MAFS525 – Computational Methods for Pricing Structured Products

4.4. Hull-White interest rate model

The Hull-White model for the instantaneous short rate r_t is

$$dr_t = [\phi(t) - \alpha r_t]\, dt + \sigma\, dZ_t.$$

- Analytic procedure of fitting the initial term structure of bond prices
- Calibration of interest rate trees against market discount curves
- Extension to other interest rate models
- Pricing of interest rate products using the calibrated interest rate trees.

Analytic procedure of fitting the initial term structures of bond prices

- In the Hull-White short rate model, $\phi(t)$ in the drift term is the only time dependent function in the model. Under the risk neutral measure Q, the short rate r_t is assumed to follow

$$dr_t = [\phi(t) - \alpha r_t]\; dt + \sigma\, dZ_t,$$

  where α and σ are constant parameters. The model possesses the mean reversion property. We illustrate the analytic procedure for the determination of $\phi(t)$ using the information of the current term structure of bond prices.

- The governing equation for the bond price $B(r, t; T)$ is given by

$$\frac{\partial B}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 B}{\partial r^2} + [\phi(t) - \alpha r]\frac{\partial B}{\partial r} - rB = 0.$$

- We assume that the bond price function is of the form

$$B(t,T) = e^{a(t,T)-b(t,T)r}.$$

Solving the pair of ordinary differential equations for $a(t,T)$ and $b(t,T)$, we obtain

$$\begin{aligned} b(t,T) &= \frac{1}{\alpha}\left[1 - e^{-\alpha(T-t)}\right] \\ a(t,T) &= \frac{\sigma^2}{2}\int_t^T b^2(u,T)\ du - \int_t^T \phi(u)b(u,T)\ du. \end{aligned}$$

Our goal is to determine $\phi(T)$ in terms of the current term structure of bond prices $B(r,t;T)$.

Applying the relation:

$$\ln B(r,t;T) + rb(t,T) = a(t,T),$$

we have

$$\int_t^T \phi(u)b(u,T)\ du = \frac{\sigma^2}{2}\int_t^T b^2(u,T)\ du - \ln B(r,t;T) - rb(t,T). \quad (1)$$

- To solve for $\phi(u)$, the first step is to obtain an explicit expression for $\int_t^T \phi(u)\ du$.

- This can be achieved by differentiating $\int_t^T \phi(u)b(u,T)\ du$ with respect to T and subtracting the terms involving $\int_t^T \phi(u)e^{-\alpha(T-t)}\ du$.

- The derivative of the left hand side of Eq. (1) with respect to T gives

$$\begin{aligned}\frac{\partial}{\partial T}\int_t^T \phi(u)b(u,T)\ du &= \phi(u)b(u,T)\Big|_{u=T} + \int_t^T \phi(u)\frac{\partial}{\partial T}b(u,T)\ du \\ &= \int_t^T \phi(u)e^{-\alpha(T-u)}\ du,\end{aligned}$$

We equate the derivatives on both sides to obtain

$$\begin{aligned}\int_t^T \phi(u)e^{-\alpha(T-u)}\,du = \; & \frac{\sigma^2}{\alpha}\int_t^T [1-e^{-\alpha(T-u)}]e^{-\alpha(T-u)}\,du \\ & - \frac{\partial}{\partial T}\ln B(r,t;T) - re^{-\alpha(T-t)}. \end{aligned} \tag{2}$$

We multiply Eq. (1) by α and add it to Eq. (2) to obtain

$$\begin{aligned}\int_t^T \phi(u)\,du \; = \; & \frac{\sigma^2}{2\alpha}\int_t^T [1-e^{-2\alpha(T-u)}]\,du - r \\ & - \frac{\partial}{\partial T}\ln B(r,t;T) - \alpha \ln B(r,t;T). \end{aligned}$$

By differentiating the above equation with respect to T again, we obtain $\phi(T)$ in terms of the current term structure of bond prices $B(r,t;T)$ as follows:

$$\begin{aligned}\phi(T) &= \frac{\sigma^2}{2\alpha}[1-e^{-2\alpha(T-t)}] - \frac{\partial^2}{\partial T^2}\ln B(r,t;T) \\ &\quad - \alpha\frac{\partial}{\partial T}\ln B(r,t;T).\end{aligned}$$

- Alternatively, one may express $\phi(T)$ in terms of the current term structure of forward rates $F(t,T)$.

- Recall that $-\frac{\partial}{\partial T}\ln B(r,t;T) = F(t,T)$ so that we may rewrite $\phi(T)$ in the form

$$\phi(T) = \frac{\sigma^2}{2\alpha}[1-e^{-2\alpha(T-t)}] + \frac{\partial}{\partial T}F(t,T) + \alpha F(t,T).$$

Calibration of interest rate trees against market discount curves

The interest rates on the Hull-White tree are Δ-period rates, not the same as the instantaneous short rate r. Let $R(t)$ denote the Δt-period rate at time t. Recall

$$B(r,t) = a(t,T)e^{-b(t,T)r}$$

so that

$$e^{-R\Delta t} = a(t,t+\Delta t)e^{-b(t,t+\Delta t)r}.$$

Hence, $r(T)$ and $R(t)$ are related by

$$r(t) = \frac{R(t)\Delta t + \ln a(t,t+\Delta t)}{b(t,t+\Delta t)}.$$

- We assume that the Δt-rate, R, follows the same process as r:

  $$dR = [\theta(t) - aR]\,dt + \sigma\,dZ.$$

  Clearly, this is reasonable in the limit as Δt tends to zero.

Tree construction procedures

Unlike the usual trinomial trees used in equity pricing, the calibrated interest rate trees are distorted. The size of the displacement is the same for all nodes at a particular time t, but it is not usually the same for nodes at two different times.

- The first stage in building a tree for this model is to construct a tree for a variable R^* that is initially zero and follow the process

$$dR^* = -aR^*\,dt + \sigma\,dZ.$$

  We build a symmetrical tree similar to Figure 2 for R^*.

- In the second stage, we build the tree for R that calibrates to the initial term structures of bond prices.

- This process is symmetrical about $R^* = 0$. The variable $R^*(t+\Delta t) - R^*(t)$ is normally distributed. If those terms of higher order than Δt are ignored, the expected value of $R^*(t+\Delta t) - R^*(t)$ is $\sigma^2 \Delta t$.

- We define ΔR as the spacing between interest rates on the tree and set

$$\Delta R = \sigma\sqrt{3\Delta t}.$$

  This proves to be a good choice of ΔR from the viewpoint of error minimization.

- Our objective during the first stage of this procedure is to build a tree similar to that shown in Figure 2 for R^*. To do this, we must resolve which of the three branching methods shown in Figure 1 will apply at each node. This will determine the overall geometry of the tree. Once this is done, the branching probabilities must also be calculated.

Which of the three branching models shown in Figure 1 will apply at each node?

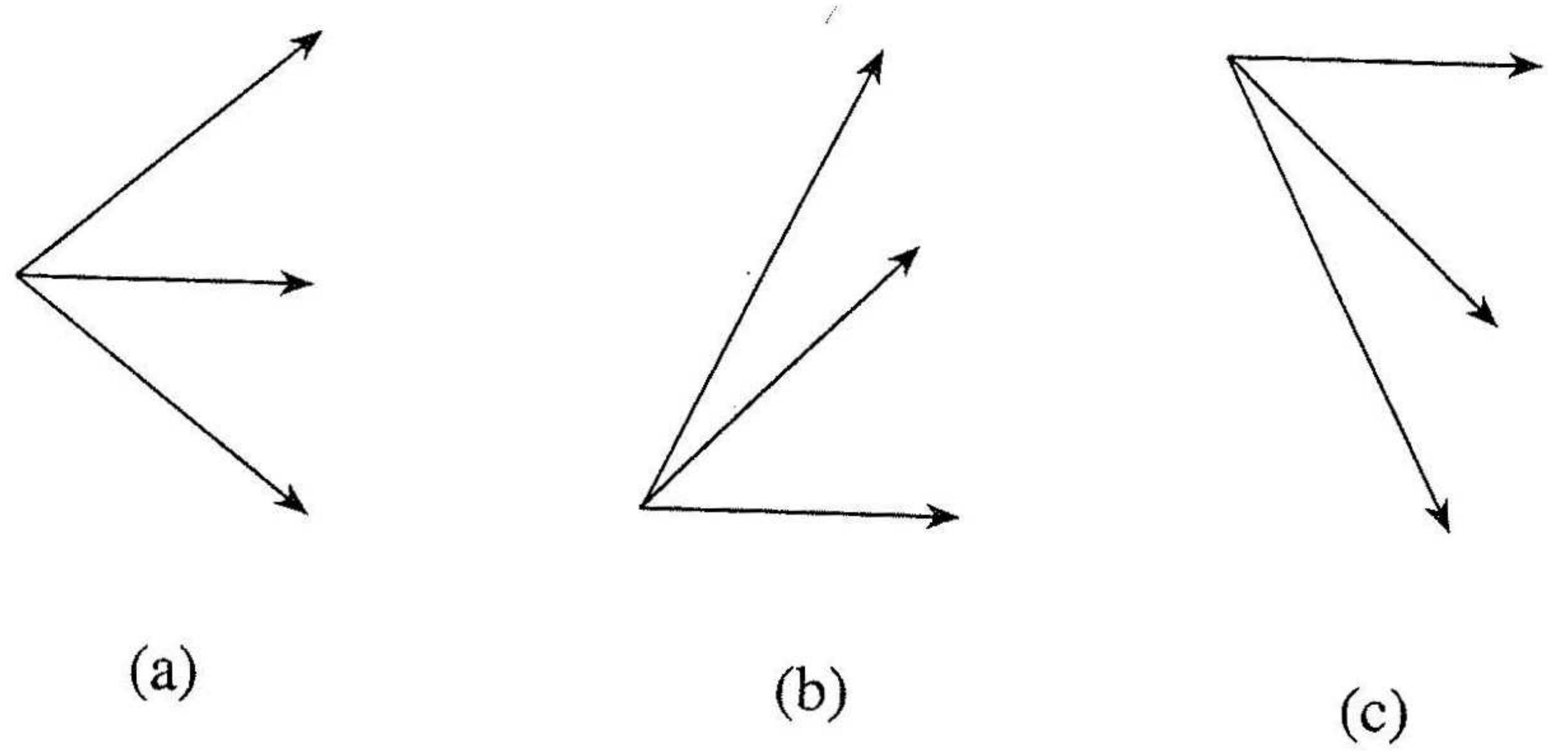


Figure 1. Alternative branching methods in a trinomial tree.

- Define (i, j) as the node where $t = i\Delta t$ and $R^* = j\Delta R$. The variable i is a positive integer and j is a positive or negative integer. The branching method used at a node must lead to the probabilities on all three branches being positive. Most of the time, the branching shown in Figure 1(a) is appropriate.

- When $a > 0$, it is necessary to switch from the branching in Figure 1(a) to the branching in Figure 1(c) for a sufficiently large j. Similarly, it is necessary to switch from the branching in Figure 1(a) to the branching in Figure 1(b) when j is sufficiently negative.

- Define j_{max} as the value of j where we switch from the Figure 1(a) branching to the Figure 1(c) branching to the j_{min} as the value of j where we switch from the Figure 1(a) branching to the Figure 1(b) branching.

- The probabilities are always positive if we set j_{max} equal to the smallest integer greater than 0.184/$(a\Delta t)$ and j_{min} equal to $-j_{max}$.

- Define p_u, p_m, and p_d as the probabilities of the highest, middle, and lowest branches emanating from the node. The probabilities are chosen to match the expected change and variance of the change in R^* over the next time interval Δt. The probabilities must also sum to unity. This leads to three equations in the three probabilities.

- The mean change in R^* in time Δt is $-aR^*\Delta t$ and the variance of the change is $\sigma^2\Delta t$. At node $(i,j), R^* = j\Delta r$.

- If the branching has the form shown in Figure 1(a), the p_u, p_m, and p_d at node (i,j) must satisfy the following three equations:

$$
\begin{aligned}
&p_u\Delta R - p_d\Delta R = -aj\Delta R\Delta t \\
&p_u\Delta R^2 + p_d\Delta R^2 = \sigma^2\Delta t + a^2j^2\Delta R^2\Delta t^2 \\
&p_u + p_m + p_d = 1.
\end{aligned}
$$

Using $\Delta R = \sigma\sqrt{3\Delta t}$, the solution to these equations is

$$\begin{aligned} p_u &= \frac{1}{6} + \frac{1}{2}(a^2 j^2 \Delta t^2 - aj\Delta t) \\ p_m &= \frac{2}{3} - a^2 j^2 \Delta t^2 \\ p_d &= \frac{1}{6} + \frac{1}{2}(a^2 j^2 \Delta t^2 + aj\Delta t). \end{aligned}$$

Similarly, if the branching has the form shown in Figure 1(b), the probabilities are

$$\begin{aligned} p_u &= \frac{1}{6} + \frac{1}{2}(a^2 j^2 \Delta t^2 + aj\Delta t) \\ p_m &= -\frac{1}{3} - a^2 j^2 \Delta t^2 - 2aj\Delta t \\ p_d &= \frac{7}{6} + \frac{1}{2}(a^2 j^2 \Delta t^2 + 3aj\Delta t). \end{aligned}$$

Finally, if the branching has the form shown in Figure 1(c), the probabilities are

$$
\begin{aligned}
p_u &= \frac{7}{6} + \frac{1}{2}(a^2 j^2 \Delta t^2 - 3aj\Delta t) \\
p_m &= -\frac{1}{3} - a^2 j^2 \Delta t^2 + 2aj\Delta t \\
p_d &= \frac{1}{6} + \frac{1}{2}(a^2 j^2 \Delta t^2 - aj\Delta t).
\end{aligned}
$$

One can readily derive conditions on j for the transition probabilities to be strictly positive.

(i) For normal branching

$$\frac{-\sqrt{2/3}}{a\Delta t} < j < \frac{\sqrt{2/3}}{a\Delta t};$$

(ii) For upward branching

$$\frac{-1-\sqrt{2/3}}{a\Delta t} < j < \frac{-1+\sqrt{2/3}}{a\Delta t};$$

(iii) For downward branching

$$\frac{1-\sqrt{2/3}}{a\Delta t} < j < \frac{1+\sqrt{2/3}}{a\Delta t}.$$

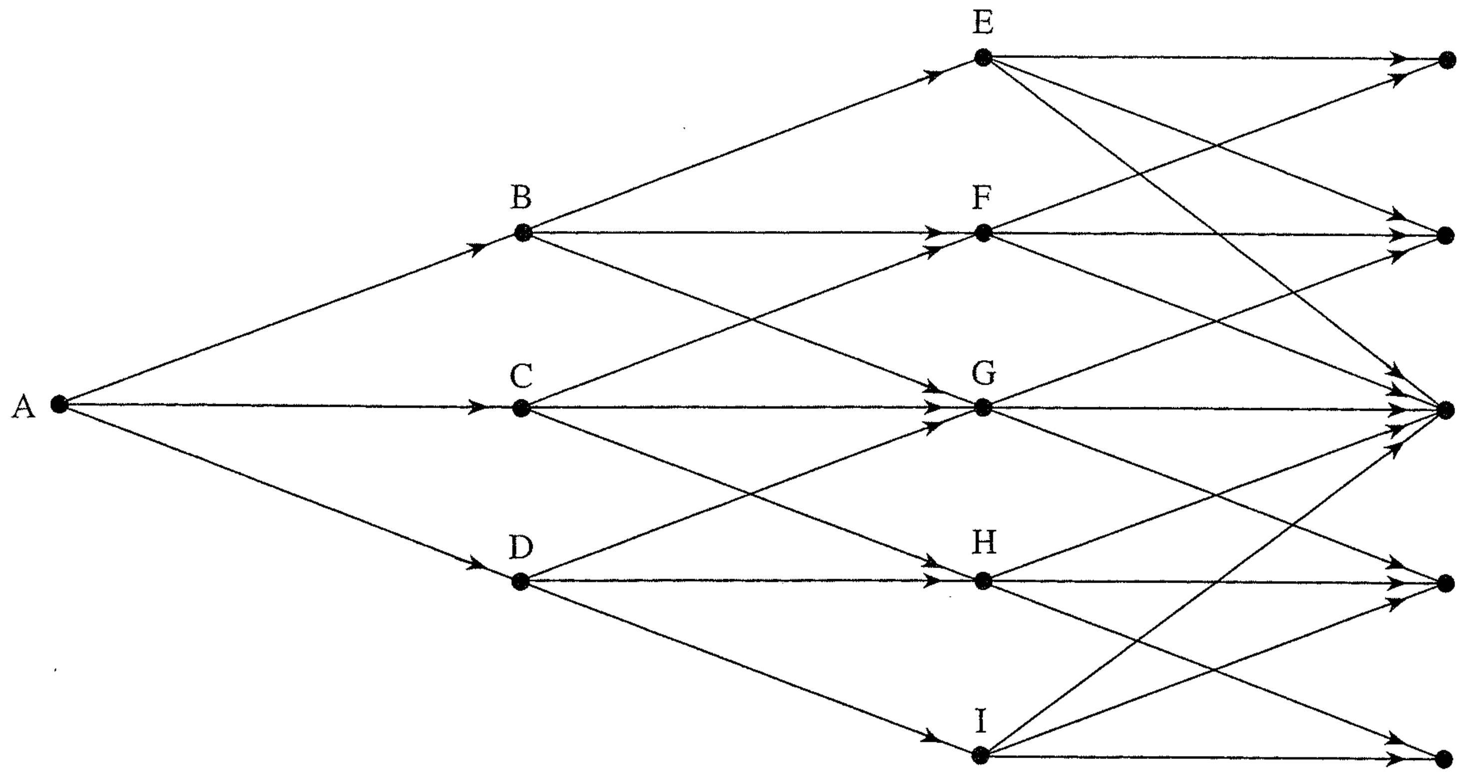
Define j_{max} as the smallest integer greater than $(1-\sqrt{2/3}/(a\Delta t) \approx 0.184/(a\Delta t)$, and take $j_{min} = -j_{max}$. Normal branching is used for $j_{min} < j < j_{max}$, downward branching is used for extreme positive value $j = j_{max}$, upward branching is used for extreme negative value $j = j_{min}$.

Numerical example – Forward induction procedure

- To illustrate the first stage of the tree construction, suppose that $\sigma = 0.01, a = 0.1$, and $\Delta t = 1$ year. In this case, $\Delta R = 0.01\sqrt{3} = 0.0173, j_{max}$ is set equal to the smallest integer greater than 0.184/0.1, and $j_{min} = -j_{max}$. This means that $j_{max} = 2$ and $j_{min} = -2$ and the tree is as shown in Figure 2. The probabilities on the branches emanating from each node are shown below the tree and are calculated using the equations above for p_u, p_m and p_d.

- Note that the probabilities at each node in Figure 2 depend only on j. For example, the probabilities at node B are the same as the probabilities at node F. Furthermore, the tree is symmetrical. The probabilities at node D are the mirror image of the probabilities at node B.

Tree for R^* in Hull–White model (first stage).



Node:	A	B	C	D	E	F	G	H	I
R(%)	0.000	1.732	0.000	−1.732	3.464	1.732	0.000	−1.732	−3.464
p_u	0.1667	0.1217	0.1667	0.2217	0.8867	0.1217	0.1667	0.2217	0.0867
p_m	0.6666	0.6566	0.6666	0.6566	0.0266	0.6566	0.6666	0.6566	0.0266
p_d	0.1667	0.2217	0.1667	0.1217	0.0867	0.2217	0.1667	0.1217	0.8867

Figure 2. Tree for R^* in Hull-White Model (first stage)

Second Stage

- The second stage in the tree construction is to convert the tree for R^* into a tree for R. This is accomplished by displacing the nodes on the R^*-tree so that the initial term structure of interest rates is exactly matched.

- Define

$$\alpha(t) = R(t) - R^*(t).$$

  We calculate the α's iteratively so that the initial term structure is matched exactly.

- Define α_i as $\alpha(i\Delta t)$, the value of R at time $i\Delta t$ on the R-tree minus the corresponding value of R^* at time $i\Delta t$ on the r^*-tree.

- Define $Q_{i,j}$ as the present value of a security that pays off \$1 if node (i,j) is reached and zero otherwise. The α_i and $Q_{i,j}$ can be calculated using forward induction in such a way that the initial term structure is matched exactly.

Illustration of the Second Stage

- The value of $Q_{0,0}$ is 1.0. The value of α_0 is chosen to give the right price for a zero-coupon bond maturing at time Δt. That is, α_0 is set equal to the initial Δt-period interest rate.

- Because $\Delta t = 1$ in this example, $\alpha_0 = 0.03824$. This defines the position of the initial node on the R-tree in Figure 3.

- The next step is to calculate the values of $Q_{1,1}, Q_{1,0}$, and $Q_{1,-1}$. There is a probability of 0.1667 that the $(1,1)$ node is reached and the discount rate for the first time step is 3.82%. The value of $Q_{1,1}$ is therefore $0.1667e^{-0.0382} = 0.1604$. Similarly, $Q_{1,0} = 0.6417$ and $Q_{1,-1} = 0.1604$.

Zero rates for the example in Figures 2 and 3

Maturity	*Rate* (%)
0.5	3.430
1.0	3.824
1.5	4.183
2.0	4.512
2.5	4.812
3.0	5.086

- Once $Q_{1,1}, Q_{1,0}$, and $Q_{1,-1}$ have been calculated, we are in a position to determine α_1. This is chosen to give the right price for a zero-coupon bond maturity at time $2\Delta t$. Because $\Delta R = 0.01732$ and $\Delta t = 1$, the price of this bond as seen at node B is $e^{-(\alpha_1 + 0.01732)}$.

- Similarly, the price as seen at node C is $e^{-\alpha_1}$ and the price as seen at node D is $e^{-(\alpha_1 - 0.01732)}$. The price as seen at the initial node A is therefore

$$Q_{1,1}e^{-(\alpha_1+0.01732)} + Q_{1,0}e^{-\alpha_1} + Q_{1,-1}e^{-(\alpha_1-0.01732)}.$$

From the initial term structure, this bond price should be $e^{-0.04512\times 2} = 0.9137$. Substituting for the Q's in the above equation, we obtain

$$0.1604e^{-(\alpha_1+0.01732)} + 0.6417e^{-\alpha_1} + 0.1604e^{-(\alpha_1-0.01732)} = 0.9137$$

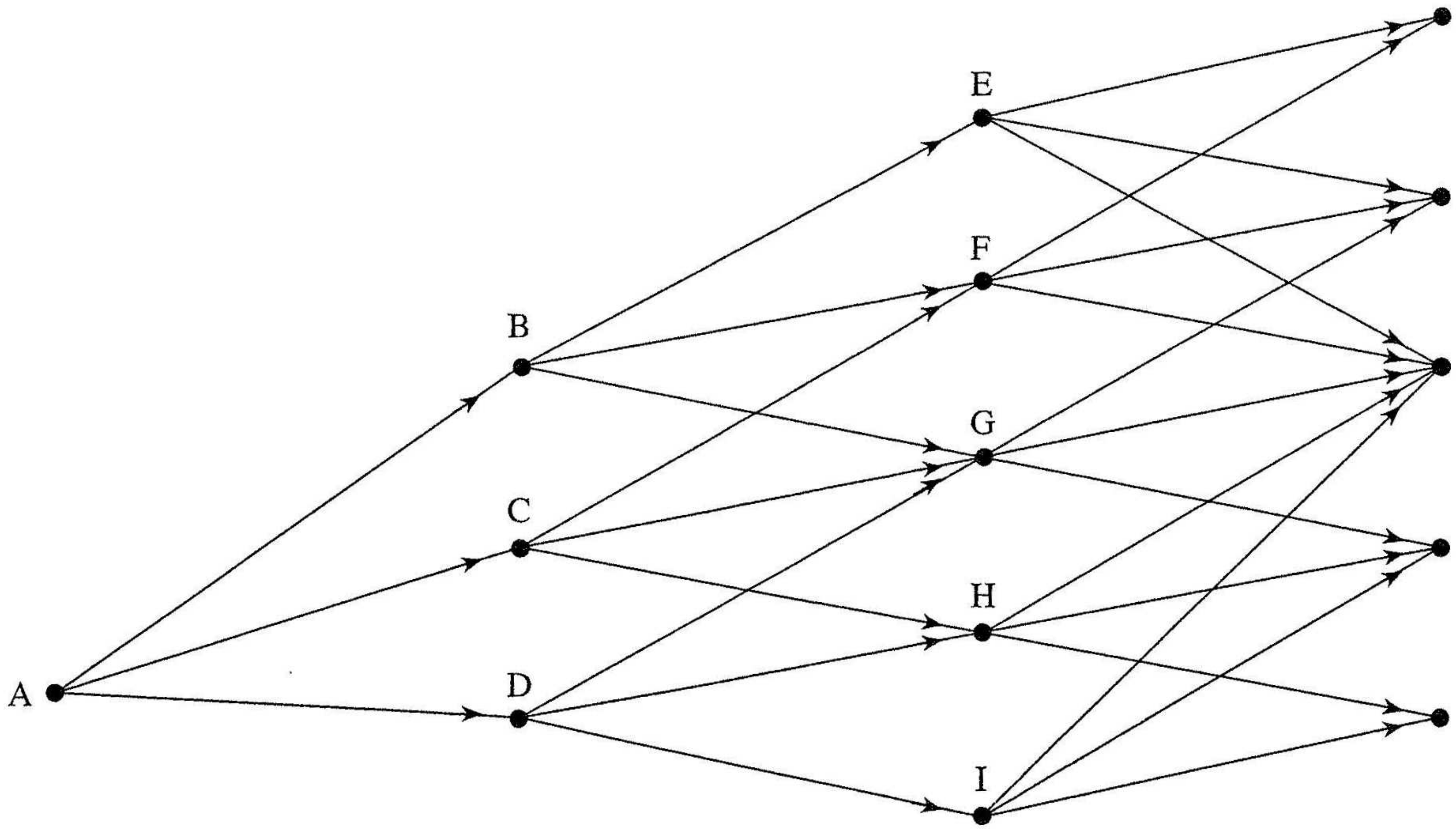
or

$$e^{\alpha_1}(0.1604e^{-0.01732} + 0.6417 + 0.1604e^{0.01832}) = 0.9137$$

or

$$\alpha_1 = \ln\left[\frac{0.1604e^{-0.01732} + 0.6417 + 0.1604e^{0.01732}}{0.9317}\right] = 0.05205.$$

This means that the central node at time Δt in the tree for R corresponds to an interest rate of 5.205% (see Figure 3).



Node:	A	B	C	D	E	F	G	H	I
R (%)	3.824	6.937	5.205	3.473	9.716	7.984	6.252	4.520	2.788
p_u	0.1667	0.1217	0.1667	0.2217	0.8867	0.1217	0.1667	0.2217	0.0867
p_m	0.6666	0.6566	0.6666	0.6566	0.0266	0.6566	0.6666	0.6566	0.0266
p_d	0.1667	0.2217	0.1667	0.1217	0.0867	0.2217	0.1667	0.1217	0.8867

Figure 3. Tree for R in Hull-White Model (second stage)

- The next step is to calculate $Q_{2,2}, Q_{2,1}, Q_{2,0}, Q_{2,-1}$, and $Q_{2,-2}$. The calculations can be shortened by using previously determined Q values.

- Consider $Q_{2,1}$ as an example. This is the value of a security that pays off $1 if node F is reached and zero otherwise. Node F can be reached only from nodes B and C. The interest rates at these nodes are 6.937% and 5.205%, respectively. The probabilities associated with the B-F and C-F branches are 0.6566 and 0.1667.

- The value at node B of a security that pays \$1 at node F is therefore $0.6566e^{-0.06937}$. The value at node C is $0.1667e^{-0.05205}$.

- The variable $Q_{2,1}$ is $0.6566e^{-0.06937}$ times the present value of \$1 received at node B plus $0.1667e^{-0.05205}$ times the present value of \$1 received at node C; that is,

  $$Q_{2,1} = 0.6566e^{-0.0693} \times 0.1604 + 0.1667e^{-0.05205} \times 0.6417 = 0.1998.$$

  Similarly, $Q_{2,2} = 0.0182, Q_{2,0} = 0.4736, Q_{2,-1} = 0.2033$, and $Q_{2,-2} = 0.0189$.

The next step in producing the R-tree in Figure 3 is to calculate α_2. After that, the $Q_{3,j}$'s can then be computed. We can then calculate α_3; and so on.

Formulas for α's and Q's

To express the approach more formally, we suppose that the $Q_{i,j}$ have been determined for $i \leq m\,(m \geq 0)$. The next step is to determine α_m so that the tree correctly prices a zero-coupon bond maturing at $(m+1)\Delta t$. The interest rate at node (m, j) is $\alpha_m + j\Delta R$, so that the price of a zero-coupon bond maturing at time $(m+1)\Delta t$ is given by

$$p_{m+1} = \sum_{j=-n_m}^{n_m} Q_{m,j}\,\text{exp}[-(\alpha_m + j\Delta R)\Delta t]$$

where n_m is the number of nodes on each side of the central node at time $m\Delta t$.

The solution to this equation is

$$\alpha_m = \frac{\ln \sum_{j=-n_m}^{n_m} Q_{m,j} e^{-j\Delta R \Delta t} - \ln P_{m+1}}{\Delta t}.$$

Once α_m has been determined, the $Q_{i,j}$ for $i = m + 1$ can be calculated using

$$Q_{m+1,j} = \sum_k Q_{m,k} q(k,j) \exp[-(\alpha_m + k\Delta R)\Delta t]$$

where $q(k,j)$ is the probability of moving from node (m,k) to node $(m+1,j)$ and the summation is taken over all values of k for which this is nonzero.

Extension to other models

The procedure that has just been outlined can be extended to more general models of the form

$$df(r) = [\theta(t) - af(r)]\, dt + \sigma\, dZ.$$

The family of models has the property that they can fit any term structure.

As before, we assume that the Δt period rate, R, follows the same process as r:

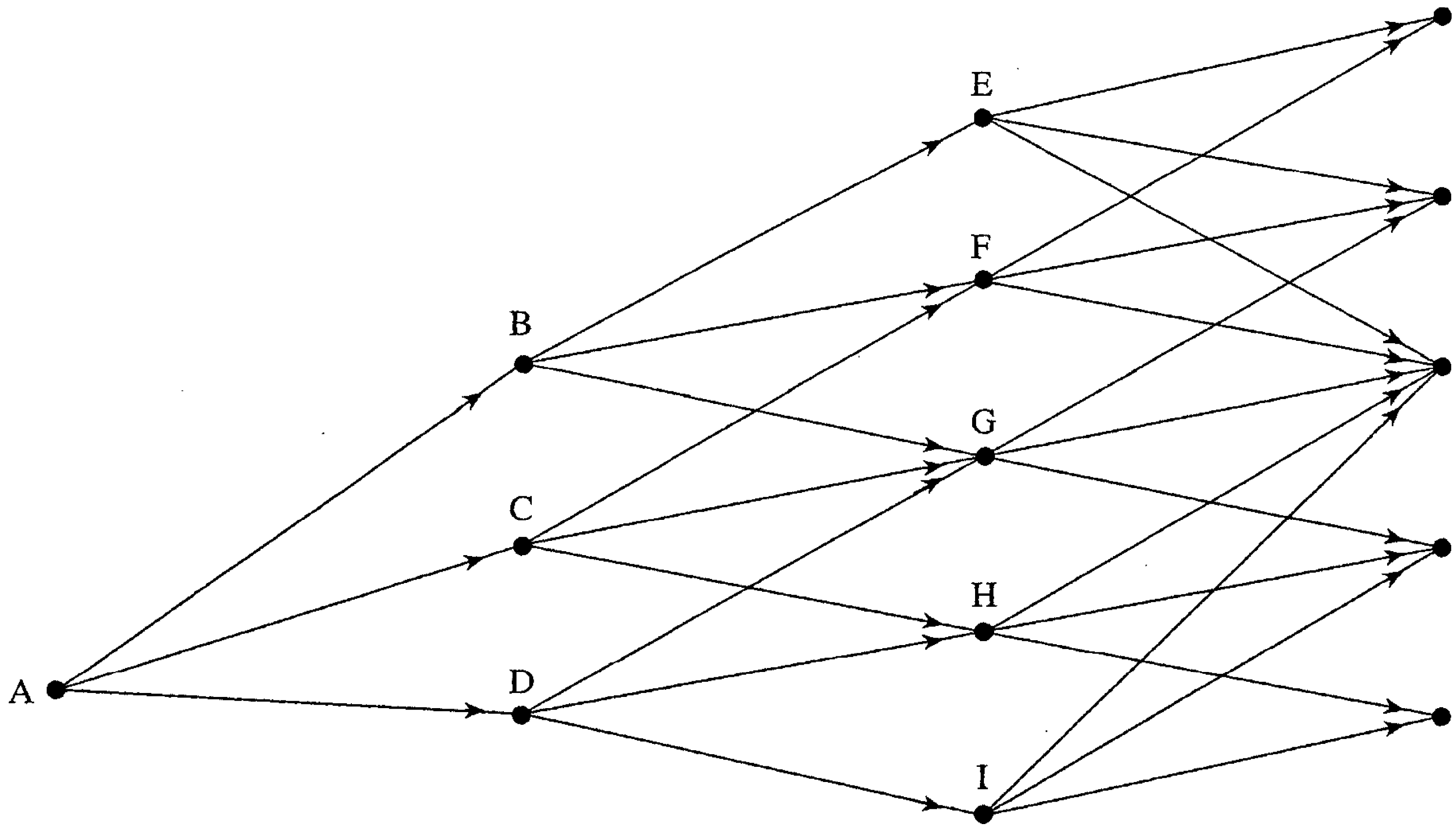
$$df(R) = [\theta(t) - af(R)]\, dt + \sigma\, dZ.$$

We start by setting $x = f(R)$, so that

$$dx = [\theta(t) - ax]\, dt + \sigma\, dZ.$$

The first stage is to build a tree for a variable x^* that follows the same process as x except that $\theta(t) = 0$ and the initial value is zero. The procedure here is identical to the procedure already outlined for building a tree such as that in Figure 2.

As in Figure 3, we then displace the nodes at time $i\Delta t$ by an amount α_i to provide an exact fit to the initial term structure. The equations for determining α_i and $Q_{i,j}$ inductively are slightly different from those for the $f(R) = R$ case.



Node:	A	B	C	D	E	F	G	H	I
x	−3.373	−2.875	−3.181	−3.487	−2.430	−2.736	−3.042	−3.349	−3.655
R (%)	3.430	5.642	4.154	3.058	8.803	6.481	4.772	3.513	2.587
p_u	0.1667	0.1177	0.1667	0.2277	0.8609	0.1177	0.1667	0.2277	0.0809
p_m	0.6666	0.6546	0.6666	0.6546	0.0582	0.6546	0.6666	0.6546	0.0582
p_d	0.1667	0.2277	0.1667	0.1177	0.0809	0.2277	0.1667	0.1177	0.8609

Figure 4. Tree for lognormal model

- The value of Q at the first node, $Q_{0,0}$, is set equal to 1.

- Suppose that the $Q_{i,j}$ have been determined for $1 \leq m (m \geq 0)$.

- The next step is to determine α_m so that the tree correctly prices an $(m+1)\Delta t$ zero-coupon bond.

- Define g as the inverse function of f so that the Δt-period interest rate at the j^{th} node at time $m\Delta t$ is

$$g(\alpha_m + j\Delta x).$$

- The price of a zero-coupon bond maturing at time $(m+1)\Delta t$ is given by

$$P_{m+1} = \sum_{j=-n_m}^{n_m} Q_{m,j} \exp[-g(\alpha_m + j\Delta x)\Delta t].$$

- This equation can be solved using a numerical procedure such as Newton-Raphson. The value α_0 of α when $m = 0$, is $f(R(0))$.

- Once α_m has been determined, the $Q_{i,j}$ for $i = m+1$ can be calculated using

$$Q_{m+1,j} = \sum_k Q_{m,k} q(k,j) \exp[-g(\alpha_m + k\Delta x)\Delta t]$$

where $q(k,j)$ is the probability of moving from node (m,k) to node $(m+1,j)$ and the summation is taken over all values of k where this is nonzero.

- Figure 4 shows the results of applying the procedure to the model

$$d\ln(r) = [\theta(t) - a\ln(r)]\,dt + \sigma\,dZ$$

when $a = 0.22, \sigma = 0.25, \Delta t = 0.5$, and the zero rates are as in the Table.

Various choices of $f(R)$

- When $f(r) = r$ we obtain the Hull-White model.
- When $f(r) = \ln r$ we obtain the Black-Karasinksi model. —in most circumstances these two models appear to perform about equally well in fitting market data on actively traded instruments such as caps and European swap options.
- The main advantage of the $f(r) = r$ model is its analytic tractability. Its main disadvantage is that negative interest rates are possible.

- In most circumstances, the probability of negative interest rates occurring under the model is very small, but some analysts are reluctant to use a model where there is any chance at all of negative interest rates.

- The $f(r) = \ln r$ model has no analytic tractability, but has the advantage that interest rates are always positive. Another advantage is that traders naturally think in terms of σ's arising from a lognormal model rather than σ's arising from a normal model.

- There is a problem in choosing a satisfactory model for countries with low interest rates.

- The normal model is unsatisfactory because, when the initial short rate is low, the probability is unsatisfactory because the volatility of rates (i.e., the σ parameter in the lognormal model) is using much greater when rates are low than when they are high.

- For example, a volatility of 100% might be appropriate when the short rate is less than 1%, while 20% might be appropriate when it is 4% or more.

- A model that appears to work well is one where $f(r)$ is chosen so that rates are lognormal for r less than 1% and normal for r greater than 1%.

Pricing of interest rate products

Once the Arrow-Debreu prices are available, it becomes straightforward to price any interest rate products based on the calibrated interest rate trees.

In the continuous version, the Arrow-Debreu price $G(r, 0; r, T)$ is defined by

$$G(r_0, 0; r, T) = E_0 \left[\exp\left(-\int_0^T r_u \, du \right) \delta(r_T - r) \Big|_{r_{t=0}=r_0} \right].$$

This corresponds to the value at time 0, given current state r_0, of a riskless security that pays one dollar if state $r_T = r$ is attained at any later time $T > 0$.

More theoretical formulas

The zero-coupon bonds are given by

$$P(0,T) = \int_0^\infty G(r_0, 0; r, T)\, dr.$$

Continuity relation:

$$G(r_0, 0; r_i, T_i) = \int_0^\infty G(r_0, 0; r_{i-1}, T_{i-1}) G(r_{i-1}, T_{i-1}; r_i, T_i)\, dr_{i-1}.$$

Price of a discount bond at time $t \geq 0$ (any time later than current time), with time to maturity of Δt, conditional on the short rate having value r at time t is given by

$$\begin{aligned} P(r, t; t + \Delta t) &= E_t \left[\exp\left(- \int_t^{t+\Delta t} r_u\, ds \right) \bigg|_{r_t = r} \right] \\ &= \int_0^\infty G(r, t; r_T, T = t + \Delta t)\, dr_T. \end{aligned}$$

Under the discrete calibrated interest rate tree, the conditional zero-coupon bonds are obtained from

$$P(j, T_i, T_{i+n}) = \sum_{k=j-n}^{j+n} G(j, T_i; k, T_{i+n}),$$

where the $2n+1$ Arrow-Debreu prices (conditional on beginning at a j^{th} node at time T_i and ending at node $k = j-n, \cdots, j+n$ at time T_{i+n}) are computed by the general forward recursion relation

$$G(i, j; k, T_{i+m}) = \sum_{S; |s| \leq i+m-1} p(k, s) e^{-r(s, i+m-1)\Delta t} G(j, T_i; s, T_{i+m-1}),$$

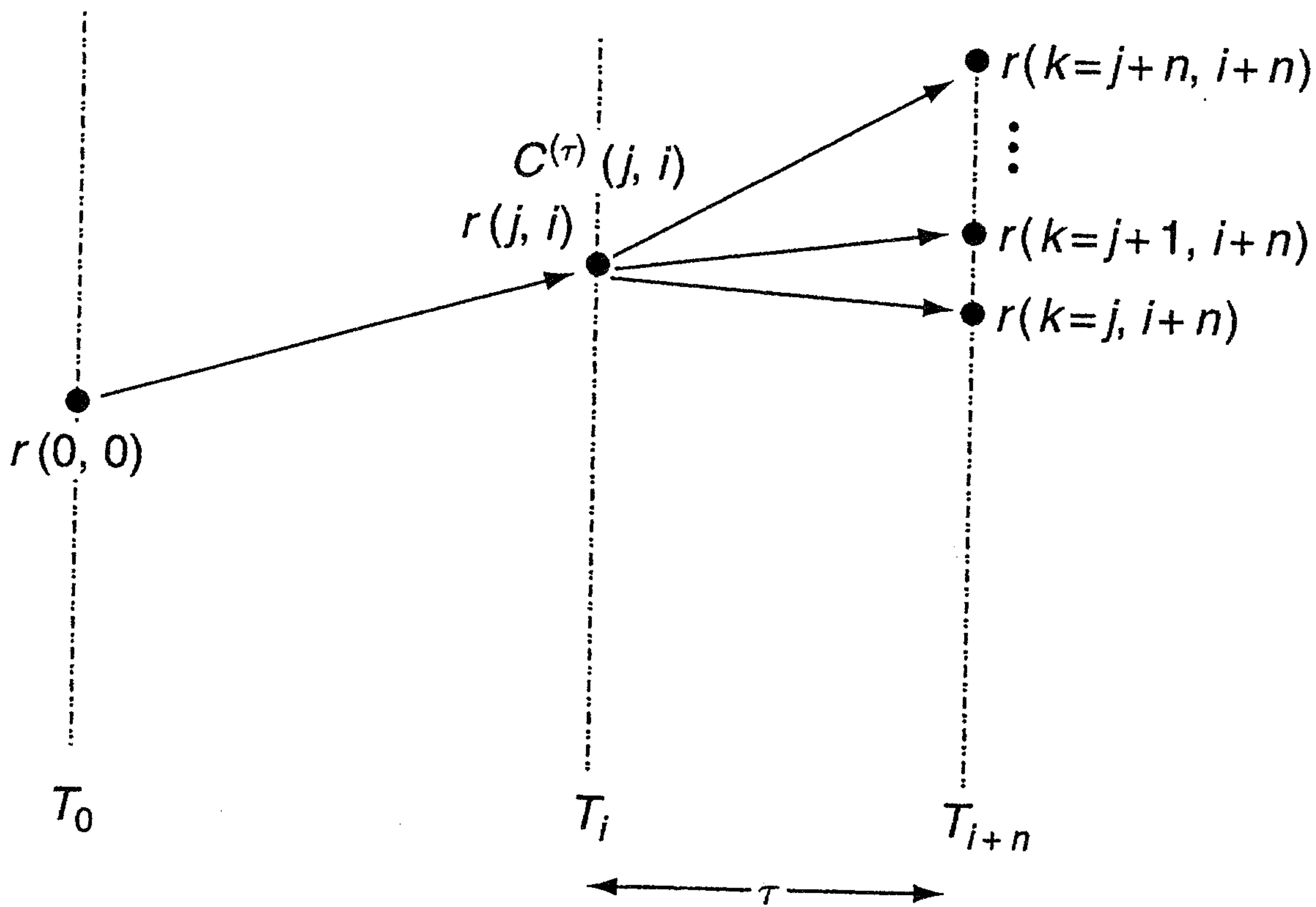
where $p(k, s)$ are the nodal transition probabilities. Note that the starting node is index s and the ending node is index k.

Pricing of a caplet

$$C^{(I)}_{PI_0}(R_K, T_i) = \sum_{j=-i}^{i} G(0,0;j,T_i)C^{(\tau)}(j,i)$$

a caplet valued at current time T_0 and maturity at time T_i of tenor $\tau = n\Delta t$.

The initial leg starting from the current time node $r(0,0)$ gives the Arrow-Debreu prices $G(0,0;j,T_i)$ at each j^{th} node $r(j,i)$ at time T_i. The payoff vector of the caplet with j^{th} component $C^{(\tau)}(j,i)$ (for the j^{th} node at time T_i) is obtained by summing all the Arrow-Debrue prices $G(j,T_i;k,T_{i+n})(k = j,\cdots,j+n)$ that are conditional on starting at the node $r(j,i)$ at time T_i and ending at nodes $r(k,i+n)$ at time T_{i+n} for the period of the caplet.


$r(k=j+n,\ i+n)$
$C^{(\tau)}(j,\ i)$
$r(j,\ i)$
$r(k=j+1,\ i+n)$
$r(k=j,\ i+n)$
$r(0,\ 0)$
T_0
T_i
T_{i+n}
τ