Analytical results for tests

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Abstract

We summarise the analytical results used in the tests for `scikit-monaco`.

Following Numerical Recipes, we calculate the standard error in the Monte-Carlo integration of the function f as:

$$\mathrm{Err}_N(f) = \Omega\sqrt{\frac{\langle f^2\rangle - \langle f\rangle^2}{N}} \quad (1)$$

where N is the number of points and $\langle g\rangle = \int_\Omega g(x)dx$, where Ω is the volume being sampled during the integration and x denotes all the variables of integration.

1 Constant function

Let $f(x) = 1$. Then, $\langle f\rangle = \Omega$, where Ω is the volume of integration. $\langle f^2\rangle = 1$, such that $\mathrm{Err}_N = 0$ for all $N > 0$.

2 Product function

Let $f(x) = \prod_i^d x_i$, where d is the dimensionality of the integration. Thus, if $d = 2$, $f(x, y) = xy$. We consider the d-dimensional hypercube with upper and lower limit b and a, respectively, such that each $a \leq x_i \leq b$.

$$\langle f\rangle = \int\cdots\int_a^b \prod_i x_i dx_i = \left(\frac{b^2 - a^2}{2}\right)^d \quad (2)$$

$$\langle f^2\rangle = \int\cdots\int_a^b \prod_i x_i^2 dx_i = \left(\frac{b^3 - a^3}{3}\right)^d \quad (3)$$

If $a = 0$ and $b = 1$, we have $\langle f\rangle = 1/2^d$ and $\langle f^2\rangle = 1/3^d$. Then, $\mathrm{Err}_N(f) = \frac{\sqrt{1/3^d - 1/4^d}}{N}$.

3 Gaussian

Let $f(x) = \prod_i^d \exp(-\beta^2 x_i^2)$. Again, we consider the hypercube such that $a \leq x_i \leq b$ for each i. Then:

$$\langle f \rangle = \frac{\sqrt{\pi}}{2\beta}(\mathrm{erf}(\beta b) - \mathrm{erf}(\beta a) \tag{4}$$

$$\langle f^2 \rangle = \frac{\sqrt{\pi/2}}{2\beta}(\mathrm{erf}(\sqrt{2}\beta b) - \mathrm{erf}(\sqrt{2}\beta a) \tag{5}$$