MODULAR DEUTSCH ENTROPIC UNCERTAINTY PRINCIPLE

K. Mahesh Krishna
School of Mathematics and Natural Sciences
Chanakya University Global Campus
Haraluru Village, Near Kempe Gowda International Airport (BIAL)
Devanahalli Taluk, Bengaluru Rural District
Karnataka 562 110 India
Email: kmaheshak@gmail.com
Date: June 14, 2024

Abstract: Khosravi, Drnovšek and Moslehian [*Filomat, 2012*] derived Buzano inequality for Hilbert C*-modules. Using this inequality we derive Deutsch entropic uncertainty principle for Hilbert C*-modules over commutative unital C*-algebras.
Keywords: Buzano inequality, Entropic uncertainty, Hilbert C*-module.
Mathematics Subject Classification (2020): 46L08, 42C15, 46L05.

1. Introduction

Let $\mathcal{H}$ be a finite dimensional Hilbert space. Given an orthonormal basis $\{\tau_j\}_{j=1}^n$ for $\mathcal{H}$, the **Shannon entropy** at a point $h \in \mathcal{H}_\tau$ is defined as

$$S_\tau(h) \coloneqq -\sum_{j=1}^{n} |\langle h, \tau_j\rangle|^2 \log |\langle h, \tau_j\rangle|^2, \tag{1}$$

where $\mathcal{H}_\tau \coloneqq \{h \in \mathcal{H} : \|h\| = 1, \langle h, \tau_j\rangle \neq 0, 1 \leq j \leq n\}$ [3]. In 1983, Deutsch derived following breakthrough entropic uncertainty principle for Shannon entropy [3].

Theorem 1.1. *[3] (**Deutsch Entropic Uncertainty Principle**) Let $\{\tau_j\}_{j=1}^n$, $\{\omega_k\}_{k=1}^n$ be two orthonormal bases for a finite dimensional Hilbert space $\mathcal{H}$. Then*

$$S_\tau(h) + S_\omega(h) \geq -2\log\left(\frac{1 + \max\limits_{1\leq j,k\leq n} |\langle \tau_j, \omega_k\rangle|}{2}\right), \quad \forall h \in \mathcal{H}_\tau \cap \mathcal{H}_\omega. \tag{2}$$

The inequality (2) is recently derived for Banach spaces [9]. It is observed very recently that using Buzano inequality (see [1, 5, 14]) one can provide a simple proof of Theorem 1.1 (see Corollary 1 in [9]). As Hilbert C*-modules became more important in noncommutative geometry, we are mainly motivated from the following problem. What is the modular version of Theorem 1.1? Hilbert C*-modules are first introduced by Kaplansky [6] for modules over commutative C*-algebras and later developed for modules over arbitrary C*-algebras by Paschke [11] and Rieffel [13].

Definition 1.2. *[6, 11, 13] Let $\mathcal{A}$ be a unital C*-algebra. A left module $\mathcal{E}$ over $\mathcal{A}$ is said to be a (left) Hilbert C*-module if there exists a map $\langle \cdot, \cdot \rangle : \mathcal{E} \times \mathcal{E} \to \mathcal{A}$ such that the following hold.*

(i) $\langle x, x\rangle \geq 0$, $\forall x \in \mathcal{E}$. *If $x \in \mathcal{E}$ satisfies $\langle x, x\rangle = 0$, then $x = 0$.*
(ii) $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$, $\forall x, y, z \in \mathcal{E}$.
(iii) $\langle ax, y\rangle = a\langle x, y\rangle$, $\forall x, y \in \mathcal{E}$, $\forall a \in \mathcal{A}$.

(iv) $\langle x, y\rangle = \langle y, x\rangle^*$, $\forall x, y \in \mathcal{E}$.
(v) *$\mathcal{E}$ is complete w.r.t. the norm* $\|x\| := \sqrt{\|\langle x, x\rangle\|}$, $\forall x \in \mathcal{E}$.

Our prime tool to derive modular Deutsch uncertainty is the following modular Buzano inequality by Khosravi, Drnovšek, and Moslehian [7].

Theorem 1.3. *[7] (Modular Buzano Inequality) If $\mathcal{E}$ is a Hilbert C*-module over a unital C*-algebra $\mathcal{A}$, then*

$$\|\langle x, z\rangle\langle z, y\rangle\| \leq \frac{1}{2}\left(\|x\|\|y\| + \|\langle x, y\rangle\|\right), \quad \forall x, y, z \in \mathcal{E}, \langle z, z\rangle = 1.$$

In this paper we derive Theorem 1.1 for Hilbert C*-modules over commutative unital C*-algebras.

2. Modular Deutsch Entropic Uncertainty Principle

We begin by recalling the definition of frames for Hilbert C*-modules.

Definition 2.1. *[4] Let $\mathcal{E}$ be a Hilbert C*-module over a C*-algebra $\mathcal{A}$. A collection $\{\tau_j\}_{j=1}^{\infty}$ in $\mathcal{E}$ is said to be a (modular)* ***Parseval frame*** *for $\mathcal{E}$ if*

$$x = \sum_{j=1}^{\infty}\langle x, \tau_j\rangle\tau_j, \quad \forall x \in \mathcal{E}.$$

A collection $\{\tau_j\}_{j=1}^{\infty}$ in a Hilbert C*-module $\mathcal{E}$ over unital C*-algebra $\mathcal{A}$ with identity 1 is said to have **unit inner product** if

$$\langle \tau_j, \tau_j\rangle = 1, \quad \forall j \in \mathbb{N}.$$

In analogy with Equation (1), given a unit inner product Parseval frame $\{\tau_j\}_{j=1}^{\infty}$ for $\mathcal{E}$, we define **modular Shannon entropy** at a point $x \in \mathcal{E}_\tau$ is defined as

$$S_\tau(x) := -\sum_{j=1}^{\infty}\langle x, \tau_j\rangle\langle \tau_j, x\rangle \log(\langle x, \tau_j\rangle\langle \tau_j, x\rangle) \tag{3}$$

where $\mathcal{E}_\tau := \{x \in \mathcal{E} : \langle x, x\rangle = 1, \langle x, \tau_j\rangle \neq 0, \forall j \in \mathbb{N}\}$.

Theorem 2.2. *(**Modular Deutsch Entropic Uncertainty Principle**) Let $\mathcal{E}$ be a Hilbert C*-module over a commutative unital C*-algebra $\mathcal{A}$. Let $\{\tau_j\}_{j=1}^{\infty}$, $\{\omega_k\}_{k=1}^{\infty}$ be two Parseval frames for $\mathcal{E}$. Then*

$$S_\tau(x) + S_\omega(x) \geq -2\log\left(\frac{1 + \sup\limits_{j,k\in\mathbb{N}}\|\langle \tau_j, \omega_k\rangle\|}{2}\right), \quad \forall x \in \mathcal{E}_\tau \cap \mathcal{E}_\omega.$$

Proof. Let $x \in \mathcal{E}_\tau \cap \mathcal{E}_\omega$. Using the Parseval frame property, the commutativity of C*-algebra, Theorem 1.3 and the result that 'function logarithm is operator monotone' [2], we get

$$\begin{aligned}
S_\tau(x) + S_\omega(x) &= -\sum_{j=1}^{\infty} \langle x, \tau_j\rangle\langle \tau_j, x\rangle \log(\langle x, \tau_j\rangle\langle \tau_j, x\rangle) - \sum_{k=1}^{\infty} \langle x, \omega_k\rangle\langle \omega_k, x\rangle \log(\langle x, \omega_k\rangle\langle \omega_k, x\rangle) \\
&= -\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \langle x, \tau_j\rangle\langle \tau_j, x\rangle\langle x, \omega_k\rangle\langle \omega_k, x\rangle \left[\log(\langle x, \tau_j\rangle\langle \tau_j, x\rangle) + \log(\langle x, \omega_k\rangle\langle \omega_k, x\rangle)\right] \\
&= -\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \langle x, \tau_j\rangle\langle \tau_j, x\rangle\langle x, \omega_k\rangle\langle \omega_k, x\rangle \log(\langle x, \tau_j\rangle\langle \tau_j, x\rangle\langle x, \omega_k\rangle\langle \omega_k, x\rangle) \\
&= -\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \langle x, \tau_j\rangle\langle \tau_j, x\rangle\langle x, \omega_k\rangle\langle \omega_k, x\rangle \log(\langle \tau_j, x\rangle\langle x, \omega_k\rangle\langle \omega_k, x\rangle\langle x, \tau_j\rangle) \\
&\geq -\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \langle x, \tau_j\rangle\langle \tau_j, x\rangle\langle x, \omega_k\rangle\langle \omega_k, x\rangle \log\left(\frac{[\|\tau_j\|\|\omega_k\| + \|\langle \tau_j, \omega_k\rangle\|][\|\omega_k\|\|\tau_j\| + \|\langle \omega_k, \tau_j\rangle\|]}{4}\right) \\
&= -\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \langle x, \tau_j\rangle\langle \tau_j, x\rangle\langle x, \omega_k\rangle\langle \omega_k, x\rangle \log\left(\frac{(\|\tau_j\|\|\omega_k\| + \|\langle \tau_j, \omega_k\rangle\|)^2}{4}\right) \\
&= -2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \langle x, \tau_j\rangle\langle \tau_j, x\rangle\langle x, \omega_k\rangle\langle \omega_k, x\rangle \log\left(\frac{\|\tau_j\|\|\omega_k\| + \|\langle \tau_j, \omega_k\rangle\|}{2}\right) \\
&\geq -2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \langle x, \tau_j\rangle\langle \tau_j, x\rangle\langle x, \omega_k\rangle\langle \omega_k, x\rangle \log\left(\frac{1 + \|\langle \tau_j, \omega_k\rangle\|}{2}\right) \\
&\geq -2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \langle x, \tau_j\rangle\langle \tau_j, x\rangle\langle x, \omega_k\rangle\langle \omega_k, x\rangle \log\left(\frac{1 + \sup_{j,k\in\mathbb{N}}\|\langle \tau_j, \omega_k\rangle\|}{2}\right) \\
&= -2\log\left(\frac{1 + \sup_{j,k\in\mathbb{N}}\|\langle \tau_j, \omega_k\rangle\|}{2}\right)\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \langle x, \tau_j\rangle\langle \tau_j, x\rangle\langle x, \omega_k\rangle\langle \omega_k, x\rangle \\
&= -2\log\left(\frac{1 + \sup_{j,k\in\mathbb{N}}\|\langle \tau_j, \omega_k\rangle\|}{2}\right)\langle x, x\rangle\langle x, x\rangle \\
&= -2\log\left(\frac{1 + \sup_{j,k\in\mathbb{N}}\|\langle \tau_j, \omega_k\rangle\|}{2}\right).
\end{aligned}$$

$\square$

In 1988, Maassen and Uffink (motivated from the conjecture by Kraus made in 1987 [8]) improved Deutsch entropic uncertainty principle.

Theorem 2.3. *[10] (**Maassen-Uffink Entropic Uncertainty Principle**) Let $\{\tau_j\}_{j=1}^n$, $\{\omega_k\}_{k=1}^n$ be two orthonormal bases for a finite dimensional Hilbert space $\mathcal{H}$. Then*

$$S_\tau(h) + S_\omega(h) \geq -2\log\left(\max_{1\leq j,k\leq n} |\langle \tau_j, \omega_k\rangle|\right), \quad \forall h \in \mathcal{H}_\tau \cap \mathcal{H}_\omega.$$

In 2013, Ricaud and Torrésani [12] showed that orthonormal bases in Theorem 2.3 can be improved to Parseval frames.

Theorem 2.4. *[12] (**Ricaud-Torrésani Entropic Uncertainty Principle**) Let $\{\tau_j\}_{j=1}^n$, $\{\omega_k\}_{k=1}^m$ be two Parseval frames for a finite dimensional Hilbert space $\mathcal{H}$. Then*

$$S_\tau(h) + S_\omega(h) \geq -2\log\left(\max_{1\leq j\leq n, 1\leq k\leq m} |\langle \tau_j, \omega_k\rangle|\right), \quad \forall h \in \mathcal{H}_\tau \cap \mathcal{H}_\omega.$$

Proofs of Theorems 2.3 and 2.4 use Riesz-Thorin interpolation (RTI). To the best of author's knowledge, RTI does not exists for abstract Hilbert C*-modules. Therefore we end by formulating the following conjecture.

Conjecture 2.5. *(**Modular Kraus Entropic Conjecture**) Let $\mathcal{E}$ be a Hilbert C*-module over a commutative unital C*-algebra $\mathcal{A}$. Let $\{\tau_j\}_{j=1}^\infty$, $\{\omega_k\}_{k=1}^\infty$ be two Parseval frames for $\mathcal{E}$. Then*

$$S_\tau(x) + S_\omega(x) \geq -2\log\left(\sup_{j,k\in\mathbb{N}} \|\langle \tau_j, \omega_k\rangle\|\right), \quad \forall x \in \mathcal{E}_\tau \cap \mathcal{E}_\omega.$$

References

[1] Maria Luisa Buzano. Generalizzazione della diseguaglianza di Cauchy-Schwarz. *Rend. Sem. Mat. Univ. e Politec. Torino*, 31:405–409 (1974), 1971/73.

[2] Pattrawut Chansangiam. A survey on operator monotonicity, operator convexity, and operator means. *Int. J. Anal.*, pages Art. ID 649839, 8, 2015.

[3] David Deutsch. Uncertainty in quantum measurements. *Phys. Rev. Lett.*, 50(9):631–633, 1983.

[4] Michael Frank and David R. Larson. Frames in Hilbert C^*-modules and C^*-algebras. *J. Operator Theory*, 48(2):273–314, 2002.

[5] Masatoshi Fujii and Fumio Kubo. Buzano's inequality and bounds for roots of algebraic equations. *Proc. Amer. Math. Soc.*, 117(2):359–361, 1993.

[6] Irving Kaplansky. Modules over operator algebras. *Amer. J. Math.*, 75:839–858, 1953.

[7] Maryam Khosravi, Roman Drnovšek, and Mohammad Sal Moslehian. A commutator approach to Buzano's inequality. *Filomat*, 26(4):827–832, 2012.

[8] K. Kraus. Complementary observables and uncertainty relations. *Phys. Rev. D (3)*, 35(10):3070–3075, 1987.

[9] K. Mahesh Krishna. Functional Deutsch uncertainty principle. *J. Class. Anal.*, 23(1):11–18, 2024.

[10] Hans Maassen and J. B. M. Uffink. Generalized entropic uncertainty relations. *Phys. Rev. Lett.*, 60(12):1103–1106, 1988.

[11] William L. Paschke. Inner product modules over B^*-algebras. *Trans. Amer. Math. Soc.*, 182:443–468, 1973.

[12] Benjamin Ricaud and Bruno Torrésani. Refined support and entropic uncertainty inequalities. *IEEE Trans. Inform. Theory*, 59(7):4272–4279, 2013.

[13] Marc A. Rieffel. Induced representations of C^*-algebras. *Advances in Math.*, 13:176–257, 1974.

[14] J. Michael Steele. *The Cauchy-Schwarz master class: An introduction to the art of mathematical inequalities*. Cambridge University Press, Cambridge, 2004.