sided holes using combined subdivision schemes

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Abstract. A new method is presented for calculating N-sided surface patches. The algorithm generates a subdivision surface which satisfies arbitrary C^1 boundary conditions. The proposed subdivision scheme is based on a Catmull-Clark type subdivision scheme that operates in the surface interior. Near the boundary we introduce new subdivision rules that involve the given boundary conditions. The generated subdivision surface is C^2-continuous except at one extraordinary point. In the neighborhood of this point the surface curvature is bounded.

§1. Background

The problem of constructing N-sided surface patches occurs frequently in computer-aided geometric design. The N-sided patch is required to connect smoothly to given surfaces surrounding a polygonal hole, as shown in Fig. 1.

Referring to [10,25,26], N-sided patches can be generated basically in two ways. Either the polygonal domain, which is to be mapped into 3D, is subdivided in the parametric plane, or one uniform equation is used to represent the entire patch. In the former case triangular or rectangular elements are put together [2,6,12,20,23] or recursive subdivision methods are applied [5,8,24]. In the latter case either the known control-point based methods are generalized or a weighted sum of 3D interpolants gives the surface equation [1,3,4,22,26].

The method presented in this paper is a recursive subdivision scheme specially designed to consider arbitrary boundary conditions. Subdivision schemes provide efficient algorithms for the design, representation and processing of smooth surfaces of arbitrary topological type. Their simplicity and their multiresolution structure make them attractive for applications in 3D surface modeling, and in computer graphics [7,9,11,13,19,27,28].

The subdivision scheme presented in this paper falls into the category of *combined subdivision schemes* [14,15,17,18], where the underlying surface is

Saint-Malo Proceedings
XXX, XXX, and Larry L. Schumaker (eds.), pp. 1–8.

ISBN 1-xxxxx-xxx-x.

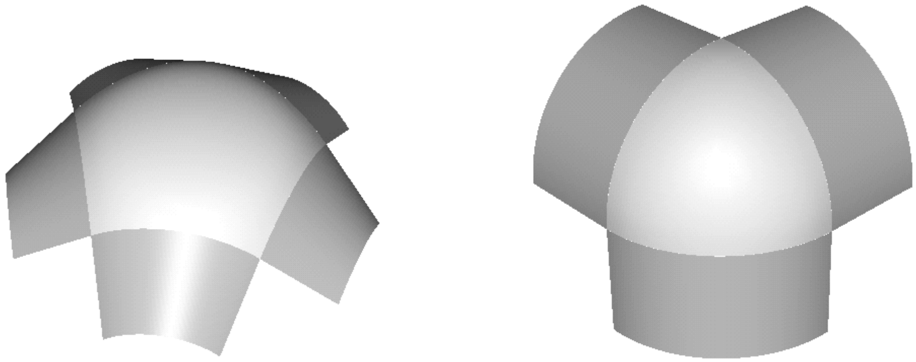

Fig. 1. A 5 sided and a 3 sided surface patch.

represented not only by a control net, but also by the given boundary conditions. The scheme repeatedly applies a subdivision operator to the control net, which becomes more and more dense. In the limit, the vertices of the control net converge to a smooth surface. Samples of the boundary conditions participate in every iteration of the subdivision, and as a result the limit surface satisfies the given conditions, regardless of their representation. Thus, our scheme performs so-called transfinite interpolation.

The motivation behind the specific subdivision rules, and the smoothness analysis of the scheme are presented in [16]. In the following sections, we describe Catmull-Clark's scheme, and we present the details of our scheme.

§2. Catmull-Clark subdivision

A net $\Sigma = (V, E)$ consists of a set of vertices V and the topological information of the net E, in terms of edges and faces. A net is closed when each edge is shared by exactly two faces.

Camull-Clark's subdivision scheme is defined over closed nets of arbitrary topology, as an extension of the tensor product bi-cubic B-spline subdivision scheme [5,8]. Variants of the original scheme were analyzed by Ball and Storry [24]. Our algorithm employs a variant of Catmull-Clark's scheme due to Sabin [21], which generates limit surfaces that are C^2-continuous everywhere except at a finite number of irregular points. In the neighborhood of those points the surface curvature is bounded. The irregular points come from vertices of the original control net that have valency other than 4, and from faces of the original control net that are not quadrilateral.

Given a net Σ, the vertices V' of the new net $\Sigma' = (V', E')$ are calculated by applying the following rules on Σ (see Fig. 2):

1. For each old face f, make a new face-vertex $v(f)$ as the weighted average of the old vertices of f, with weights W_m that depend on the valency m of each vertex.

2. For each old edge e, make a new edge-vertex $v(e)$ as the weighted average of the old vertices of e and the new face vertices associated with the two faces originally sharing e. The weights W_m (which are the same as the weights used in rule 1) depend on the valency m of each vertex.

3. For each old vertex v, make a new vertex-vertex $v(v)$ at the point given by the following linear combination, whose coefficients $\alpha_m, \beta_m, \gamma_m$ depend on the valency m of v:

$\alpha_m\cdot$ (the centroid of the new edge vertices of the edges meeting at v) + $\beta_m\cdot$ (the centroid of the new face vertices of the faces sharing those edges) + $\gamma_m \cdot v$.

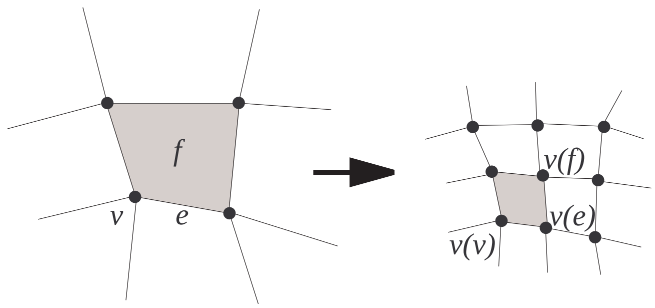

Fig. 2. Catmull-Clark's scheme.

The topology E' of the new net is calculated by the following rule: For each old face f and for each vertex v of f, make a new quadrilateral face whose edges join $v(f)$ and $v(v)$ to the edge vertices of the edges of f sharing v (see Fig. 2).

We present the procedure for calculating the weights mentioned above, as formulated by Sabin in [21]: Let $m > 2$ denote a vertex valency. Let $k := \cos(\pi/m)$. Let x be the unique real root of

$$x^3 + (4k^2 - 3)x - 2k = 0,$$

satisfying $x > 1$. Then

$$W_m = x^2 + 2kx - 3, \quad \alpha_m = 1,$$

$$\gamma_m = \frac{kx + 2k^2 - 1}{x^2(kx+1)}, \quad \beta_m = -\gamma_m.$$

The weights W_m and γ_m for $m = 3, \ldots, 7$ are given by

$$\begin{array}{ll}
W_3 = 1.23606797749979\ldots & \gamma_3 = 0.06524758424985\ldots \\
W_4 = 1.00000000000000000 & \gamma_4 = 0.25000000000000000 \\
W_5 = 0.71850240323974\ldots & \gamma_5 = 0.40198344690335\ldots \\
W_6 = 0.52233339335931\ldots & \gamma_6 = 0.52342327689253\ldots \\
W_7 = 0.39184256502794\ldots & \gamma_7 = 0.61703187134796\ldots
\end{array}$$

Remark: The original paper by Sabin [21] contains a mistake: the formulas for the parameters α, β and γ that appear in §4 there, are $\beta := 1,\ \gamma := -\alpha$.

§3. The boundary conditions

The input to our scheme consists of N smooth curves given in a parametric representation $c_j : [0,2] \to \mathbb{R}^3$ over the parameter interval $[0,2]$, and corresponding cross-boundary derivative functions $d_j : [0,2] \to \mathbb{R}^3$ (see Fig. 3). We say that the boundary conditions are C^0-compatible at the j-th corner if

$$c_j(2) = c_{j+1}(0).$$

We say that the boundary conditions are C^1-compatible if

$$d_j(0) = -c'_{j-1}(2),$$
$$d_j(2) = c'_{j+1}(0).$$

We say that the boundary conditions are C^2-compatible if the curves c_j have Hölder continuous second derivatives, the functions d_j have Hölder continuous derivatives, and the following twist compatibility condition is satisfied:

$$d'_j(2) = -d'_{j+1}(0). \tag{1}$$

The requirement of Hölder continuity is used in [16] for the proof of C^2-continuiuty in case the boundary conditions are C^2-compatible.

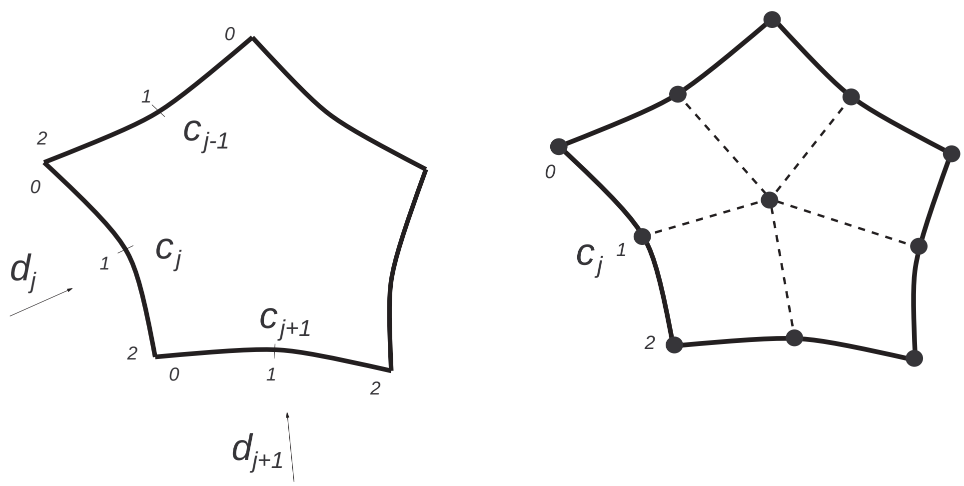

Fig. 3. The input data (left) and the initial control net (right).

§4. The algorithm

In this section we describe our algorithm for the design of an N-sided patch satisfying the boundary conditions described in §3. The key ingredients of the algorithm are two formulas for calculating the boundary vertices of the net. These formulas are given in §4.3 and §4.4.

§4.1. Constructing an initial control net

The algorithm starts by constructing an initial control net, whose faces are all quadrilateral, with $2N$ boundary vertices and one middle vertex, as shown in Fig. 3. The boundary vertices are placed at the parameter values $0, 1, 2$ on the given curves. The middle vertex can be arbitrarily chosen by the designer, and it controls the shape of the resulting surface.

§4.2. A single iteration of subdivision

We denote by n denote the iteration number, where $n = 0$ corresponds to the first iteration. In the n-th iteration we perform three steps: First, we relocate the boundary vertices according to the rules given below in §4.3 - §4.4. Then, we apply Sabin's variant of Catmull-Clark's scheme to calculate the new net topology and the position of the new internal vertices. For the purpose of choosing appropriate weights in the averaging process, we consider the boundary vertices as if they all have valency 4. This makes up for the fact that the net is not closed. In the third and final step, we sample the boundary vertices from the given curves at uniformly spaced parameter values with interval length $2^{-(n+1)}$.

§4.3. A smooth boundary rule

Let v denote a boundary vertex corresponding to the parameter $0 < u < 2$ on the curve c_j. Let w denote the unique internal vertex which shares an edge with v (see Fig. 4 (left)). In the first step of the n-th iteration we calculate the position of the v by the following formula:

$$v = 2c_j(u) - \frac{1}{2}w - \frac{1}{4}\left(c_j\left(u + 2^{-n}\right) + c_j\left(u - 2^{-n}\right)\right) -$$
$$2^{-n}\frac{1}{12}\left(d_j\left(u + 2^{-n}\right) + d_j\left(u - 2^{-n}\right)\right) + 2^{-n}\frac{2}{3}d_j(u).$$

§4.4. A corner rule

Let v denote a boundary vertex corresponding to the point $c_{j-1}(2) = c_j(0)$. Let w be the unique internal vertex sharing a face with v (see Fig. 4 (right)). In the first step of the n-th iteration we calculate the position of v by the following formula:

$$v = \frac{5}{2}c_j(0) + \frac{1}{4}w - \left(c_j(2^{-n}) + c_{j-1}(2 - 2^{-n})\right) + \frac{1}{8}c_j(2^{1-n}) +$$
$$\frac{1}{8}c_{j-1}(2 - 2^{1-n}) + 2^{-n}\frac{29}{48}\left(d_j(0) + d_{j-1}(2)\right) - 2^{-n}\frac{1}{12}d_j(2^{-n}) -$$
$$2^{-n}\frac{1}{12}d_{j-1}(2 - 2^{-n}) - 2^{-n}\frac{1}{48}\left(d_j(2^{1-n}) + d_{j-1}(2 - 2^{1-n})\right).$$

§5. Properties of our scheme

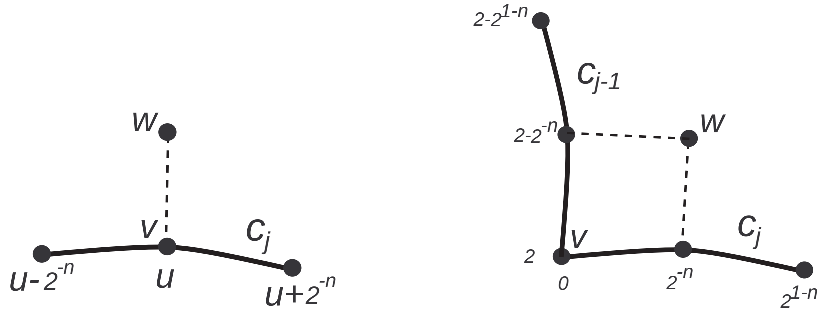

Fig. 4. The stencils for the smooth boundary rule (left) and the corner rule (right).

In [16] we prove that the vertices generated by the above procedure converge to a surface which is C^2-continuous almost everywhere, provided that the boundary conditions are C^2-compatible (as defined in §3). The only point where the surface is not C^2-continuous is a middle-point (corresponding to the middle vertex, which has valency N), where the surface is only G^1-continuous. In the neighborhood of this extraordinary point, the surface curvature is bounded.

The limit surface interpolates the given curves, for C^0-compatible boundary conditions. For C^1-compatible boundary conditions, the tangent plane of the limit surface at the point $c_j(u)$ is spanned by the vectors $c_j'(u)$ and $d_j(u)$, thus the surface satisfies C^1-boundary conditions. Furthermore, due to the locality of this scheme, the limit surface is C^2 near the boundaries except at points where the C^2-compatibility condition is not satisfied.

The surfaces in Fig. 5 demonstrate that the limit surface behaves moderately even in the presence of wavy boundary conditions. The limit surfaces are C^2-continuous near the boundary except at corners where the twist compatibility condition (1) is not satisfied.

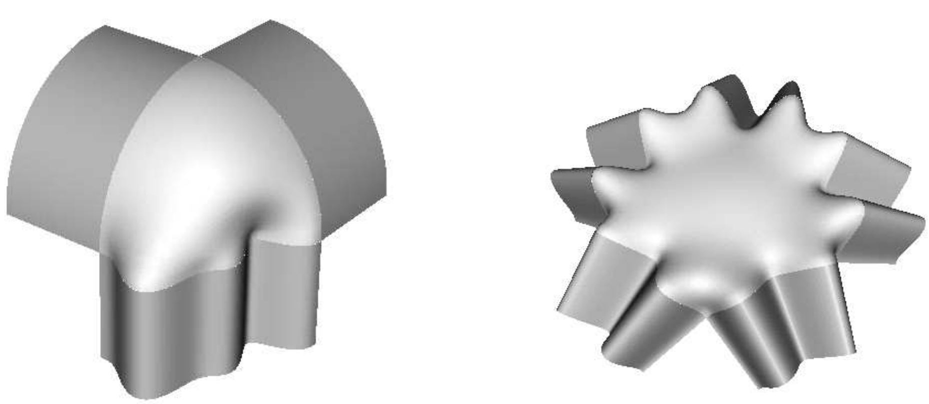

Fig. 5. A 5-sided and a 3-sided surface patch with wavy boundary curves.

Acknowledgments. This work is sponsored by the Israeli Ministry of Science. The author wishes to thank Nira Dyn for her guidance and many helpful comments.

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