Journal of International Money and Finance (1983), **2**, 231–237

Foreign Currency Option Values

Mark B. Garman and Steven W. Kohlhagen*

School of Business Administration, University of California at Berkeley, Berkeley CA 94720, USA

Foreign exchange options are a recent market innovation. The standard Black–Scholes option-pricing model does not apply well to foreign exchange options, since multiple interest rates are involved in ways differing from the Black–Scholes assumptions. The present paper develops alternative assumptions leading to valuation formulas for foreign exchange options. These valuation formulas have strong connections with the commodity-pricing model of Black (1976) when forward prices are given, and with the proportional-dividend model of Samuelson and Merton (1969) when spot prices are given.

Foreign exchange options (hereafter 'FX options') are an important new market innovation. They provide a significant expansion in the available risk-control and speculative instruments for a vital source of risk, namely foreign currency values. The purpose of this paper is to develop the relevant pricing formulas for FX options.

The deliverable instrument of an FX option is a fixed amount of underlying foreign currency. In the standard Black–Scholes (1973) option-pricing model, the underlying deliverable instrument is a non-dividend-paying stock. The difference between the two underlying instruments is readily seen when we compare their equilibrium forward prices. When interest rates are constant (as in the Black–Scholes assumptions), the forward price of the stock must, by arbitrage, command a forward premium equal to the interest rate. But in the foreign currency markets, forward prices can involve either forward premiums or discounts. This is because the forward value of a currency is related to the ratio of the prices of riskless bonds traded in each country. The familiar arbitrage relationship ('interest rate parity') correspondingly asserts that the forward exchange premium must equal the interest rate *differential*, which may be either positive or negative. Thus both foreign and domestic interest rates play a role in the valuation of these forward contracts, and it is therefore logical to expect that such a role extends to options as well. That this is indeed the case we shall see below.

* The authors gratefully acknowledge comments contributed during the course of this research by Fischer Black, Robert Geske, Richard Roll, and Terry Turner, without implicating them in any errors contained herein.

00261-5606/83/03/0231-07$03 00

I. Development

We use notation as follows:

S	= the spot price of the deliverable currency (domestic units per foreign unit)
F	= the forward price of the currency delivered at option maturity
K	= exercise price of option (domestic units per foreign unit)
T	= time remaining until maturity of option
$C(S,T)$	= the price of an FX call option (domestic units per foreign unit)
$P(S,T)$	= the price of an FX put option (domestic units per foreign unit)
r_D	= the domestic (riskless) interest rate
r_F	= the foreign (riskless) interest rate
σ	= volatility of the spot currency price
μ	= drift of the spot currency price
$N(.)$	= cumulative normal distribution function
α	= the expected rate of return on a security
δ	= the standard deviation of the security rate of return

Our assumptions are the usual ones for an option-pricing model, that:

1. Geometric Brownian motion governs the currency spot price: i.e., the differential representation of spot price movements is $dS = \mu S dt + \sigma S dz$, where z is the standard Wiener process.
2. Option prices are a function of only one stochastic variable, namely S.
3. Markets are frictionless.
4. Interest rates, both in the domestic and foreign markets, are constant.[1]

As is also usual, our analysis shall pertain to European FX options: options which can be exercised only on their maturity date. The American options, which may be exercised at any time prior to maturity, are discussed later.

The key to understanding FX option pricing is to properly appreciate the role of foreign and domestic interest rates. We do this by comparing the advantages of holding an FX option with those of holding its underlying currency. As is well known, the risk-adjusted expected excess returns of securities governed by our assumptions must be identical in an arbitrage-free continuous-time economy.[2] That is, we must have

$$\langle 1 \rangle \qquad \frac{\alpha_i - r_D}{\delta_i} = \lambda, \textit{ for all } i$$

where λ does not depend on the security considered.[3] Applying this fact to the ownership of foreign currency, we have[4]

$$\langle 2 \rangle \qquad \frac{(\mu + r_f) - r_D}{\sigma} = \lambda$$

That is, the expected return from holding the foreign currency is μ, the 'drift' of the exchange rate (domestic units per foreign unit), plus the riskless capital growth arising from holding the foreign currency in the form of an asset (e.g., foreign treasury notes and CD's) paying interest at the rate of r_F. The denominator of the left-hand-side of equation $\langle 2 \rangle$ is σ, since this is the standard deviation of the rate of return on holding the currency. (Note that μ, σ, r_F, and r_D are all dimensionless quantities, so there is no issue of conversion between foreign and domestic terms.)

Next, letting $C(S,T)$ be the price of a European call option with time T left to maturity, $\langle 1 \rangle$ implies

$$\langle 3 \rangle \qquad \frac{\alpha_C - r_D}{\delta_C} = \lambda$$

where α_C and δ_C are the call option's expected rate of return and standard deviation of same, respectively. By Ito's lemma, we have

$$\langle 4 \rangle \qquad \alpha_C C = \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + \mu S \frac{\partial C}{\partial S} - \frac{\partial C}{\partial T}$$

and

$$\langle 5 \rangle \qquad \delta_C S = \sigma S \frac{\partial C}{\partial S}$$

Substituting $\langle 4 \rangle$ and $\langle 5 \rangle$ into $\langle 3 \rangle$ yields

$$\langle 3' \rangle \qquad \frac{\tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + \mu S \frac{\partial C}{\partial S} - \frac{\partial C}{\partial T} - r_D C}{\sigma S \frac{\partial C}{\partial S}} = \lambda$$

Thus equating $\langle 2 \rangle$ and $\langle 3 \rangle$ we have

$$\langle 6 \rangle \qquad \frac{\sigma^2}{2} S^2 \frac{\partial^2 C}{\partial S^2} - r_D C + (r_D S - r_F S)\frac{\partial C}{\partial S} = \frac{\partial C}{\partial T}$$

The latter equation is reminiscent of models proposed by Samuelson (1965) and Samuelson and Merton (1969), in which the dividend rate of a stock is presumed to be proportional to the level of the stock price. Indeed, there is a similar interpretation for foreign currency options. Consider r_F as the 'dividend rate' of the foreign currency. However, this rate is in foreign terms, so to convert to domestic terms, one would naturally multiply it by the spot exchange rate S. The Samuelson–Merton model has not received a great deal of attention in the literature, probably because of its rather strained assumption of a proportional dividend policy. That is, under their model, a firm must constantly monitor its stock price and adjust a continuously-paid dividend as a fixed fraction of that price. This is rather impractical as a realistic dividend policy. But in the foreign exchange context, the 'adjustment of dividends' takes place in an automatic fashion, since the conversion from foreign to domestic currency terms at the market exchange rate is natural for dimensional consistency within $\langle 6 \rangle$.

II. Solutions

The solution to $\langle 6 \rangle$ for a European FX call option must obey the further boundary condition that $C(S,0) = \max[0, S-K]$, yielding[5] the valuation formula

$$\langle 7 \rangle \qquad C(S,T) = e^{-r_F T} S\, N(x + \sigma\sqrt{T}) - e^{-r_D T} K\, N(x)$$

where

$$x \equiv \frac{\ln(S/K) + \{r_D - r_F - (\sigma^2/2)\}\, T}{\sigma\sqrt{T}}$$

Note that both the foreign interest rate r_F, and the interest differential, $r_D - r_F$, play distinct roles in the solution.

Of course, equation ⟨6⟩ governs all securities satisfying our original assumptions. Thus the European FX put option also satisifes that differential equation, but with the boundary condition $P(S,0)=\max[0,K-S]$. Hence the solution to the European FX put option is given as

$$\langle 8\rangle \qquad P(S,T) = e^{-r_F T} S\,[N(x+\sigma\sqrt{T})-1] - e^{-r_D T} K\,[N(x)-1]$$

where x is as defined for the call option.[6]

III. Comparative Statics

The partial derivatives of formula ⟨7⟩ are also of interest, and these are computed below. Foremost in significance is the 'hedge ratio':

$$\langle 9\rangle \qquad \frac{\partial C}{\partial S} = e^{-r_F T} N(x+\sigma\sqrt{T}) > 0$$

Other partial derivatives are:

$$\langle 10\rangle \qquad \frac{\partial C}{\partial K} = -e^{-r_D T} N(x) < 0$$

$$\langle 11\rangle \qquad \frac{\partial C}{\partial \sigma} = e^{-r_D T} K\sqrt{T}\, N'(x) > 0$$

$$\langle 12\rangle \qquad \frac{\partial C}{\partial r_D} = T e^{-r_D T} K N(x) > 0$$

$$\langle 13\rangle \qquad \frac{\partial C}{\partial r_F} = -T e^{-r_F T} S\, N(x+\sigma\sqrt{T}) < 0$$

and

$$\langle 14\rangle \quad \frac{\partial C}{\partial T} \equiv -r_F e^{-r_F T} S\, N(x+\sigma\sqrt{T}) + r_D e^{-r_D T} K N(x) + \frac{e^{-r_D T}\sigma}{2\sqrt{T}} K N'(x)$$

Interpreting, when other variables (significantly the spot rate) are held constant, FX European call values rise when the domestic interest rate increases, and fall when the foreign rate increases. Increases in volatility uniformly give rise to increases in FX option prices, while increases in the strike price cause FX call option prices to decline. However, the sign of the time derivative is ambiguous. In-the-money calls tend to have negative signs for this derivative when the time to maturity is short. The situation is exacerbated when the calls become deep-in-the-money or when foreign interest rates rise well above domestic rates. Of course, a negative time derivative could not pertain to an American FX option, and so we see that the European formulas for calls (and puts) are clearly inadequate descriptions of their American counterparts in these cases. (See also the discussion by Merton (1973) for the proportional-dividend case.)

The derivatives of the European FX put options are obtained analogously from ⟨8⟩, with the obvious changes in sign for the derivatives involved.

IV. Relationship to Contemporaneous Forward Price

Asserting the familiar relationship known as 'interest rate parity' (Keynes, 1923), the forward price of currency deliverable contemporaneously[7] with the maturation of the option is[8]

$$F = e^{(r_D - r_F)T} S \tag{15}$$

Substituting this relation into the solution ⟨7⟩ gives the alternate solution[9]

$$C(F,T) = \{F N(x+\sigma\sqrt{T}) - K N(x)\} e^{-r_D T} \tag{16}$$

where

$$x \equiv \frac{\ln(F/K) - (\sigma^2/2)\,T}{\sigma\sqrt{T}}$$

Note that with this substitution the call value depends only upon F and r_D; it does not depend independently upon S and r_F. That is, given the current domestic rate of interest, all option-relevant information concerning the foreign interest rate and the spot currency price is reflected in the forward price.

The European put value formula is analogous:

$$P(F,T) = \{F[N(x+\sigma\sqrt{T})-1] - K[N(x)-1]\} e^{-r_D T} \tag{17}$$

We now augment some conclusions regarding comparative statics, this time using the forward-based formula ⟨16⟩. The derivative of the call value with respect to forward price is given as

$$\frac{\partial C}{\partial F} = e^{-r_D T} N(x+\sigma\sqrt{T}) > 0 \tag{18}$$

However, some caution should be observed in applying this latter derivative as a 'hedge ratio'. This is because the forward price is not equivalent to the value of a forward contract, the latter being the important determinant of current wealth at risk. Rather, the forward price is a parameter, not unlike a strike price, which is continuously adjusted so as to make the value of the forward contract identically zero. Consequently, the forward price must be discounted by the factor $e^{(r_F - r_D)T}$ to properly reflect current values, and hence the correct 'hedge ratio' between wealth at risk in forward and option contract positions is as given previously in ⟨9⟩.

With regard to other partial derivatives, we have

$$\frac{\partial C}{\partial K} = -e^{-r_D T} N(x) < 0 \tag{19}$$

and

$$\frac{\partial C}{\partial \sigma} = e^{-r_D T} K\sqrt{T}\, N'(x) > 0 \tag{20}$$

exactly as before. However, the sign of the domestic interest rate partial derivative is just the opposite of the previous section:

$$\frac{\partial C}{\partial r_D} = -e^{-r_D T} T\{F N(x+\sigma\sqrt{T}) - K N(x)\} = -TC < 0 \tag{21}$$

That is, if the contemporaneous forward rate is held constant, an increase in domestic interest rates results in a decrease in FX call values. Finally, we have

$$\langle 22\rangle \qquad \frac{\partial C}{\partial T} = -r_D C + e^{-r_D T}\frac{\sigma}{2\sqrt{T}} K N'(x)$$

Again the last derivative is ambiguous in sign, reflecting the European, as opposed to American, nature of the options treated.

V. Comments on American FX Options

As noted previously, the European formulas will not serve to adequately price American FX options. (See also Samuelson (1965), Samuelson and Merton (1969), and Merton (1973).) Early exercise is decidedly a factor in pricing the American options,[10] and affects primarily the deep-in-the-money options (particularly calls on currencies with negative forward premiums and puts on currencies with positive forward premiums). Of course, American FX options must conform to the basic differential equation ⟨6⟩. However, the boundary conditions differ from the European case inasmuch as the option prices must never be less than the immediate conversion value, e.g.

$$C(S,T) \geq \max[0, S-K]$$

for all T. Following the methodology of Merton (1973), it can also be shown that

$$C(S,T) \geq \max[0, Se^{-r_F T} - Ke^{-r_D T}]$$

for both the European and American cases.

Analytic solutions for the above type of boundary conditions problem seem quite difficult to derive. Therefore numerical methods, such as proposed by Brennan and Schwartz (1977), Parkinson (1977), or Cox, Ross and Rubinstein (1979) (all recently reviewed by Geske and Shastri (1982)), are indicated for the evaluation of such American options.

VI. Conclusions

The appropriate valuation formulas for European FX options depend importantly on both foreign and domestic interest rates. The present paper has developed such formulas, and these are closely related to the proportional-dividend model when the spot prices are given, and to the commodity-pricing model when contemporaneous forward prices are given. The comparative statics are as might be expected, with two exceptions: the reaction of FX option prices to interest rate changes depends upon the nature of the concommitant changes required in either the spot or forward currency markets. Finally, American FX option values exceed the European FX option values most markedly for deep-in-the-money options, particularly for calls on currencies with negative forward premiums and puts on currencies with positive forward premiums.

Notes

1. The analysis could be extended without much difficulty to stochastic interest rates, by assuming that the market is 'neutral' towards the sources of uncertainty driving such rates. In this case, volatility parameters must be redefined to incorporate the variances and covariances of interest

rate movements as well as spot price movements. However, we forego this extension in the interest of clarity.

2. This is true, however, for only the case where there is a single source of uncertainty considered; multiple sources give rise to multiple volatility factors and risk premia, which are better expressed in alternative forms. Also, it is important to emphasize that the invariance of the risk-adjusted excess return is a pure arbitrage result, and does not depend upon any specific asset pricing model in a continuous-time (diffusion) setting.
3. In general, λ may depend on time and the state variables involved; however, in this particular case it is a constant.
4. The more usual presentation of our formula ⟨2⟩ would be $\mu=(r_D-r_F)+\lambda\sigma$, emphasizing that the expected return can be decomposed into an interest-rate-related drift and a risk premium. The form given emphasizes the invariance of risk premia across securities, in order to compare these.
5. The solution proceeds analogously to Merton's (1973) description of the proportional-dividend model, replacing his dividend rate d by the foreign interest rate, as noted previously.
6. Alternatively, we could use put-call parity to determine the put option formula without resolving ⟨6⟩.
7. At the current writing, FX options are traded on the Philadelphia Stock Exchange and were designed to mature concurrently with the IMM currency futures contracts, in March, June, September, and December.
8. For an introduction to exchange rate relationships, see for example the recent text by Shapiro (1982). This particular relationship is a pure-arbitrage result which employs riskless bonds of maturity identical to the forward contract, which of course can be created when instantaneous interest rates are constant.
9. This solution, although derived in a somewhat different fashion, is equivalent to Black's (1976) commodity option-pricing formula, showing that FX options may be treated on the same basis as commodity options generally, provided that the contemporaneous forward instruments exist.
10. At typical currency parameter values, it is not unusual to see a 10–20% difference between American and European values for certain in-the-money options.

References

BRENNAN, M. AND E. SCHWARTZ, 'The Valuation of American Put Options', *J. Finance*, May 1977, **32:** 449–462.

BLACK, F., 'The Pricing of Commodity Contracts', *J. Financial Econ.*, January 1976, **3:** 167–179.

BLACK, F. AND M. SCHOLES, 'The Pricing of Options and Corporate Liabilities', *J. Pol. Econ.* May/June 1973, **81:** 637–654.

COX, J., S. ROSS AND M. RUBINSTEIN, 'Option Pricing: A Simplified Approach', *J. Financial Econ.* September 1979, **7:** 229–263.

GESKE, R. AND K. SHASTRI, 'Valuation by Approximation: A Comparison of Alternative Option Valuation Techniques', working paper, Graduate School of Management, UCLA, 1982.

KEYNES, J., *A Tract on Monetary Reform*, London: Macmillan, 1923.

MERTON, R.C., 'Theory of Rational Option Pricing', *Bell J. Econ. and Management Sci.*, Spring 1973, **4:** 141–183.

PARKINSON, M. 'Option Pricing: The American Put', *J. Business*, January 1977; **50:** 21–36

SAMUELSON, P.A., 'Rational Theory of Warrant Pricing', *Ind. Management Rev.*, Spring 1965, **6:** 13–31.

SAMUELSON, P. AND R. MERTON, 'A Complete Model of Warrant Pricing that Maximizes Utility', *Ind. Management Rev.*, Winter 1969, **10:** 17–46.

SHAPIRO, A., *Multinational Financial Management*, Boston: Allyn and Bacon, 1982.