Adjoint kernels for a self-gravitating, hydrostatic earth model from a weak formulation

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Abstract

We derive sensitivity kernels for a non-rotating, self-gravitating earth model in hydrostatic equilibrium, and restrict the perturbations to those that keep it so: in three dimensions, perturbations of the elastic tensor (including its anelastic dispersion) at fixed density, and of the source; for spherically symmetric perturbations, also the density and the radii of discontinuities. Laterally varying density and topography would introduce a deviatoric initial stress that the hydrostatic equations do not determine, and we show that equivalent forms of those equations then give different first-order sensitivities. The derivation starts from the weak form of the equations of motion, in which the boundary conditions are natural, and treats the equilibrium gravitational potential Φ as an independent variable constrained by Poisson's equation, so that the effect of a density perturbation on the equilibrium gravity is carried by a single, static adjoint potential λ whose source is formed, once, from the forward and adjoint fields integrated over frequency. Every kernel is then local, and for spherically symmetric perturbations λ reduces to a radial quadrature. The method is formulated in the frequency domain at complex frequencies, as in Adourian et al. (2024); the adjoint problem is derived in continuous form and then discretised with the rules that form the synthetic seismograms, which for these linear operations coincides with the discrete adjoint.

1 Setting and notation

The earth model occupies a ball V of radius a, divided into regions by internal discontinuities $\Sigma = \Sigma_{SS} \cup \Sigma_{FS}$ (solid–solid and fluid–solid), with free surface ∂V. The unit normal $\hat{\mathbf{n}}$ points from the $-$ side of a boundary to the $+$ side, and $[f]^+_- = f_+ - f_-$; on ∂V the $+$ side is outside. With this convention, for a tensor field $\mathbf{v}$ smooth in each region,

$$\int_V \nabla \cdot \mathbf{v} \, \mathrm{d}V = -\int_{\Sigma \cup \partial V} [\hat{\mathbf{n}} \cdot \mathbf{v}]^+_- \, \mathrm{d}\Sigma, \tag{1}$$

with $\mathbf{v} = 0$ outside V. For vector fields we use the bilinear pairing $\langle \mathbf{a}, \mathbf{b} \rangle = \int_V \mathbf{a} \cdot \mathbf{b} \, \mathrm{d}V$, without complex conjugation; all forms below are bilinear, and the frequency-domain operators are complex symmetric rather than Hermitian.

The equilibrium model has density ρ, elastic tensor $\mathbf{c}$ (the incremental tensor of the pre-stressed state, with the major and minor symmetries) and gravitational potential Φ, with

$$\nabla^2 \Phi = 4\pi G \rho \quad \text{in } \mathbb{R}^3, \qquad \Phi \to 0 \text{ as } |\mathbf{x}| \to \infty. \tag{2}$$

It is in hydrostatic equilibrium: the initial stress is a pressure p_0 with $\nabla p_0 = -\rho \nabla \Phi$, so that $\nabla \rho \parallel \nabla \Phi$ in each region and the boundaries are level surfaces of Φ. For a spherically symmetric model, $\nabla \Phi = g \hat{\mathbf{r}}$ with $g = 4\pi G r^{-2} \int_0^r \rho r'^2 \, \mathrm{d}r' \geq 0$.

Scope. The background is spherically symmetric. A perturbation of $\mathbf{c}$, or of its quality factors, at fixed ρ leaves ρ, Φ and p_0 unchanged, so the perturbed model is still in hydrostatic equilibrium, whatever the perturbation's form; so does a spherically symmetric perturbation of ρ or of the radius of a discontinuity. These are the perturbations considered: three-dimensional ones of the elastic tensor at fixed density, and spherically symmetric ones of every parameter. A laterally varying density or topography is excluded, for the reason given in Section 7. The derivation below is written for general perturbations of ρ and Φ where that costs nothing, and specialised at the end.

2 The forward problem

2.1 The hydrostatic equations

In the hydrostatic form (Dahlen & Tromp, 1998, eq. 3.259), the initial pressure has been eliminated with $\nabla p_0 = -\rho\nabla\Phi$, and the displacement $\mathbf{s}$ and the perturbation ϕ of the gravitational potential satisfy, without rotation,

$$\rho\ddot{\mathbf{s}} - \nabla\cdot\mathbf{T} + \nabla(\rho\mathbf{s}\cdot\nabla\Phi) - \nabla\cdot(\rho\mathbf{s})\nabla\Phi + \rho\nabla\phi = \mathbf{f}, \qquad \mathbf{T} = \mathbf{c} : \nabla\mathbf{s}, \tag{3}$$

$$(4\pi G)^{-1}\nabla^2\phi + \nabla\cdot(\rho\mathbf{s}) = 0, \tag{4}$$

the first in each region of V, the second in $\mathbb{R}^3$. The boundary conditions are $[\hat{\mathbf{n}}\cdot\mathbf{T}]^+_- = \mathbf{0}$ on $\Sigma\cup\partial V$ (with $\mathbf{T}=0$ outside), $[\mathbf{s}]^+_- = \mathbf{0}$ on Σ_{SS}, $[\hat{\mathbf{n}}\cdot\mathbf{s}]^+_- = 0$ and $\hat{\mathbf{n}}\cdot\mathbf{T} \parallel \hat{\mathbf{n}}$ on Σ_{FS}, and $[\phi]^+_- = 0$, $[\hat{\mathbf{n}}\cdot\boldsymbol{\xi}]^+_- = 0$ with $\boldsymbol{\xi} = (4\pi G)^{-1}\nabla\phi + \rho\mathbf{s}$ on every boundary, with $\phi\to 0$ at infinity (Liu & Tromp, 2008). These are the equations that YSpec solves, written in terms of the radial variables of Section 9.

Integrating (3) through a thin layer across a boundary, the distributional parts of $\nabla(\rho\mathbf{s}\cdot\nabla\Phi)$ and $-\nabla\cdot(\rho\mathbf{s})\nabla\Phi$ are $\hat{\mathbf{n}}\,[\rho]^+_-\,g\,\hat{\mathbf{n}}\cdot\mathbf{s}$ and $-\,[\rho]^+_-\,(\hat{\mathbf{n}}\cdot\mathbf{s})g\,\hat{\mathbf{n}}$, which cancel because $\nabla\Phi = g\hat{\mathbf{n}}$ there and $\hat{\mathbf{n}}\cdot\mathbf{s}$ is continuous. The equations therefore hold in each region with only the stated conditions on the boundaries.

2.2 The weak form

Let $\mathcal{U}$ be the displacements that honour the kinematic conditions ($\mathbf{s}$ continuous on Σ_{SS}, $\hat{\mathbf{n}}\cdot\mathbf{s}$ continuous on Σ_{FS}) and $\mathcal{P}$ the potentials in $H^1(\mathbb{R}^3)$ that decay at infinity. Take $\mathbf{s}'\in\mathcal{U}$, multiply (3) by it, integrate over V, and use (1):

- $-\langle\mathbf{s}', \nabla\cdot\mathbf{T}\rangle = \int_V \nabla\mathbf{s}' : \mathbf{T}\,\mathrm{d}V + \int [\mathbf{s}'\cdot\hat{\mathbf{n}}\cdot\mathbf{T}]^+_-\,\mathrm{d}\Sigma$, and the surface term vanishes: on Σ_{SS} both $\mathbf{s}'$ and $\hat{\mathbf{n}}\cdot\mathbf{T}$ are continuous, on Σ_{FS} the traction is $\tau\hat{\mathbf{n}}$ with τ and $\hat{\mathbf{n}}\cdot\mathbf{s}'$ continuous, and on ∂V the traction is zero.
- $\langle\mathbf{s}', \nabla(\rho\mathbf{s}\cdot\nabla\Phi)\rangle - \langle\mathbf{s}', \nabla\cdot(\rho\mathbf{s})\nabla\Phi\rangle = \int_V \rho\,[\mathbf{s}\cdot\nabla(\mathbf{s}'\cdot\nabla\Phi) - (\nabla\cdot\mathbf{s}')(\mathbf{s}\cdot\nabla\Phi)]\,\mathrm{d}V$ plus the surface terms $\int [-(\hat{\mathbf{n}}\cdot\mathbf{s}')\rho\,\mathbf{s}\cdot\nabla\Phi + (\mathbf{s}'\cdot\nabla\Phi)\rho\,\hat{\mathbf{n}}\cdot\mathbf{s}]^+_-\,\mathrm{d}\Sigma$, which vanish because $\nabla\Phi = g\hat{\mathbf{n}}$ on the boundaries.

Similarly, multiplying (4) by $\phi'\in\mathcal{P}$ and integrating over $\mathbb{R}^3$, the surface terms vanish by the continuity of $\hat{\mathbf{n}}\cdot\boldsymbol{\xi}$. With

$$A_g(\mathbf{s}', \mathbf{s}) = \int_V \rho\left[\mathbf{s}\cdot\nabla\nabla\Phi\cdot\mathbf{s}' + (\mathbf{s}\cdot\nabla\mathbf{s}')\cdot\nabla\Phi - (\nabla\cdot\mathbf{s}')(\mathbf{s}\cdot\nabla\Phi)\right]\mathrm{d}V, \tag{5}$$

$$B\big((\mathbf{s}', \phi'), (\mathbf{s}, \phi)\big) = \int_V \nabla\mathbf{s}' : \mathbf{c} : \nabla\mathbf{s}\,\mathrm{d}V + A_g(\mathbf{s}', \mathbf{s}) + \int_V \rho\,(\mathbf{s}'\cdot\nabla\phi + \mathbf{s}\cdot\nabla\phi')\,\mathrm{d}V + \frac{1}{4\pi G}\int_{\mathbb{R}^3}\nabla\phi'\cdot\nabla\phi\,\mathrm{d}V, \tag{6}$$

where $(\mathbf{s}\cdot\nabla\mathbf{s}')\cdot\nabla\Phi = s_i\,\partial_i s'_j\,\partial_j\Phi$, the forward problem is to find $(\mathbf{s},\phi)$ in $\mathcal{U}\times\mathcal{P}$ such that

$$\langle \mathbf{s}', \rho\ddot{\mathbf{s}}\rangle + B\big((\mathbf{s}',\phi'),(\mathbf{s},\phi)\big) = \langle \mathbf{s}', \mathbf{f}\rangle \quad \text{for all } (\mathbf{s}',\phi')\in\mathcal{U}\times\mathcal{P}. \tag{7}$$

The equation with $\phi'=0$ is the weak form of (3); that with $\mathbf{s}'=0$ is minus that of (4). All the boundary conditions on tractions and fluxes are natural. The gravitational form A_g is eq. B26 of Al-Attar & Tromp (2014), derived there for quasi-static loading, where the fluid is described by ϕ alone and A_g is accordingly an integral over the solid with surface terms on the fluid–solid boundaries; with a fluid displacement, as here, those become part of the volume integral.

Lemma 1. *For a hydrostatic background, A_g, and so B, is symmetric.*

Proof. With $\mathbf{N} = \rho(\mathbf{s}\otimes\mathbf{s}' - \mathbf{s}'\otimes\mathbf{s})$,

$$A_g(\mathbf{s}',\mathbf{s}) - A_g(\mathbf{s},\mathbf{s}') = \int_V \left[\partial_i N_{ij}\,\partial_j\Phi - (\partial_i\rho)(s_i s'_j - s'_i s_j)\partial_j\Phi\right]\,\mathrm{d}V.$$

The second term vanishes because $\nabla\rho \parallel \nabla\Phi$. In the first, integrating by parts leaves $-\int N_{ij}\partial_i\partial_j\Phi$, which vanishes because $\mathbf{N}$ is antisymmetric, and the surface term $\int [n_i N_{ij}\partial_j\Phi]^+_-\,\mathrm{d}\Sigma$, which is $\int [g\,\hat{\mathbf{n}}\cdot\mathbf{N}\cdot\hat{\mathbf{n}}]^+_-\,\mathrm{d}\Sigma = 0$. □

The symmetry gives reciprocity, and with it the identification of the adjoint problem with a forward one (Section 4). It uses the hydrostatic conditions; we return to this in Section 7.

2.3 Complex frequencies

As in YSpec and Adourian et al. (2024), the equations are transformed with time dependence $e^{i\omega t}$ and solved at the complex frequencies $\omega_\varepsilon = \omega - i\varepsilon$, ω real, with ε chosen so that the response decays by a factor e^{-5} over the length of the record; the time series is recovered by an inverse transform followed by multiplication by $e^{\varepsilon t}$. For each real ω the fields $(\mathbf{s}_\omega,\phi_\omega)$, which we label by ω although they are evaluated at ω_ε, satisfy

$$S_\omega\big((\mathbf{s}',\phi'),(\mathbf{s}_\omega,\phi_\omega)\big) := -\omega_\varepsilon^2\langle \mathbf{s}', \rho\mathbf{s}_\omega\rangle + B_\omega\big((\mathbf{s}',\phi'),(\mathbf{s}_\omega,\phi_\omega)\big) = \langle \mathbf{s}', \mathbf{f}_\omega\rangle, \tag{8}$$

where B_ω is B with $\mathbf{c}$ replaced by the complex tensor $\mathbf{c}_\omega = \mathbf{c}(\omega_\varepsilon)$, which has the same symmetries. Each S_ω is a complex-symmetric bilinear form. Because the time series are real, the fields at $-\omega$ are the complex conjugates of those at ω, and for such Hermitian functions of frequency we write

$$\langle\!\langle F\rangle\!\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\,\mathrm{d}\omega = \frac{1}{\pi}\int_0^\infty \mathrm{Re}\,F(\omega)\,\mathrm{d}\omega. \tag{9}$$

3 Observables and adjoint sources

3.1 The continuous adjoint source

Let $X(\omega) = \hat{\boldsymbol{\nu}}\cdot\mathbf{s}_\omega(\mathbf{x}_r)$ be the spectrum of the component $\hat{\boldsymbol{\nu}}$ at the receiver $\mathbf{x}_r$. The processed synthetic is

$$u(t) = \frac{e^{\varepsilon t}}{2\pi}\int_{-\infty}^{\infty} w(\omega)R(\omega)X(\omega)\,e^{i\omega t}\,\mathrm{d}\omega, \tag{10}$$

with w a real taper and R the instrument response at the complex frequency, which includes any time derivatives of the displacement. For a real functional $J[u]$ with Fréchet derivative $j(t) = \delta J/\delta u(t)$,

$$\delta J = \int j(t)\,\delta u(t)\,\mathrm{d}t = \langle\!\langle A\,\delta X\rangle\!\rangle, \qquad A(\omega) = w(\omega)R(\omega)\int j(t)\,e^{\varepsilon t}e^{i\omega t}\,\mathrm{d}t, \tag{11}$$

the processed adjoint source of Adourian et al. (2024, eqs 37–38): the factor $e^{\varepsilon t}$ reflects the undamping of the synthetic.

3.2 Cross-correlation traveltimes

For the delay τ of a phase in a window $W(t)$, measured by cross-correlation with a reference that is the synthetic itself (Dahlen et al., 2000), a delay $\delta\tau$ changes the synthetic by $\delta u = -\dot{u}\,\delta\tau$, and so

$$\delta\tau = -\frac{\int W\dot{u}\,\delta u\,\mathrm{d}t}{\int W\dot{u}^2\,\mathrm{d}t}, \qquad j(t) = -\frac{W(t)\dot{u}(t)}{\int W\dot{u}^2\,\mathrm{d}t}, \tag{12}$$

with $\dot{u}$ the velocity synthetic: the adjoint source of Liu & Tromp (2008, eqs 82–83), here at complex frequency.

4 The Lagrangian

The model parameters are ρ, $\mathbf{c}$ (through its elastic values and quality factors) and the source. The equilibrium potential depends on ρ through (2); rather than eliminate it, we treat Φ as an independent variable and impose (2) in weak form with a multiplier λ:

$$P(\Phi,\lambda;\rho) := \frac{1}{4\pi G}\int_{\mathbb{R}^3}\nabla\lambda\cdot\nabla\Phi\,\mathrm{d}V + \int_V \rho\lambda\,\mathrm{d}V = 0 \quad \text{for all } \lambda. \tag{13}$$

The Lagrangian is

$$\mathcal{L} = J - \Big\langle\!\Big\langle S_\omega\big((\mathbf{s}^\circ_\omega,\phi^\circ_\omega),(\mathbf{s}_\omega,\phi_\omega)\big) - \langle\mathbf{s}^\circ_\omega,\mathbf{f}_\omega\rangle\Big\rangle\!\Big\rangle + P(\Phi,\lambda;\rho), \tag{14}$$

with $(\mathbf{s}^\circ_\omega,\phi^\circ_\omega)$ the multipliers for the forward problems, Hermitian in ω like the fields. Where the forward and equilibrium problems hold, $\mathcal{L} = J$ whatever the multipliers, so $\delta J = \partial_m\mathcal{L}\,\delta m$ once the multipliers make $\mathcal{L}$ stationary with respect to $\mathbf{s}_\omega$, ϕ_ω and Φ.

The adjoint fields. Stationarity with respect to $(\mathbf{s}_\omega,\phi_\omega)$ at each $\omega > 0$, using (11), requires $\mathrm{Re}[A\,\hat{\boldsymbol{\nu}}\cdot\delta\mathbf{s}(\mathbf{x}_r)] = \mathrm{Re}[S_\omega((\mathbf{s}^\circ_\omega,\phi^\circ_\omega),(\delta\mathbf{s},\delta\phi))]$ for all complex $(\delta\mathbf{s},\delta\phi)$, and so, replacing $\delta\mathbf{s}$ by $i\delta\mathbf{s}$ for the imaginary part, and by the symmetry of S_ω (Lemma 1),

$$S_\omega\big((\delta\mathbf{s},\delta\phi),(\mathbf{s}^\circ_\omega,\phi^\circ_\omega)\big) = \langle\delta\mathbf{s}, A(\omega)\hat{\boldsymbol{\nu}}\,\delta(\mathbf{x}-\mathbf{x}_r)\rangle \quad \text{for all } (\delta\mathbf{s},\delta\phi). \tag{15}$$

The adjoint field at ω is the forward response, at the same complex frequency, to the point force $A(\omega)\hat{\boldsymbol{\nu}}$ at the receiver. No time reversal or change of frequency is needed, and the solver is the forward one.

The adjoint potential. Stationarity with respect to Φ requires

$$\frac{1}{4\pi G}\int_{\mathbb{R}^3}\nabla\lambda\cdot\nabla\psi\,\mathrm{d}V = \int_V (\mathbf{M}:\nabla\nabla\psi + \mathbf{m}\cdot\nabla\psi)\,\mathrm{d}V \quad \text{for all } \psi, \tag{16}$$

where, from (5), the forward and adjoint fields enter through the static fields

$$\mathbf{M} = \langle\!\langle\rho\ \mathrm{sym}(\mathbf{s}_\omega\otimes\mathbf{s}^\circ_\omega)\rangle\!\rangle, \qquad \mathbf{m} = \langle\!\langle\rho\,((\mathbf{s}_\omega\cdot\nabla)\mathbf{s}^\circ_\omega - \mathbf{s}_\omega\,\nabla\cdot\mathbf{s}^\circ_\omega)\rangle\!\rangle. \tag{17}$$

This is a single, static Poisson problem, solved once after the frequency integral; for the spherically symmetric density perturbations considered here only its spherical average is needed, which is a radial quadrature (Section 6).

5 The kernels

With the multipliers so chosen,

$$\delta J = -\Big\langle\!\Big\langle \partial_m S_\omega\big((\mathbf{s}^\circ_\omega,\phi^\circ_\omega),(\mathbf{s}_\omega,\phi_\omega)\big)\delta m\Big\rangle\!\Big\rangle + \langle\!\langle\langle\mathbf{s}^\circ_\omega,\delta\mathbf{f}_\omega\rangle\rangle\!\rangle + \int_V \lambda\,\delta\rho\,\mathrm{d}V. \tag{18}$$

Elastic tensor. $\delta J = \int_V \mathbf{K}_c :: \delta\mathbf{c}\,\mathrm{d}V$ with

$$\mathbf{K}_c = -\left\langle\!\!\left\langle \frac{\partial \mathbf{c}_\omega}{\partial \mathbf{c}}^{\mathsf{T}} : \left(\nabla \mathbf{s}_\omega^\circ \otimes \nabla \mathbf{s}_\omega\right) \right\rangle\!\!\right\rangle, \tag{19}$$

the chain rule through the dispersion. For transverse isotropy with a radial axis, in terms of the strains $\boldsymbol{\epsilon} = \operatorname{sym} \nabla \mathbf{s}_\omega$ and $\boldsymbol{\epsilon}^\circ = \operatorname{sym} \nabla \mathbf{s}_\omega^\circ$, with h the horizontal components, the elastic energy density is $C\epsilon_{rr}^\circ \epsilon_{rr} + A \operatorname{tr}_h \boldsymbol{\epsilon}^\circ \operatorname{tr}_h \boldsymbol{\epsilon} + F(\epsilon_{rr}^\circ \operatorname{tr}_h \boldsymbol{\epsilon} + \operatorname{tr}_h \boldsymbol{\epsilon}^\circ \epsilon_{rr}) + 4L\, \boldsymbol{\epsilon}_{rh}^\circ \cdot \boldsymbol{\epsilon}_{rh} + N(2\boldsymbol{\epsilon}_{hh}^\circ : \boldsymbol{\epsilon}_{hh} - 2 \operatorname{tr}_h \boldsymbol{\epsilon}^\circ \operatorname{tr}_h \boldsymbol{\epsilon})$, every term invariant under a common rotation about $\hat{\mathbf{r}}$. The two fields may therefore be expressed in different horizontal frames provided one is turned into the other's: with spin components $a_\pm = a_\theta \mp i a_\phi$, a frame turned by γ about $\hat{\mathbf{r}}$ multiplies the components of spin weight N by $e^{iN\gamma}$, so only the terms of L and N need the turn. With $D(\omega)$ the dispersion of YSpec, so that a modulus X with inverse quality factor q_X is $X(1 + q_X D)$ at ω_ε,

$$K_X = -\langle\!\langle (1 + q_X D)\, e_X(\boldsymbol{\epsilon}^\circ, \boldsymbol{\epsilon}) \rangle\!\rangle, \qquad K_{q_X} = -\langle\!\langle X D\, e_X(\boldsymbol{\epsilon}^\circ, \boldsymbol{\epsilon}) \rangle\!\rangle, \tag{20}$$

for $X \in \{A, C, F, L, N\}$ and e_X its coefficient above. Nothing in the derivation depends on the form of the dispersion: for any rheology whose moduli $\mathbf{c}(\omega; \mathbf{p})$ are holomorphic in the lower half of the complex plane, as causality requires, so that they may be evaluated at ω_ε, the kernel for a rheological parameter p is $K_p = -\langle\!\langle \partial_p \mathbf{c}_\omega : (\nabla \mathbf{s}_\omega^\circ \otimes \nabla \mathbf{s}_\omega) \rangle\!\rangle$: for a power-law Q, its exponent, or for standard linear solids, their relaxation strengths and times. For isotropy, $K_\kappa = -\langle\!\langle (1 + q_\kappa D)\, \nabla \cdot \mathbf{s}_\omega^\circ\, \nabla \cdot \mathbf{s}_\omega \rangle\!\rangle$ and $K_\mu = -\langle\!\langle (1 + q_\mu D)\, 2\mathbf{d}^\circ : \mathbf{d} \rangle\!\rangle$ with $\mathbf{d}$ the deviatoric strain, as in Liu & Tromp (2008, eqs 63–64). Attenuation kernels come at no extra cost. At fixed density the velocity kernels follow from these: for isotropy, in relative perturbations, $K_\alpha = 2(\kappa + \frac{4}{3}\mu) K_\kappa$ and $K_\beta = 2\mu(K_\mu - \frac{4}{3} K_\kappa)$ with K_κ, K_μ the absolute kernels above (cf. Liu & Tromp, 2008, eq. 68), and similarly for the five velocities of transverse isotropy.

Density, at fixed moduli. For the spherically symmetric perturbations to which density is restricted, the angular integral of the following is needed; we write it pointwise. From (8), (5), (6) and (18),

$$\begin{aligned} K_\rho = \Big\langle\!\!\Big\langle\, & \omega_\varepsilon^2\, \mathbf{s}_\omega^\circ \cdot \mathbf{s}_\omega - \mathbf{s}_\omega^\circ \cdot \nabla \phi_\omega - \mathbf{s}_\omega \cdot \nabla \phi_\omega^\circ \\ & - \mathbf{s}_\omega \cdot \nabla\nabla\Phi \cdot \mathbf{s}_\omega^\circ - (\mathbf{s}_\omega \cdot \nabla \mathbf{s}_\omega^\circ) \cdot \nabla\Phi + (\nabla \cdot \mathbf{s}_\omega^\circ)(\mathbf{s}_\omega \cdot \nabla\Phi) \Big\rangle\!\!\Big\rangle + \lambda. \end{aligned} \tag{21}$$

Every term is local; the equilibrium gravity's dependence on the density is carried entirely by λ. For a spherically symmetric background the gravitational terms simplify to

$$\mathbf{s} \cdot \nabla\nabla\Phi \cdot \mathbf{s}^\circ + (\mathbf{s} \cdot \nabla \mathbf{s}^\circ) \cdot \nabla\Phi - (\nabla \cdot \mathbf{s}^\circ)(\mathbf{s} \cdot \nabla\Phi) = \left(4\pi G\rho - \frac{4g}{r}\right) s_r s_r^\circ + g\left(\mathbf{s}_h \cdot \nabla_h s_r^\circ - s_r \nabla_h \cdot \mathbf{s}_h^\circ\right), \tag{22}$$

using $\nabla\nabla\Phi = \Phi'' \hat{\mathbf{r}}\hat{\mathbf{r}} + (g/r)(\mathbf{I} - \hat{\mathbf{r}}\hat{\mathbf{r}})$ with $\Phi'' = 4\pi G\rho - 2g/r$, and $(\mathbf{s} \cdot \nabla)\hat{\mathbf{r}} = \mathbf{s}_h / r$, with ∇_h the horizontal gradient. The coefficient $-4g/r$ is that of the radial equations.

Density, at fixed velocities. Again for spherically symmetric perturbations: each modulus is proportional to the density at fixed velocities, so $\rho K_\rho' = \rho K_\rho + \sum_X X K_X$: for isotropy $\rho K_\rho + \kappa K_\kappa + \mu K_\mu$ (Liu & Tromp, 2008, eq. 68), for transverse isotropy $\rho K_\rho + A K_A + C K_C + F K_F + L K_L + N K_N$. The velocity kernels follow as usual.

Source. For a moment tensor $\mathbf{M}_s$ at $\mathbf{x}_s$, stepped in time, $\langle \mathbf{s}_\omega^\circ, \delta\mathbf{f}_\omega \rangle = \delta\mathbf{M}_s : \boldsymbol{\epsilon}^\circ(\mathbf{x}_s)/(i\omega_\varepsilon)$, giving the source kernels by reciprocity at no extra cost.

At the source. The strain of a point moment tensor is not locally integrable: besides its R^{-3} near field, whose angular average vanishes, it has a part concentrated at the source, as $\nabla\nabla R^{-1}$ has the local part of Section 6. In the expansion in spherical harmonics this part is explicit. At each degree the displacement jumps across $r = r_s$, by $\mathbf{M}_s \cdot \hat{\mathbf{r}}$ divided by C or L there, and its radial derivative has a part $[\mathbf{s}]^+_- \, \delta(r - r_s)$, so that

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}_{\mathrm{reg}} + \mathrm{sym}\left(\hat{\mathbf{r}} \otimes [\mathbf{s}]^+_-\right) \delta(r - r_s). \tag{23}$$

The kernels of the moduli that multiply ϵ_{rr} and $\boldsymbol{\epsilon}_{rh}$, namely C, F and L, and their quality factors, have accordingly a part concentrated on the sphere $r = r_s$; those of A, N and ρ do not. For spherically symmetric perturbations it is a point term $P_X \, \delta X(r_s)$ in δJ, with

$$P_X = -r_s^2 \left\langle\!\!\left\langle (1 + q_X D) \oint e_X\left(\boldsymbol{\epsilon}^\circ, \mathrm{sym}(\hat{\mathbf{r}} \otimes [\mathbf{s}]^+_-)\right) \mathrm{d}\Omega \right\rangle\!\!\right\rangle, \tag{24}$$

evaluated with the adjoint field at r_s, which is regular there unless the receiver is at the depth of the source. Finite differences that perturb the moduli around the source need it.

Discretisation. YSpec computes the spectra at $f_k = k\,\Delta f$ and forms the samples $u_n = u(n\Delta t)$ by a fast Fourier transform. Discretised with the same rules, $\langle\!\langle F \rangle\!\rangle$ becomes $2\Delta f \sum_k \mathrm{Re}\, F(\omega_k)$ and the time integral in (11) a sum over the samples, so that $A_k = w_k R_k \sum_n j(t_n)\, e^{\varepsilon t_n} e^{2\pi i f_k t_n} \Delta t$. For these linear operations this coincides with the derivative of the discrete misfit, with $\partial J / \partial u_n = j(t_n)\Delta t$, so the kernels predict finite differences of the computed J to the accuracy of the solver.

6 The adjoint potential

6.1 Strong form

Where $\mathbf{M}$ and $\mathbf{m}$ are smooth, (16) is $-(4\pi G)^{-1}\nabla^2\lambda = \nabla\cdot\nabla\cdot\mathbf{M} - \nabla\cdot\mathbf{m}$, and λ is harmonic outside V and decays at infinity. The source contains double layers where $\mathbf{M}_{nn}$ jumps (as ρ does at discontinuities), so λ is itself discontinuous there, with $[\lambda]^+_- = -4\pi G\,[\mathbf{M}_{nn}]^+_-$; the weak form handles this without comment. For the perturbations considered here only the spherical average of λ is needed, which reduces to the radial quadrature of Section 6.3; the integral representation below is given for comparison with Eaton et al. (2026).

6.2 Integral representation and comparison with Eaton et al. (2026)

Taking $\psi = -G/|\mathbf{x} - \mathbf{x}_0|$ in (16), for which $\nabla^2\psi = 4\pi G\delta(\mathbf{x} - \mathbf{x}_0)$, gives $\lambda(\mathbf{x}_0) = -\int_V (\mathbf{M} : \nabla\nabla\psi + \mathbf{m}\cdot\nabla\psi)\,\mathrm{d}V$. Now $\nabla\psi = G\,\mathbf{R}/R^3$ with $\mathbf{R} = \mathbf{x} - \mathbf{x}_0$, and

$$\nabla\nabla\psi = G\,\Pi + \frac{4\pi G}{3}\mathbf{I}\,\delta(\mathbf{x} - \mathbf{x}_0), \qquad \Pi = \frac{\mathbf{I}}{R^3} - \frac{3\mathbf{R}\otimes\mathbf{R}}{R^5}, \tag{25}$$

the integral over Π being a principal value; the local term is needed for $\nabla^2\psi = 4\pi G\delta$, since Π is traceless. Hence

$$\lambda(\mathbf{x}_0) = -G\int_V \left[\mathbf{M} : \Pi + \mathbf{m}\cdot\frac{\mathbf{x} - \mathbf{x}_0}{|\mathbf{x} - \mathbf{x}_0|^3}\right] \mathrm{d}V - \frac{4\pi G}{3}\,\mathrm{tr}\,\mathbf{M}(\mathbf{x}_0). \tag{26}$$

With $\mathbf{M} \leftrightarrow \tilde{\mathbf{K}}_\rho$ and $\mathbf{m} \leftrightarrow \tilde{\mathbf{k}}_\rho$ (Eaton et al., 2026, eqs 23–24), the first term is the non-local part of their density kernel (their eq. 29 is for relative perturbations, and so carries an extra factor ρ), so the two results agree, given one remark: their eq. A4 writes $\nabla\nabla\delta\Phi = G\int\delta\rho\,\Pi$ without the local term, which is needed for $\nabla^2\delta\Phi = 4\pi G\delta\rho$. Their numerical route, which takes derivatives of solutions of Poisson's equation, includes it automatically. Here a single scalar Poisson problem replaces their nine.

6.3 One-dimensional perturbations

For $\delta\rho(r)$ the perturbation of the equilibrium potential is radial and explicit, $\delta\Phi' = 4\pi G\,\delta m(r)/r^2$ with $\delta m(r) = \int_0^r \delta\rho\, r'^2\,\mathrm{d}r'$, and $\delta\Phi'' = 4\pi G\delta\rho - 2\delta\Phi'/r$, so λ need not be formed. With $\nabla\nabla\delta\Phi = \delta\Phi''\hat{\mathbf{r}}\hat{\mathbf{r}} + (\delta\Phi'/r)(\mathbf{I} - \hat{\mathbf{r}}\hat{\mathbf{r}})$ and the angular integrals

$$\bar{M}_{rr}(r) = \oint \mathbf{M}_{rr}\,\mathrm{d}\Omega, \quad \bar{M}_h(r) = \oint (\mathbf{M}_{\theta\theta} + \mathbf{M}_{\phi\phi})\,\mathrm{d}\Omega, \quad \bar{m}_r(r) = \oint \mathbf{m}_r\,\mathrm{d}\Omega, \tag{27}$$

the gravitational part of the one-dimensional density kernel, defined by $\delta J = \int_0^a K_\rho^{1D}\,\delta\rho\, r^2\,\mathrm{d}r$, is

$$K_\rho^{1D,\Phi}(r) = -4\pi G\left[\bar{M}_{rr}(r) + \int_r^a \left(\frac{\bar{M}_h - 2\bar{M}_{rr}}{r'} + \bar{m}_r\right)\mathrm{d}r'\right], \tag{28}$$

non-local only through an integral over the radii above. The local part is the angular integral of the bracket in (21). YSpec computes g from ρ, so finite differences in the density test this kernel completely, gravitational part included.

7 Why density is not perturbed laterally

A background in hydrostatic equilibrium cannot remain so under a laterally varying $\delta\rho$, or lateral topography: the initial stress acquires a deviatoric part that the density does not determine, and the hydrostatic equations no longer define the problem. One symptom: for a hydrostatic background

$$\nabla(\rho\mathbf{s}\cdot\nabla\Phi) - \nabla\cdot(\rho\mathbf{s})\nabla\Phi = \rho\,[\nabla(\mathbf{s}\cdot\nabla\Phi) - (\nabla\cdot\mathbf{s})\nabla\Phi] \tag{29}$$

(Al-Attar & Tromp, 2014, eq. B23), the two sides differing by $\nabla\rho\,(\mathbf{s}\cdot\nabla\Phi) - (\nabla\rho\cdot\mathbf{s})\nabla\Phi$, which vanishes when $\nabla\rho \parallel \nabla\Phi$. Perturbed by a laterally varying $\delta\rho$ and the $\delta\Phi$ it produces, with $\nabla\Phi = g\hat{\mathbf{r}}$ and $\nabla\rho = \rho'\hat{\mathbf{r}}$, perturbing the left-hand form rather than the right-hand one changes δJ by

$$\left\langle\!\!\left\langle \int_V \left[\rho'\,\nabla_h\delta\Phi - g\,\nabla_h\delta\rho\right]\cdot(s_r\mathbf{s}_h^\circ - s_r^\circ\mathbf{s}_h)\;\mathrm{d}V \right\rangle\!\!\right\rangle, \tag{30}$$

where ρ' includes $[\rho]_-^+\,\delta_\Sigma$ at every discontinuity. The three-dimensional density kernel therefore depends at first order on which of two equivalent forms of the equations is perturbed; equivalently, A_g is symmetric only for a hydrostatic background (Lemma 1), and $A_g(\mathbf{s}^\circ, \mathbf{s})$ and $A_g(\mathbf{s}, \mathbf{s}^\circ)$ have different perturbations. Liu & Tromp (2008), and so Eaton et al. (2026), perturb the left-hand form, and also omit the surface term $-\int_\Sigma [\rho(s_n^\circ\mathbf{s}_h - s_n\mathbf{s}_h^\circ)]_-^+\cdot\nabla_h\delta\Phi\,\mathrm{d}\Sigma$, which perturbing it generates on every boundary across which ρ jumps (it is the δ_Σ part of the expression above). None of this arises for the perturbations considered here: at fixed density, and for spherically symmetric $\delta\rho$, $\delta\Phi$ is radial and every term vanishes.

8 Topography on discontinuities

This section is an outline, to be completed, and is restricted to changes in the radius of a discontinuity, the only topography that keeps the model hydrostatic. Moving a discontinuity changes only the domains of the integrals in (14). Mapping the perturbed domain onto the reference one and using that every field satisfies its equations and transmission conditions, the first variation reduces to the jump across the boundary of the normal–normal component of the energy–momentum tensor of the Lagrangian density, $\mathbb{E} = \ell\mathbf{I} - \sum_v (\nabla v)^\mathsf{T}\cdot\partial\ell/\partial(\nabla v)$, summed over the fields v on which ℓ depends through first derivatives, with an additional tangential term on Σ_{FS}, where the boundary may move relative to the slipping fluid. For the elastic and inertial terms, $\ell = \rho\omega^2\mathbf{s}^\circ\cdot\mathbf{s} - \nabla\mathbf{s}^\circ : \mathbf{c} : \nabla\mathbf{s}$, and $\hat{\mathbf{n}}\cdot\mathbb{E}\cdot\hat{\mathbf{n}} = \ell + (\hat{\mathbf{n}}\cdot\mathbf{T})\cdot\partial_n\mathbf{s}^\circ + (\hat{\mathbf{n}}\cdot\mathbf{T}^\circ)\cdot\partial_n\mathbf{s}$, which corresponds, up to their sign convention and the time domain, to the kernel of Liu & Tromp

(2008, eq. 60). The constraint contributes $\left[\rho\lambda - (4\pi G)^{-1} g\,\partial_n\lambda\right]^+_-$ at a spherical boundary, and the gravitational terms of A_g involve $\nabla\nabla\Phi$, so that their energy–momentum tensor needs the second-order form. The kernel is one number per discontinuity, and finite differences that move a discontinuity will check it.

9 Evaluation in YSpec

For a spherically symmetric background, YSpec gives the forward field of each source and the field of a point force at each receiver at every radius wanted, in the frame with the source (or receiver) at the pole, as coefficients of generalised spherical harmonics with $|m| \leq 2$ (or 1), and their strains as the spin components $N = -2, \ldots, 2$.

- One-dimensional kernels need only angular integrals of products of the two fields. The adjoint field is expanded in the frame of the source, where a force away from the pole excites every order but only $|m| \leq 2$ meet the forward field: its weights are those of the force at the pole with $X^N_{lm}(0)$ replaced by $X^N_{lm}(\Delta)\, e^{-im\phi_r}$. By orthogonality, $\oint Y^N_{lm} Y^{-N}_{l,-m}\,\mathrm{d}\Omega = (-1)^{N+m}$, each integral is then a sum over degree and order of products of radial functions, and for displacements $\oint \mathbf{s}\cdot\mathbf{s}^\circ\,\mathrm{d}\Omega = \sum_{lm}(-1)^m[U_m U^\circ_{-m} + \zeta^2(V_m V^\circ_{-m} + W_m W^\circ_{-m})]$; there is no spatial mesh. The kernels are evaluated at Gauss–Legendre nodes in elements between the discontinuities of the model, the source and the receiver, and the radial integral in $K^{1D,\Phi}_\rho$ is taken on the same nodes. With $\bar{M}_h - 2\bar{M}_{rr} + r\bar{m}_r = \rho\big[\zeta^2\sum(-1)^m(V_m U^\circ_{-m} + U_m V^\circ_{-m}) - 4\sum(-1)^m U_m U^\circ_{-m}\big]$ summed over degree, its integrand is the same combination as the gravitational term $g(\mathbf{s}_h\cdot\nabla_h s^\circ_r - s_r\nabla_h\cdot\mathbf{s}^\circ_h) - (4g/r)s_r s^\circ_r$ of the local kernel, divided by g/ρ.

- Three-dimensional kernels, for the elastic parameters, need the fields at the points of a mesh. Each field is evaluated in its own frame, the forward field's with the source at the pole, where it has the orders $|m| \leq 2$ alone, and the adjoint field's with the receiver at the pole, where it has $|m| \leq 1$: at each point a sum over degree of a few terms, with X^N_{lm} at the point's distances from the source and from the receiver. The adjoint strain's spin components are then turned into the forward field's frame by $e^{-iN\gamma}$, with γ the angle between the two frames' $\hat{\boldsymbol{\theta}}$ at the point, and the kernels are the pointwise products integrated over frequency. No rotation of expansions, and no λ, is needed. The mesh is that of a planetmodel model file: nodes in each layer, both sides of each discontinuity included, and a Gauss–Legendre grid at each. Its cost is that of the sums at every point, of order the number of points times the number of frequencies times the largest degree.

- Checks: finite differences of J for spherically symmetric perturbations of every parameter, density with gravity included; the integrals of the three-dimensional elastic kernels over every sphere against the one-dimensional ones; their values at single points against those computed in the source's frame alone, where the adjoint field, its force away from the pole, has every order; and their symmetry for an explosion and a vertical component.

One restriction: a receiver must be where YSpec can place a source, that is, in a solid.

References

Adourian, S., Dursun, M.S., Lau, H.C.P. & Al-Attar, D., 2024. Adjoint sensitivity kernels for free oscillation spectra, *Geophys. J. Int.*, 238, 257–271.

Al-Attar, D. & Tromp, J., 2014. Sensitivity kernels for viscoelastic loading based on adjoint methods, *Geophys. J. Int.*, 196, 34–77.

Dahlen, F.A., Hung, S.-H. & Nolet, G., 2000. Fréchet kernels for finite-frequency traveltimes—I. Theory, *Geophys. J. Int.*, 141, 157–174.

Dahlen, F.A. & Tromp, J., 1998. *Theoretical Global Seismology*, Princeton University Press.

Eaton, W., Gharti, H.N. & Tromp, J., 2026. Sensitivity of prompt elasto-gravitational signals to 3-D Earth structure, *Geophys. J. Int.*, 245, 1–25.

Liu, Q. & Tromp, J., 2008. Finite-frequency sensitivity kernels for global seismic wave propagation based upon adjoint methods, *Geophys. J. Int.*, 174, 265–286.