Description Logic Syntax and Semantics

1 Preliminaries

Let $\mathcal{I}$ be an interpretation with non-empty domain $\Delta^{\mathcal{I}}$. Each atomic concept A is interpreted as a set $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$, each atomic role R as a binary relation $R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$, and each individual name a as an element $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$.

Concept descriptions denote subsets of $\Delta^{\mathcal{I}}$. Role descriptions denote binary relations over $\Delta^{\mathcal{I}}$. For planning states, let s denote the set of atoms true in the current state, and let γ denote the set of goal literals.

2 Concept Constructors

Name	Syntax	Semantics
Top	$\top$	$\top^{\mathcal{I}} = \Delta^{\mathcal{I}}$
Bottom	$\bot$	$\bot^{\mathcal{I}} = \emptyset$
Positive atomic state concept	P	$P^{\mathcal{I}} = \{a \in \Delta^{\mathcal{I}} \mid P(a) \in s\}$
Negative atomic state concept	$\neg P$	$\neg P^{\mathcal{I}} = \Delta^{\mathcal{I}} \setminus P^{\mathcal{I}}$
Positive atomic goal concept	G	$G^{\mathcal{I}} = \{a \in \Delta^{\mathcal{I}} \mid G(a) \in \gamma\}$
Negative atomic goal concept	$\neg G$	$\neg G^{\mathcal{I}} = \{a \in \Delta^{\mathcal{I}} \mid \neg G(a) \in \gamma\}$
Intersection	$C \sqcap D$	$C \sqcap D^{\mathcal{I}} = C^{\mathcal{I}} \cap D^{\mathcal{I}}$
Union	$C \sqcup D$	$C \sqcup D^{\mathcal{I}} = C^{\mathcal{I}} \cup D^{\mathcal{I}}$
Negation	$\neg C$	$\neg C^{\mathcal{I}} = \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$
Value restriction	$\forall R.C$	$\forall R.C^{\mathcal{I}} = \{a \in \Delta^{\mathcal{I}} \mid \forall b.\, (a,b) \in R^{\mathcal{I}} \Rightarrow b \in C^{\mathcal{I}}\}$
Existential quantification	$\exists R.C$	$\exists R.C^{\mathcal{I}} = \{a \in \Delta^{\mathcal{I}} \mid \exists b.\, (a,b) \in R^{\mathcal{I}} \wedge b \in C^{\mathcal{I}}\}$
At-least number restriction	$\geq n\, R$	$\geq n\, R^{\mathcal{I}} = \{a \in \Delta^{\mathcal{I}} \mid \lvert\{b \in \Delta^{\mathcal{I}} \mid (a,b) \in R^{\mathcal{I}}\}\rvert \geq n\}$
At-most number restriction	$\leq n\, R$	$\leq n\, R^{\mathcal{I}} = \{a \in \Delta^{\mathcal{I}} \mid \lvert\{b \in \Delta^{\mathcal{I}} \mid (a,b) \in R^{\mathcal{I}}\}\rvert \leq n\}$
Exact number restriction	$= n\, R$	$= n\, R^{\mathcal{I}} = \{a \in \Delta^{\mathcal{I}} \mid \lvert\{b \in \Delta^{\mathcal{I}} \mid (a,b) \in R^{\mathcal{I}}\}\rvert = n\}$
Qualified at-least number restriction	$\geq n\, R.C$	$\geq n\, R.C^{\mathcal{I}} = \{a \in \Delta^{\mathcal{I}} \mid \lvert\{b \in \Delta^{\mathcal{I}} \mid (a,b) \in R^{\mathcal{I}} \wedge b \in C^{\mathcal{I}}\}\rvert \geq n\}$

Name	Syntax	Semantics
Qualified at-most number restriction	$\leq n\, R.C$	$\leq n\, R.C^{\mathcal{I}} = \{a \in \Delta^{\mathcal{I}} \mid \lvert\{b \in \Delta^{\mathcal{I}} \mid (a,b) \in R^{\mathcal{I}} \wedge b \in C^{\mathcal{I}}\}\rvert \leq n\}$
Qualified exact number restriction	$= n\, R.C$	$= n\, R.C^{\mathcal{I}} = \{a \in \Delta^{\mathcal{I}} \mid \lvert\{b \in \Delta^{\mathcal{I}} \mid (a,b) \in R^{\mathcal{I}} \wedge b \in C^{\mathcal{I}}\}\rvert = n\}$
Role-value map inclusion	$R \subseteq S$	$R \subseteq S^{\mathcal{I}} = \{a \in \Delta^{\mathcal{I}} \mid \forall b.\, (a,b) \in R^{\mathcal{I}} \Rightarrow (a,b) \in S^{\mathcal{I}}\}$
Role-value map equality	$R = S$	$R = S^{\mathcal{I}} = \{a \in \Delta^{\mathcal{I}} \mid \forall b.\, (a,b) \in R^{\mathcal{I}} \Leftrightarrow (a,b) \in S^{\mathcal{I}}\}$
Agreement	$u_1 \doteq u_2$	$u_1 \doteq u_2^{\mathcal{I}} = \{a \in \Delta^{\mathcal{I}} \mid \exists b \in \Delta^{\mathcal{I}}.\, u_1^{\mathcal{I}}(a) = b = u_2^{\mathcal{I}}(a)\}$
Disagreement	$u_1 \dot{\neq} u_2$	$u_1 \dot{\neq} u_2^{\mathcal{I}} = \{a \in \Delta^{\mathcal{I}} \mid \exists b_1, b_2 \in \Delta^{\mathcal{I}}.\, u_1^{\mathcal{I}}(a) = b_1 \neq b_2 = u_2^{\mathcal{I}}(a)\}$
Nominal	I	$I^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$ and $\lvert I^{\mathcal{I}}\rvert = 1$

Here $n \in \mathbb{N}$. The expressions u_1, u_2 denote chains of functional roles, also called features. If $u = f_1 \cdots f_m$, then $u^{\mathcal{I}}$ is the composition of the partial functions interpreting the features.

3 Concrete Syntax for Concept Constructors

Name	Concrete syntax	Abstract syntax
Top	`c_top`	$\top$
Bottom	`c_bot`	$\bot$
Atomic state concept	`(c_atomic_state "p" polarity)`	P or $\neg P$
Atomic goal concept	`(c_atomic_goal "p" polarity)`	G or $\neg G$
Intersection	`(c_and C1 ... Cn)`	$C_1 \sqcap \cdots \sqcap C_n$
Union	`(c_or C1 ... Cn)`	$C_1 \sqcup \cdots \sqcup C_n$
Negation	`(c_not C)`	$\neg C$
Value restriction	`(c_all R C)`	$\forall R.C$
Existential quantification	`(c_some R C)`	$\exists R.C$
At-least number restriction	`(c_at_least n R)`	$\geq n\, R$
At-most number restriction	`(c_at_most n R)`	$\leq n\, R$
Exact number restriction	`(c_exactly n R)`	$= n\, R$
Qualified at-least restriction	`(c_at_least n R C)`	$\geq n\, R.C$
Qualified at-most restriction	`(c_at_most n R C)`	$\leq n\, R.C$
Qualified exact restriction	`(c_exactly n R C)`	$= n\, R.C$
Same-as / agreement	`(c_same_as u1 u2)`	$u_1 \doteq u_2$
Role-value map	`(c_subset R1 R2)`	$R_1 \subseteq R_2$

Name	Concrete syntax	Abstract syntax
Role fillers	`(c_fillers R I1 ... In)`	$\exists R.I_1 \sqcap \cdots \sqcap \exists R.I_n$
One-of	`(c_one_of I1 ... In)`	$I_1 \sqcup \cdots \sqcup I_n$

4 Restrictions on Role Interpretations

Functional roles. A feature f is interpreted as a functional binary relation:

$$\forall a, b, c.\, (a, b) \in f^{\mathcal{I}} \wedge (a, c) \in f^{\mathcal{I}} \Rightarrow b = c.$$

Equivalently, features may be viewed as partial functions.

Transitive roles. A transitive role R is interpreted as a transitive binary relation:

$$\forall a, b, c.\, (a, b) \in R^{\mathcal{I}} \wedge (b, c) \in R^{\mathcal{I}} \Rightarrow (a, c) \in R^{\mathcal{I}}.$$

5 Role Constructors

Name	Syntax	Semantics
Universal role	U	$U^{\mathcal{I}} = \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$
Positive atomic state role	P	$P^{\mathcal{I}} = \{(a, b) \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \mid P(a, b) \in s\}$
Negative atomic state role	$\neg P$	$\neg P^{\mathcal{I}} = (\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}) \setminus P^{\mathcal{I}}$
Positive atomic goal role	G	$G^{\mathcal{I}} = \{(a, b) \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \mid G(a, b) \in \gamma\}$
Negative atomic goal role	$\neg G$	$\neg G^{\mathcal{I}} = \{(a, b) \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \mid \neg G(a, b) \in \gamma\}$
Intersection	$R \sqcap S$	$R \sqcap S^{\mathcal{I}} = R^{\mathcal{I}} \cap S^{\mathcal{I}}$
Union	$R \sqcup S$	$R \sqcup S^{\mathcal{I}} = R^{\mathcal{I}} \cup S^{\mathcal{I}}$
Complement	$\neg R$	$\neg R^{\mathcal{I}} = (\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}) \setminus R^{\mathcal{I}}$
Inverse	R^-	$(R^-)^{\mathcal{I}} = \{(b, a) \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \mid (a, b) \in R^{\mathcal{I}}\}$
Composition	$R \circ S$	$R \circ S^{\mathcal{I}} = \{(a, c) \mid \exists b.\, (a, b) \in R^{\mathcal{I}} \wedge (b, c) \in S^{\mathcal{I}}\}$
Transitive closure	R^+	$(R^+)^{\mathcal{I}} = \bigcup_{n \geq 1} (R^{\mathcal{I}})^n$
Reflexive-transitive closure	R^*	$(R^*)^{\mathcal{I}} = \bigcup_{n \geq 0} (R^{\mathcal{I}})^n$
Role restriction	$R \vert C$	$R \vert C^{\mathcal{I}} = R^{\mathcal{I}} \cap (\Delta^{\mathcal{I}} \times C^{\mathcal{I}})$
Identity	$\mathrm{id}(C)$	$\mathrm{id}(C)^{\mathcal{I}} = \{(d, d) \mid d \in C^{\mathcal{I}}\}$

The iterated composition $(R^{\mathcal{I}})^n$ is defined by

$$(R^{\mathcal{I}})^0 = \{(d, d) \mid d \in \Delta^{\mathcal{I}}\}, \qquad (R^{\mathcal{I}})^{n+1} = (R^{\mathcal{I}})^n \circ R^{\mathcal{I}}.$$

6 Concrete Syntax for Role Constructors

Name	Concrete syntax	Abstract syntax
Universal role	`r_universal`	U
Atomic state role	`(r_atomic_state "p" polarity)`	P or $\neg P$
Atomic goal role	`(r_atomic_goal "p" polarity)`	G or $\neg G$
Intersection	`(r_and R1 ... Rn)`	$R_1 \sqcap \cdots \sqcap R_n$
Union	`(r_or R1 ... Rn)`	$R_1 \sqcup \cdots \sqcup R_n$
Complement	`(r_complement R)`	$\neg R$
Inverse	`(r_inverse R)`	R^-
Composition	`(r_composition R1 ... Rn)`	$R_1 \circ \cdots \circ R_n$
Transitive closure	`(r_transitive_closure R)`	R^+
Reflexive-transitive closure	`(r_reflexive_transitive_closure R)`	R^*
Role restriction	`(r_restriction R C)`	$R\|C$
Identity	`(r_identity C)`	$\mathrm{id}(C)$

7 Boolean Constructors

Name	Syntax	Semantics
Positive atomic state predicate	$\mathrm{holds}(p)$	true iff there exists an atom $p() \in s$
Negative atomic state predicate	$\neg\mathrm{holds}(p)$	true iff no current atom with predicate p is true in the state
Positive atomic goal predicate	$\mathrm{goal}(p)$	true iff $p() \in \gamma$
Negative atomic goal predicate	$\neg\mathrm{goal}(p)$	true iff $\neg p() \in \gamma$
Non-empty concept	$C \neq \emptyset$	true iff $C^{\mathcal{I}} \neq \emptyset$
Non-empty role	$R \neq \emptyset$	true iff $R^{\mathcal{I}} \neq \emptyset$

8 Concrete Syntax for Boolean Constructors

Name	Concrete syntax	Abstract syntax
Atomic state predicate	`(b_atomic_state "p" polarity)`	$\mathrm{holds}(p)$ or $\neg\mathrm{holds}(p)$
Atomic goal predicate	`(b_atomic_goal "p" polarity)`	$\mathrm{goal}(p)$ or $\neg\mathrm{goal}(p)$
Non-empty concept	`(b_nonempty C)`	$C \neq \emptyset$

Name	Concrete syntax	Abstract syntax
Non-empty role	`(b_nonempty R)`	$R \neq \emptyset$

9 Numerical Constructors

Name	Syntax	Semantics
Concept count	$\|C\|$	$\|C^{\mathcal{I}}\|$
Role count	$\|R\|$	$\|R^{\mathcal{I}}\|$
Role distance	$d_R(C, D)$	$\min\{n \in \mathbb{N} \mid C^{\mathcal{I}} \times D^{\mathcal{I}} \cap (R^{\mathcal{I}})^n \neq \emptyset\}$, or ∞ if empty

10 Concrete Syntax for Numerical Constructors

Name	Concrete syntax	Abstract syntax
Concept count	`(n_count C)`	$\|C\|$
Role count	`(n_count R)`	$\|R\|$
Role distance	`(n_distance C R D)`	$d_R(C, D)$

Source

This document summarizes Appendix 1, Section A1.2 of *The Description Logic Handbook*, edited by Franz Baader, Deborah L. McGuinness, Daniele Nardi, and Peter F. Patel-Schneider.