Recovering a Harmonic Surface from a Boundary Frame Loop

A cotangent-weighted Dirichlet height field for spatial curve interpolation

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Abstract

We present a deterministic method that turns a sampled closed spatial curve into a topologically disk-shaped triangle mesh whose boundary coincides exactly with the input loop. The loop is projected onto a local orthonormal chart chosen by the polygon area vector; the projected boundary is then triangulated with constrained refinement, edge-flipping, and Laplacian relaxation; and the interior is solved as a Dirichlet problem for a cotangent-weighted discrete Laplace–Beltrami operator with fixed boundary heights. We prove two analytical invariants—zero boundary drift and the exact planar-loop solution—and report numerical verification on circle, concave, and non-planar loops. The method is intended as the surface-layer counterpart of frame-field curve interpolation: local boundary frames are the only input, and the interior geometry is recovered by integration rather than by explicit parametrization.

1 Introduction

Frame-based geometric modeling splits naturally into two layers. The *curve layer* takes a sequence of local frames and reconstructs a smooth spatial curve by Hermite or Catmull–Romke composition [**?**]. The *surface layer*, when it exists, usually starts from an explicit analytic parametrization $S(u, v)$ and solves for coord poses on that host surface [**?**]. What is missing is the inverse problem: given only a closed spatial curve—a boundary frame loop—recover a coherent surface spanning it.

A non-planar closed loop does not determine a unique surface. We therefore fix a deterministic choice: project the loop onto a stable plane and solve a harmonic height field over that projection. This is not a NURBS patch and does not claim to be an exact interpolant; it is the simplest surface that (a) passes through every boundary vertex, (b) has no interior curvature imposed by the user, and (c) has a disk topology.

2 Method

2.1 Area-vector normal and translation invariance

Let $\{p_i\}_{i=0}^{N-1}$ be the sampled loop, closed with $p_N = p_0$. We estimate the projection normal as

$$\mathbf{n} = \sum_{i=0}^{N-1} p_i \times p_{i+1}. \tag{1}$$

For a planar polygon this is twice the oriented area vector, hence the exact plane normal. For a non-planar loop it is the area-weighted average direction.

Lemma 1 (Translation invariance)**.** *If every vertex is shifted by a constant vector t, the area-vector sum is unchanged.*

Proof. Writing $p_i \mapsto p_i + t$,

$$\sum_i (p_i + t) \times (p_{i+1} + t) = \sum_i p_i \times p_{i+1} + \Big(\sum_i p_i\Big) \times t + t \times \Big(\sum_i p_{i+1}\Big).$$

Since $\sum_i p_i = \sum_i p_{i+1} = C$, the last two terms are $C \times t + t \times C = 0$. □

This means the chosen plane does not depend on an arbitrary origin, which matters when the loop is later re-sampled around a different frame.

2.2 Local orthonormal chart

Given $\mathbf{n}$, pick $\mathbf{u}$ as the first non-zero projection of an edge from p_0 onto the plane perpendicular to $\mathbf{n}$, and set $\mathbf{v} = \mathbf{n} \times \mathbf{u}$. The triple $(\mathbf{u}, \mathbf{v}, \mathbf{n})$ is right-handed orthonormal. Every boundary vertex decomposes as

$$p_i - p_0 = u_i\,\mathbf{u} + v_i\,\mathbf{v} + h_i\,\mathbf{n}, \tag{2}$$

with $(u_i, v_i) \in \mathbb{R}^2$ the projected point and h_i the height. The reverse map

$$\tilde{p}_i = p_0 + u_i\,\mathbf{u} + v_i\,\mathbf{v} + h_i\,\mathbf{n} \tag{3}$$

is the exact inverse, because the basis is orthonormal. This is the source of the zero-drift property below.

2.3 Constrained triangulation and mesh quality

The projected boundary (u_i, v_i) is triangulated by a constrained ear-clipping routine. Long edges are split at their midpoints; interior edges are flipped whenever the minimum circumcircle-based quality of the two adjacent triangles improves; and interior vertices are relaxed by a Laplacian smoothing step accepted only when (i) no incident triangle flips orientation and (ii) the minimum quality improves. Boundary vertices are never moved.

2.4 Cotangent-weighted Dirichlet height field

Over the triangulated domain we assemble the symmetric discrete Laplace–Beltrami operator. For edge (i, j) shared by triangles with opposite vertices k and ℓ, the weight is

$$w_{ij} = \max\bigl(0, \cot\angle(ikj)\bigr) + \max\bigl(0, \cot\angle(i\ell j)\bigr). \tag{4}$$

Negative cotangents are clamped to zero for numerical stability; this degrades the operator locally at obtuse triangles but preserves the symmetric positive-semidefinite structure needed for a Dirichlet solve.

Interior heights solve the linear system

$$\sum_{j \sim i} w_{ij}\,(h_j - h_i) = 0, \qquad i \in \mathcal{I}, \tag{5}$$

with h_i fixed to the projected boundary values for $i \in \partial\Omega$. The system is solved by Gauss–Seidel iteration until the maximum interior update drops below a tolerance relative to the height span.

2.5 Normal-handle slope perturbation

A user-specified frame normal at a boundary anchor p_c is converted to a planar slope

$$s = -\frac{(n_c \cdot \mathbf{u},\; n_c \cdot \mathbf{v})}{n_c \cdot \mathbf{n}}, \tag{6}$$

applied as a Gaussian-localized perturbation $s \cdot (x - x_c)\, e^{-\|x - x_c\|^2/(2r^2)}$ to interior heights, with the denominator required to exceed 0.25 in magnitude. Handles whose normal is nearly parallel to the projection plane are skipped rather than clamped, because clamping the denominator would inflate the slope and push the surface in the wrong direction.

3 Correctness

Proposition 1 (Zero boundary drift)**.** *Every boundary vertex is reconstructed exactly:* $\tilde{p}_i = p_i$.

Proof. The decomposition $p_i - p_0 = u_i\mathbf{u} + v_i\mathbf{v} + h_i\mathbf{n}$ is an orthogonal decomposition in an orthonormal basis. The reverse map reconstructs the same vector algebraically. The triangulation step never moves boundary vertices, so their (u_i, v_i, h_i) are unchanged. □

Proposition 2 (Planar-loop analytic solution)**.** *If the input loop lies in a plane parallel to the projection plane (so all boundary heights are equal to* h_0*), then the unique solution of the Dirichlet problem is* $h_i \equiv h_0$ *for every interior vertex.*

Proof. The constant field $h \equiv h_0$ satisfies $\sum_j w_{ij}(h_j - h_i) = 0$ for every interior i. The cotangent-weighted Laplacian is symmetric positive-semidefinite, and with fixed boundary values the Dirichlet problem has a unique solution. Hence the constant field is the only solution. □

We verify both invariants numerically in Table **??**.

Loop	Boundary drift	Interior deviation from analytic	Euler char.
Planar circle ($z = 0$)	0.000000	0.000000 (target $h = 0$)	1
Tilted square ($z = 0.5$)	0.000000	0.000000 (target $h = 0.5$)	1
Wavy circle ($z = 0.3 \sin 3\theta$)	0.000000	n/a (non-planar boundary)	1
Self-intersecting bowtie	rejected (self-intersection)	—	—

Table 1: Numerical verification on analytically controlled loops. Boundary drift is the max distance between input loop vertices and reconstructed boundary vertices. Interior deviation is the max absolute error from the known analytic constant solution on planar loops.

4 Topological verification

Every produced mesh passes four structural checks before being accepted: (i) no edge is shared by more than two triangles (no non-manifold edge); (ii) every boundary vertex has degree exactly 2 (closed manifold loop); (iii) the summed signed triangle area matches the projected boundary area to 10^{-4} relative tolerance; and (iv) the Euler characteristic $V - E + F = 1$ (topological disk). A loop whose projection self-intersects is rejected before triangulation rather than recovered.

5 Limitations

The method makes three deliberate choices the user must be aware of. **First**, the projection plane is chosen by the area vector, not by a best-fit criterion; for a severely warped loop this may not be the visually natural plane, but it is deterministic and origin-independent. **Second**, negative cotangent weights are clamped, so the operator is not the exact Laplace–Beltrami operator at obtuse triangles. **Third**, normal-handle perturbations are added after the harmonic solve and are not re-balanced through the Dirichlet system; they are local slope nudges, not a new boundary-value problem. These are trade-offs for determinism and speed in an interactive editor, not mathematical errors.

6 Conclusion

Combined with frame-field curve interpolation, this method closes the loop from local boundary frames to a coherent interior surface. The two analytical invariants—zero boundary drift and the exact planar-loop solution—give the procedure a measurable correctness guarantee that a pure mesh heuristic would not. The companion implementation is intended as a module of the `coordinate-system` library, sitting between the existing curve interpolation layer and the analytic host-surface layer.

References

[1] Pan Guojun. *coordinate-system: a mathematical library for computable coordinate systems, quaternionic frames, and object-level spatial calculus.* Zenodo, v15.3.0. `https://github.com/panguojun/Coordinate-System`.

[2] Pan Guojun. `surface_constraints.py`: analytic parametric surfaces, coord pose solving, and Newton continuation of surface intersections. In *coordinate-system* library.

[3] Meyer, M., Desbrun, M., Schröder, P., and Barr, A. H. *Discrete differential-geometry operators for triangulated 2-manifolds.* Visualization and Mathematics III, Springer, 2003.