

When Can You Trust an Equilibrium Counterfactual? A Certified Identification Trichotomy for Interventions on Learning Systems

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Abstract

Equilibrium models answer interventional queries by mutilate-and-re-solve — the do-semantics of cyclic structural causal models; learning models answer the same query by unrolling adaptive dynamics. The two operators can disagree, and the disagreement is the formal content of a fault line running from the Lucas critique to complexity economics. We make the question — *when do the two operators return the same answer?* — a machine-checkable identification problem. Our main theorem is a trichotomy for global interventional queries on populations of adaptive learners: the equilibrium counterfactual is *point-identified* exactly under expectational stability (E-stability $\Leftrightarrow$ causal validity of the σ -solution, with the sharp learning-rate threshold); it is *set-identified* by a linear program over the ε -coarse-correlated-equilibrium polytope of the intervened game, at the regret ε_T actually measured at the horizon actually run, with no further assumptions; and otherwise it is *not identified*, but the failure itself is certifiable, with diagnostics — including multiplicity, where the honest global object is a basin-mass mixture no plain SCM can represent. Every condition is computable by instruments we release as an open-source artifact, each returning a typed certificate, and four experiments exercise every rung — including a monetary-policy toy where the *sign* of a policy conclusion flips between the two semantics exactly where the certificate says not to trust the equilibrium, and a bistable system with locally correct certificates but a measured global selection gap.

Keywords: causal identification; cyclic structural causal models; equilibrium; learning in games; coarse correlated equilibrium; E-stability; certificates

1. Introduction

Ask a New-Keynesian toy model what happens to inflation after a unit demand shock under a passive interest-rate rule, and the equilibrium answer is a *fall* of 8.2 points: mutilate the structural equations, re-solve the fixed point, read off the counterfactual. Ask the same structural equations populated by adaptive learners, and inflation *rises without bound* — +250 points within thirty learning steps (Section 6.3). Same model, same intervention, same query; opposite sign. Under an active rule the two answers agree to machine precision, and nothing in the equilibrium computation warns which case one is in.

The pattern is old. The Lucas critique (Lucas, 1976) says interventional predictions computed from an equilibrium description can fail when the underlying adaptation re-equilibrates; the adaptive-learning program in macroeconomics showed that stability of learning — E-stability — governs whether rational-expectations equilibria are attainable at all (Marcet and Sargent, 1989; Evans and Honkapohja, 2001); Dogra (2024) argues the causal interpretation of equilibrium economics is paradox-ridden precisely because

equilibrium models suppress the adjustment process; complexity economics has long insisted that “equations” and “maps” are different objects (Brock and Hommes, 1997; Pangallo et al., 2019). On the causal-inference side, cyclic SCMs give equilibrium counterfactuals an exact semantics (Bongers et al., 2021), the ODE-to-SCM construction says when an SCM abstracts an underlying dynamical system (Mooij et al., 2013; Rubenstein et al., 2017), and Blom et al. (2019) prove that equilibria depending on initial conditions are what no plain SCM can represent. What none of these provides is an *identification theory*: exactly when does the equilibrium do-operator answer the interventional question posed of the adaptive system, what can be salvaged when it does not, and a procedure that checks and says so.

This paper provides that theory for two well-delimited classes — cyclic SCMs with smooth mean fields, and finite games — with instruments that make every condition checkable:

1. **A bridge theorem (Theorems 2 and 4).** For linear cyclic SCMs, point identification of the learning-limit counterfactual by the equilibrium counterfactual holds *if and only if* the intervened system is E-stable (Evans and Honkapohja, 2001), with the sharp constant-gain threshold γ^* in closed form: *the σ -solution of the cyclic SCM is the causally correct description of the learning population exactly on the E-stable class*. A corollary formalizes the equations-vs-maps gap (naive unrolling demands the strictly stronger $\rho(B) < 1$; divergent map with convergent learners is a nonempty, certifiable regime), and Theorem 4 extends both directions to hyperbolic equilibria of smooth systems, with explicit noise conditions, at the price of basin-local scope.
2. **An identification trichotomy (Theorem 13).** For global interventional queries posed of learning populations in the two classes, exactly one of three cases holds, each decidable by a finite computation and mapped to a typed certificate: *point* identification (IDENTIFIED), *set* identification by LP bounds over the ε_T -CCE polytope of the intervened game at the *measured* realized regret (BOUNDED; Theorem 6, assumption-free at every finite horizon), and *certified non-identification* (EMPIRICAL) with diagnostics.
3. **Certified failure rather than silent failure.** In the non-identified case the instruments report *why*: a negative margin no learning rate can rescue; abstention in place of a vacuous bound; or multiplicity, where the global object is a basin-mass mixture — the causal-constraint-model caveat of Blom et al. (2019) operationalized and, in our experiments, *measured*.
4. **Instruments and evidence.** Every condition is computed by exact, dependency-free instruments released as an open-source artifact (stability-margin diagnostic and damped comparator for cyclic SCMs; CCE polytope, LP bounds, measured-regret certificates, and dual sensitivities for games); four experiments exercise every rung of the resulting ladder (Section 6).

What this paper is not. The mathematical ingredients are standard — stochastic approximation on one side, LP duality over equilibrium polytopes on the other — and Theorem 6 is a rearrangement of the definition of regret. The contribution is the synthesis: the two-semantics framing, the bridge, the trichotomy, and the instruments plus experiments

demonstrating the ladder end to end. We state each theorem’s provenance next to it and flag every place where our experimental claims stop short (Section 8).

2. Two do semantics for one query

2.1. Equilibrium do: cyclic SCMs

A cyclic SCM over endogenous $x \in \mathbb{R}^n$ and exogenous u specifies structural equations $x_i = f_i(x_{\text{pa}(i)}, u_i)$ whose dependency graph may contain cycles; its solutions are the fixed points $x = f(x, u)$, given interventional semantics by the solution-function (σ -solution) theory of Bongers et al. (2021). The **equilibrium** do is mutilate-and-re-solve: replace the targeted equations, then solve the intervened fixed-point problem. Throughout, our exact linear class is

$$x = Bx + u, \quad \text{do}(x_i = \xi) : \text{zero row } i \text{ of } B, \text{ pin } u_i = \xi, \quad (1)$$

with intervened pair $(B_{\text{do}}, u_{\text{do}})$ and, whenever $I - B_{\text{do}}$ is invertible, the unique equilibrium counterfactual

$$x^* = (I - B_{\text{do}})^{-1} \mathbb{E}[u_{\text{do}}]. \quad (2)$$

We write $\text{do}^{\text{eq}}(\cdot)$ for this operator; for finite games (Section 4) the equilibrium object is a set rather than a point, with the correlated-equilibrium polytope playing the role of $(I - B)^{-1}$.

2.2. Learning do: adaptive dynamics

Populated instead by agents who adapt rather than solve fixed points, the same equations define a process: impose the same mutilation on the *process* and unroll,

$$x_{k+1} = x_k + \gamma_k (f_{\text{do}}(x_k) - x_k + M_{k+1}), \quad (3)$$

where γ_k is a gain sequence and M_{k+1} is martingale-difference noise. Constant gain $\gamma_k \equiv \gamma \in (0, 1]$ is the Euler discretization of the **mean dynamics**

$$\dot{x} = F_{\text{do}}(x) := f_{\text{do}}(x) - x, \quad (4)$$

and decreasing gain ($\sum_k \gamma_k = \infty$, $\sum_k \gamma_k^2 < \infty$) is Robbins–Monro stochastic approximation (Robbins and Monro, 1951), the regime of least-squares learning in macroeconomics (Marcet and Sargent, 1989; Evans and Honkapohja, 2001). Equation (4) is the E-stability ODE of Evans and Honkapohja (2001) in mean-dynamics form: f_{do} is the temporary-equilibrium map from expectations to realized outcomes, and E-stability of an equilibrium x^* is local asymptotic stability of x^* under (4). The **learning** do, written $\text{do}^{\text{lr}}(\cdot)$, is the long-run behavior of (3): its almost-sure limit when one exists, and its time-averaged play otherwise.

Two spectral quantities decide everything in the linear class. With $\nu_i = \lambda_i(B_{\text{do}}) - 1$ the eigenvalues of the mean-dynamics Jacobian $B_{\text{do}} - I$,

$$\text{margin}(B_{\text{do}}) = 1 - \max_i \text{Re } \lambda_i(B_{\text{do}}), \quad \gamma^* = \min_i \frac{-2 \text{Re } \nu_i}{|\nu_i|^2}, \quad (5)$$

the *stability margin* (positive iff $B_{\text{do}} - I$ is Hurwitz) and the *maximal stable learning rate* (the sharp constant-gain threshold, Lemma 1).

2.3. Queries and certificates

A *query* is the long-run population value $Q = \mathbb{E}_\mu[h]$ of a functional h under the intervention, where μ is the population's long-run occupation law. We distinguish *global* queries — the population's long-run behavior, full stop — from *basin-local* queries conditioned on initialization within the basin of a stable solution; Section 6.4 measures a system where every basin-local query is point-identified while the global query is not. The equilibrium counterfactual *point-identifies* the query when $\text{do}^{\text{lr}}(\cdot)$ exists as a point and equals $\text{do}^{\text{eq}}(\cdot)$; *set-identifies* it when a computable set provably contains the realized value; and *fails* otherwise. Every instrument returns a typed certificate: a rung in $\{\text{IDENTIFIED}, \text{BOUNDED}, \text{EMPIRICAL}\}$, a claim, a machine-checkable *witness* (spectra, LP values, dual multipliers, measured regrets), and *hedges* — what blocked a stronger rung and, when possible, the action that would unblock it. The rung ordering is a soundness contract: IDENTIFIED only under a proof of point identification, BOUNDED only under a proof of containment, everything else degrading to EMPIRICAL with diagnostics rather than fabricating a bound. The do-comparator for cyclic SCMs and the game certifier for finite games implement the ladder in Table 1.

3. Point identification: E-stability is causal validity

Lemma 1 (gain window) *Let $\nu \in \mathbb{C}$. If $\text{Re } \nu < 0$ then $|1 + \gamma\nu| < 1$ iff $0 < \gamma < -2\text{Re } \nu/|\nu|^2$. If $\text{Re } \nu \geq 0$ then $|1 + \gamma\nu| \geq 1$ for every $\gamma > 0$.*

Proof. $|1 + \gamma\nu|^2 - 1 = \gamma(2\text{Re } \nu + \gamma|\nu|^2)$, which is negative iff $\gamma < -2\text{Re } \nu/|\nu|^2$, a positive threshold iff $\text{Re } \nu < 0$; if $\text{Re } \nu \geq 0$ the right factor is nonnegative for all $\gamma > 0$. ■

Theorem 2 (T1: point identification $\Leftrightarrow$ E-stability; linear class) *Let $x = Bx + u$ be a linear cyclic SCM with intervened pair $(B_{\text{do}}, u_{\text{do}})$, $I - B_{\text{do}}$ invertible, x^* as in (2), and learning dynamics (3) with $f_{\text{do}}(x) = B_{\text{do}}x + u_{\text{do},k}$, $\mathbb{E}[u_{\text{do},k}] = \mathbb{E}[u_{\text{do}}]$, and noise with bounded conditional second moments. The following are equivalent:*

- (1) E-stability: $\text{margin}(B_{\text{do}}) > 0$, i.e. $B_{\text{do}} - I$ is Hurwitz;
- (2) constant gain: for every $\gamma \in (0, \gamma^*)$ the mean iteration converges to x^* from every initial condition, and the stochastic iterates satisfy $\limsup_k \mathbb{E}\|x_k - x^*\|^2 = O(\gamma)$;
- (3) decreasing gain: Robbins–Monro learning converges to x^* almost surely (boundedness of the iterates has probability one under (1) for the linear recursion).

Under any of these, $\text{do}^{\text{lr}}(\cdot) = \text{do}^{\text{eq}}(\cdot)$: the σ -solution of the intervened cyclic SCM is the causally correct description of the learning population's interventional behavior. Conversely, if $\text{margin}(B_{\text{do}}) < 0$: for every $\gamma > 0$ the mean iteration diverges from all initial conditions outside a proper subspace, and with noise nondegenerate in the unstable directions, decreasing-gain learning converges to x^ with probability zero. The learning-limit do then does not exist as a point, and the equilibrium prediction has no dynamical counterpart.*

Proof (algebraic part; stochastic parts in Appendix A). The mean iteration is $x_{k+1} = M_\gamma x_k + \gamma \mathbb{E}[u_{\text{do}}]$ with $M_\gamma = I + \gamma(B_{\text{do}} - I)$, whose fixed point is x^* for every γ —

damping never moves the equilibrium, only the convergence class. Its eigenvalues are $1 + \gamma\nu_i$; by Lemma 1, $\rho(M_\gamma) < 1$ for all $\gamma \in (0, \gamma^*)$ iff every $\operatorname{Re} \nu_i < 0$, i.e. iff (1), and γ^* in (5) is sharp. If some $\operatorname{Re} \nu_i > 0$ then $|1 + \gamma\nu_i| > 1$ for all $\gamma > 0$, giving divergence along that eigenspace. The noise bound in (2), the ODE-method equivalence (1) $\Leftrightarrow$ (3) (Ljung, 1977; Kushner and Yin, 2003; Benaïm, 1999), and the probability-zero converse (Pemantle, 1990; Brandière and Duflo, 1996) are in Appendix A. ■

Theorem 2 is a bridge, and we claim it as such: each direction is classical in its home literature, but the equivalence has, to our knowledge, never been stated as a statement about the validity of SCM do-semantics — “E-stability” becomes “the σ -solution is the causally correct object,” a sentence neither literature could previously parse.

Corollary 3 (naive unrolling is a strictly stronger demand) *The literal period-1 unrolling $x_{k+1} = B_{\text{do}}x_k + u$ converges iff $\rho(B_{\text{do}}) < 1$. Since $\rho(B) < 1 \Rightarrow \max_i \operatorname{Re} \lambda_i(B) < 1$ but not conversely, the regime $\{\rho(B_{\text{do}}) \geq 1, \text{margin}(B_{\text{do}}) > 0\}$ is nonempty (e.g. $B = [-2]$: $\rho = 2$, $\text{margin} = 3$): the map diverges while every sufficiently damped adaptive population converges to the equilibrium counterfactual — the formal content of the equations-vs-maps gap in the linear class, certified by the comparator at any $\gamma < \gamma^*$.*

3.1. The nonlinear local statement

Let $F = f - \text{id}$ with $f \in C^1$, and let x^* be a *hyperbolic* zero of F (no eigenvalue of $J = DF(x^*)$ on the imaginary axis); assumptions (A1)–(A4) are stated in Appendix B.

Theorem 4 (T1': hyperbolic local version) *Under (A1)–(A3):*

- (i) *If J is Hurwitz, then for every $\gamma \in (0, \gamma^*(J))$ (the formula in (5) evaluated on the spectrum of J) the constant-gain iteration is locally exponentially stable at x^* with asymptotic mean-square error $O(\gamma)$, and decreasing-gain iterates converge to x^* almost surely on the event of remaining in a compact subset of the basin of attraction of x^* . Hence $\text{do}^{\text{lr}}(\cdot) = \text{do}^{\text{eq}}(\cdot)$ locally, on the basin of x^* .*
- (ii) *If J has an eigenvalue with positive real part, then under the additional excitation condition (A4), $\mathbb{P}(x_k \rightarrow x^*) = 0$: the equilibrium counterfactual at x^* has no learning-limit counterpart.*

Proof sketch. Hyperbolicity reduces (i) to the linear case via Hartman–Grobman plus the SA limit-set theorem (Kushner and Yin, 2003; Benaïm, 1999; Borkar, 2008); (ii) is nonconvergence-to-unstable-points (Pemantle, 1990; Brandière and Duflo, 1996). Full proof with explicit noise conditions: Appendix B. ■

Remark 5 (the basin qualifier binds) *T1' is local by nature. With multiple stable σ -solutions, every basin-local certificate can be simultaneously correct while the global interventional outcome is a basin-mass mixture across them — and interventions move both the solutions and the basin boundaries. Blom et al. (2019) prove that equilibria depending on initial conditions are exactly what plain SCM semantics cannot represent; equilibrium selection by learning is that phenomenon, not an analogy to it. Section 6.4 measures the resulting gap; hence the trichotomy below classifies global queries and treats multiplicity as certified non-identification.*

4. Set identification: the measured-regret CCE rung

The second exact class is finite games, where the equilibrium object is a set and the honest counterpart of (2) is a polytope. Let G be a finite game with agents N , action sets A_i , and utilities u_i ; an intervention pins the actions of a subset of agents, inducing G_{do} (profiles restricted to the consistent ones, no-deviation constraints only for free agents) — in the causal-games taxonomy, pre-policy for the free agents, whose play re-adapts, and post-policy for the pinned ones (Hammond et al., 2023; Mishra et al., 2024). For $\varepsilon = (\varepsilon_i)_i$, the ε -coarse-correlated-equilibrium polytope is

$$\text{CCE}_\varepsilon(G_{\text{do}}) = \left\{ \mu \in \Delta(\text{profiles}) : \mathbb{E}_\mu[u_i(a'_i, s_{-i}) - u_i(s)] \leq \varepsilon_i \quad \forall i \text{ free}, \forall a'_i \in A_i \right\}. \quad (6)$$

A population plays T rounds of G_{do} , producing the empirical joint distribution μ_T ; agent i 's realized regret $\varepsilon_T(i) = \max_{a'_i} \frac{1}{T} \sum_{t=1}^T [u_i(a'_i, s_{-i,t}) - u_i(s_t)]$, clipped at zero, is computable from the trajectory log.

Theorem 6 (T2: finite-time containment, assumption-free) *For every horizon T and every profile functional h ,*

$$\mathbb{E}_{\mu_T}[h] \in \left[\min_{\mu} \mathbb{E}_{\mu}[h], \max_{\mu} \mathbb{E}_{\mu}[h] \right] \quad \text{over } \mu \in \text{CCE}_{\varepsilon_T}(G_{\text{do}}),$$

where ε_T is the vector of measured realized regrets. Both endpoints are linear programs over a polytope contained in the simplex.

Proof. $\mu_T \in \text{CCE}_{\varepsilon_T}(G_{\text{do}})$ is the definition of realized regret rearranged: the expected deviation gain of switching to a fixed a'_i under μ_T is the time-averaged realized gain, which is at most $\varepsilon_T(i)$ by construction. Containment of $\mathbb{E}_{\mu_T}[h]$ between the min and max over the feasible set is then immediate. ■

Theorem 6 is definitionally true, and we present it as such: the ex post, finite-time character of measured-regret sets is folklore in the full-feedback case — the rationalizable set of Nekipelov et al. (2015) already holds for the realized play, as Hartline (2026) emphasizes — and the containment step is textbook. Its role here is to be the *set-identification rung of the trichotomy*: binding at the horizon actually run, for whatever the learners actually did, abstaining when vacuous, and horizon-uniform (deterministic given the log at every T). Three corollaries complete the rung.

Corollary 7 (asymptotic route) *If the population is no-regret ($\varepsilon_T \rightarrow 0$; Foster and Vohra, 1997; Hart and Mas-Colell, 2000), every limit point of $\{\mu_T\}$ lies in $\text{CCE}_0(G_{\text{do}})$ and the ε_T -bounds converge to the exact-CCE bounds (the LP value is Lipschitz in the right-hand side): the classical statement is the $\varepsilon \rightarrow 0$ limit of the certified one.*

Corollary 8 (degeneracy: when the equilibrium point prediction is safe) *The equilibrium point prediction of h is valid for every no-regret population iff h is constant over $\text{CCE}_0(G_{\text{do}})$ (LP width 0) — a checkable condition, certified IDENTIFIED. Example: every CCE of a two-player zero-sum game achieves the value, so value functionals are point-identified; generic anti-coordination games are not.*

Corollary 9 (ε -sensitivity from LP duals) *Each endpoint is a concave (max side) or convex (min side) piecewise-linear function of ε , with one-sided derivative at the measured ε_T equal to the sum of the optimal dual multipliers on the perturbed constraints — reported in the certificate witness. It prices the marginal tightness bought by running the population longer; an interval that is wide while its sensitivity is tiny is structurally loose (the game), not statistically loose (the learners) — a distinction no asymptotic statement makes.*

Remark 10 (width is the operational content of “equilibrium”) $\text{width}(h, \text{do}) = \max - \min$ over $\text{CCE}_0(G_{\text{do}})$ measures how much predictive content equilibrium analysis has under that intervention: zero is point identification; the full payoff range is none — and the certificate abstains rather than report a vacuous interval.

Remark 11 (scope: the stage-game polytope) $\text{CCE}_\varepsilon(G_{\text{do}})$ is the stage-game polytope. Populations with history-dependent strategies and intertemporal objectives — empirically, the value-bootstrapping learners of the algorithmic-collusion literature (Calvano et al., 2020) — can sustain time-averages outside it. Theorem 6 is not violated: such populations are detected, not missed — their measured ε_T stays bounded away from zero and the certificate inflates or abstains. Mapping which families escape is companion work; the trichotomy only needs that containment itself is unconditional.

Provenance. Every atomic ingredient of this section exists in some form, and the delta is exactly the conjunction of four choices no adjacent work makes — ε from *measured* play, containment on the *intervened* game, abstention, dual prices — plus the trichotomy and the width reading; Section 7 states the claim against each neighbor, nearest of which is Lomys and Magnolfi (2025). The LP lives in joint-profile space (exponential in the number of players; hardness results exist for succinct games, Papadimitriou and Roughgarden, 2008), so exact bounds are for small games; sampled and relaxed bounds are the scale path.

5. The main theorem: an identification trichotomy

We now assemble the pieces. The theorem classifies *global* queries; per Remark 5, point identification requires *global* uniqueness of the stable σ -solution, and everything outside the stated regularity routes to diagnostics.

Assumption 12 (regularity for the nonlinear SCM class) *The intervened mean field $F_{\text{do}} = f_{\text{do}} - \text{id}$ is C^1 with (R1) finitely many zeros, all hyperbolic; (R2) SA iterates remain in a compact set almost surely (dissipativity); (R3) the chain-recurrent set of the mean flow consists of its zeros only (no cycles or chaotic sets). (R1)–(R3) hold automatically in the linear class with margin $\neq 0$; (R3) is automatic for scalar systems with hyperbolic zeros. (R2)–(R3) are not decidable a priori: the instruments guard them at runtime (divergence detection, largest-Lyapunov estimate), and their failure routes to case 3 below.*

Theorem 13 (identification trichotomy for learning populations) *Let $Q = \mathbb{E}_\mu[h]$ be a global interventional query posed of a population of adaptive learners (3), where the intervened system is (a) a linear cyclic SCM, (b) a smooth mean field satisfying Assumption 12, or (c) a finite game played over T rounds with measured realized regrets ε_T . Then*

exactly one of the following holds, each condition computable by a shipped instrument, and the corresponding certificate is warranted:

1. **Point (Identified).** *Either (i) the intervened system (a)/(b) has a globally unique stable σ -solution x^* with $\text{margin} > 0$ (Hurwitz Jacobian): then for every $\gamma \in (0, \gamma^*)$, and almost surely under decreasing gain, the learning do equals the equilibrium do at x^* (Theorems 2 and 4); or (ii) the system is a finite game and h is constant over $\text{CCE}_0(G_{\text{do}})$ ($\text{LP width} \leq \text{tol}$): the equilibrium point prediction holds for every no-regret population (Corollary 8).*
2. **Set (Bounded).** *The system is a finite game with $\text{LP width} > \text{tol}$ and a nonvacuous interval at the measured ε_T : the realized time-average of h lies in the LP interval over $\text{CCE}_{\varepsilon_T}(G_{\text{do}})$, with no further assumptions (Theorem 6), and the reported dual multipliers price the interval’s growth per unit of additional regret (Corollary 9).*
3. **None (Empirical, hedged with diagnostics).** *One of: every σ -solution has unstable mean dynamics ($\text{margin} < 0$ at each; no admissible gain rescues — Theorem 2 converse); Assumption 12 fails at runtime (divergence, cycles, or a positive Lyapunov estimate); the ε_T -interval is vacuous (abstention; Remark 10); or the intervened system has multiple stable σ -solutions, so the global learning outcome is a basin-mass mixture over them — an object plain SCM semantics cannot represent (Blom et al., 2019) — reported as a selection hedge (each stable σ -solution still carries a correct basin-local case-1 certificate; the trichotomy classifies the global query).*

Proof sketch (details in Appendix D). In classes (a)/(b), enumerate the σ -solutions (finitely many by (R1)) and count the stable ones: zero (or (R) failing) $\rightarrow$ case 3; exactly one $\rightarrow$ case 1(i) via Theorem 2 (globality is automatic in (a); in (b) it follows from (R2)–(R3) plus the limit-set and nonconvergence theorems); two or more $\rightarrow$ case 3 (multiplicity). In class (c): $\text{width} \leq \text{tol} \rightarrow$ case 1(ii); else a nonvacuous measured- ε_T interval $\rightarrow$ case 2, vacuous $\rightarrow$ case 3. Exhaustiveness and exclusivity are by construction (1(i) requires global uniqueness, which case 3’s multiplicity clause negates; 1(ii) and 2 split on width), and each condition is a finite computation given the enumerated solutions. ■

Certified failure, not silent failure. Case 3 is where most of the practical value lies, because it is the case existing practice silently mislabels. The instruments report: with $\text{margin} < 0$, the divergence rate and the proof that no learning rate rescues it; with $\rho(B_{\text{do}}) \geq 1$ but $\text{margin} > 0$, the actionable hedge of Corollary 3; with cycles or chaos, a largest-Lyapunov diagnostic (time-averaged play of a finite-game population still satisfies Theorem 6); with multiplicity or singularity, a typed refusal carrying all stable solutions and ensemble basin masses (Section 6.4). The ladder is Dash (2005)’s warning — manipulating the equilibrated model misleads — made operational.

6. Experiments

Four experiments exercise every rung of Table 1; all numbers are outputs of the released scripts with fixed seeds (Appendix E), and every LP value carries an independent weak-duality certificate (slack $\sim 10^{-14}$).

condition	instrument	certificate
$\rho(B_{\text{do}}) < 1$, gap $\leq \text{tol}$	comparator (naive unrolling)	IDENTIFIED
margin > 0 , $\gamma < \gamma^*$, gap $\leq \text{tol}$	comparator (damped, T1)	IDENTIFIED
margin > 0 but naive unrolling diverges	comparator hedge + γ^*	EMPIRICAL (actionable)
h constant over $\text{CCE}_0(G_{\text{do}})$	game certifier (T2 degeneracy)	IDENTIFIED
no-regret assumed or ε_T measured	game certifier (T2)	BOUNDED
ε_T -interval vacuous	game certifier	EMPIRICAL (abstention)
margin < 0 at every σ -solution	comparator hedge (no rescuing γ)	EMPIRICAL (certified failure)
$I - B_{\text{do}}$ singular / margin ≈ 0 / multiplicity	typed refusal + selection hedge	EMPIRICAL (selection / CCM cave)

Table 1: The certificate ladder: every rung of Theorem 13, its instrument, and the certificate warranted. “Actionable” hedges name the re-run that would upgrade the rung; the last row is a typed refusal to fabricate a solution.

regime	certificate	key witness values
A: $s/d = 0.5$, naive	IDENTIFIED	gap 8.9×10^{-16} ; $p^* = 6.3333$
B: $s/d = 1.5$, naive	EMPIRICAL (actionable)	$\rho = 1.225$ diverges; margin = +1.0; hedge $\gamma^* = 0.8$
B: $s/d = 1.5$, $\gamma = 0.4$	IDENTIFIED	gap 0; $p^* = 4.2000$
C: trend $b = 1.2$	EMPIRICAL (certified failure)	margin = -0.095 ; $\gamma^* = 0$: no gain rescues
D: nonlinear supply, gain 0.14	EMPIRICAL (hedge: $\gamma > \gamma^* = 0.073$)	Lyapunov +0.222; $f'(p^*) = -26.3$

Table 2: E1: one market, four regimes, the full ladder. Regime B is Corollary 3 live: the naive cobweb oscillation diverges while the margin is positive, the hedge names γ^* , and the damped re-run certifies IDENTIFIED — Nerlove’s adaptive-expectations rescue, certified. Regime C is the certified-failure rung. Regime D is chaotic (positive Lyapunov exponent).

6.1. E1: the certificate ladder on the canonical stage

The cobweb market — demand $D(p) = a - dp$, supply $S(p^e) = c + sp^e$ under naive expectations, per-unit tax $\text{do}(\tau=1)$ — is the oldest stage for equilibrium-vs-dynamics questions (Nerlove, 1958; Hommes, 1994); Table 2 runs one market through four regimes.

Regime D’s chaos enters through the adaptive gain itself (a monotone naive-expectations cobweb map admits only fixed points and 2-cycles; Hommes, 1994): the device that rescues regime B produces chaos in the nonlinear class — two EMPIRICAL hedges, one instrument (further honesty note in Appendix E).

6.2. E2: non-convergent learning and the Bounded rung

The perturbed rock-paper-scissors of Sato et al. (2002) (tie payoffs $\epsilon_X = 0.5$, $\epsilon_Y = -0.1$) is the canonical game in which learning fails to converge to Nash. We tax X ’s first action ($\tau = 0.3$), build G_{do} explicitly, and run a Hedge population for $T = 200,000$ rounds. The Nash point prediction of X ’s payoff is 0.0667; the realized time-average is 0.1034, a miss of 0.037. The measured regret is $\varepsilon_T = 0.0021$ and the measured- ε interval $[0.0646, 0.3120]$ contains the realized value, as Theorem 6 guarantees. Both readings are honest: the

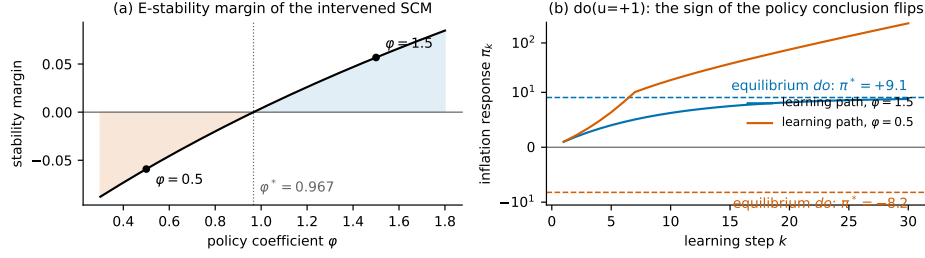


Figure 1: E4. Left: stability margin of the intervened SCM vs the policy coefficient φ , crossing zero at $\varphi^* = 0.9667$. Right: under $\text{do}(u=+1)$, learning paths vs the equilibrium predictions (dashed): at $\varphi = 1.5$ they agree (IDENTIFIED, gap 7×10^{-15}); at $\varphi = 0.5$ the equilibrium predicts $\pi^* = -8.21$ while learning explodes upward ($\pi_{30} = +249.6$); the certificate reports margin = -0.059 , $\gamma^* = 0$.

certificate catches what Nash misses, and the width (0.24 against a payoff range ≈ 2.3) is the finding — the measured predictive content of “equilibrium” here (Remark 10). We write “non-convergent learning,” not chaos: Sato et al. (2002) prove chaos for replicator families, and we did not verify a positive Lyapunov exponent for this Hedge configuration.

6.3. E4: the certified policy sign flip

A New-Keynesian-style toy in the spirit of Dogra (2024) reduces to the expectations recursion $\pi = T(\varphi) \pi^e + u'$ with $T(\varphi) = (\beta + \kappa\sigma)/(1 + \kappa\sigma\varphi)$, closed by naive expectations into the cyclic SCM $B = \begin{bmatrix} 0 & T \\ 1 & 0 \end{bmatrix}$ ($\beta = 0.99$, $\sigma = 1$, $\kappa = 0.3$). E-stability holds iff $\varphi > \varphi^* = 1 - (1 - \beta)/(\kappa\sigma) = 0.9667$ — a Taylor-principle-*type* threshold of this bespoke static-timing toy (recovering the textbook $\varphi > 1$ as $\beta \rightarrow 1$; *not* the condition of Bullard and Mitra, 2002, whose timing differs). Figure 1 shows the exhibit: the margin crosses zero at φ^* (∓ 0.0059 at $\varphi^* \mp 0.05$), and on the passive side the sign of the policy conclusion flips between the two semantics. To the objection that nobody trusts comparative statics at an E-unstable equilibrium: stability is routinely *not checked* (Dogra, 2024; Evans and Ramey, 2006), and the contribution is the machine check. A standard-timing calibrated New-Keynesian exercise is deferred to the economics-facing companion.

6.4. E7: locally certified, globally selected

The bistable mean field $F(x) = \tanh(3x) - x + u$ has two stable equilibria and one unstable one. At $u = 0$ the stable roots ± 0.995 carry margins $+0.969$ *each*: both basin-local T1' certificates are IDENTIFIED, correctly. Under $\text{do}(u=0.2)$ the roots move to -0.782 and $+1.199$ (margins $+0.892$, $+0.991$), the basin boundary moves to -0.105 , and a stochastic-approximation ensemble ($N = 10^5$, gain 0.05, noise 0.1, $x_0 \sim \mathcal{N}(0, 1.5^2)$) re-partitions from 50.2/49.8 to 47.4/52.6: 2.8% of the population crosses the boundary. Equilibrium-tracking at the positive root predicts $+1.199$; the realized mixture mean is $+0.261$ — a gap of 0.938 no basin-local certificate can see (tracking the negative root gives 1.04; the conclusion is

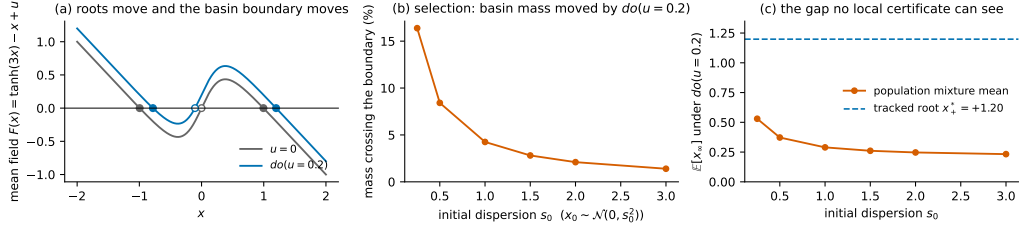


Figure 2: E7. (a) Bistable mean field: under $do(u=0.2)$ both stable roots move *and* the basin boundary moves ($0 \rightarrow -0.105$). (b) Ensemble mass crossing the boundary vs initial dispersion s_0 . (c) Population mixture mean vs the equilibrium-tracking prediction $x_+^* = +1.199$: the gap persists at every dispersion.

root-invariant). This is Theorem 13’s multiplicity branch, and the Blom et al. (2019) caveat, *measured*.

Because the basin masses depend on the (arbitrary) initial distribution, we report the whole curve (Figure 2b,c): as s_0 ranges over 0.25–3.0 the crossing mass falls from 16.4% to 1.4%, while the gap to the tracked root persists at every dispersion (0.67–0.97) and is largest when the population starts concentrated near the old boundary — the near-indifference case. The selection effect is not an artifact of the initial-condition choice; its *size* is the curve. The multiplicity diagnostic this motivates — all stable σ -solutions plus ensemble basin masses — is Table 1’s last row.

7. Related work

Cyclic SCMs and dynamical abstraction. Bongers et al. (2021) give cyclic SCMs their solution-function semantics; Mooij et al. (2013) and Rubenstein et al. (2017) characterize when an SCM consistently abstracts a dynamical system at equilibrium, and Bongers et al. (2018) prove that equilibration commutes with intervention under convergence conditions; Blom et al. (2019) prove the representational impossibility our multiplicity branch operationalizes; Iwasaki and Simon (1994), Dash (2005), and Weinberger (2023) warn that equilibrated models mispredict interventions. Our delta: not “when does the SCM abstract the dynamics” but “when do the dynamics validate the SCM’s do,” as an equivalence for expectations-based adaptive learning, with the E-stability characterization and sharp gain threshold. White and Chalak (2009) extend Pearl’s model for optimization, equilibrium, and learning (framework-level prior art; our delta is theorem-level). On the macro side, E-stability is due to Marcet and Sargent (1989); Evans and Honkapohja (2001), applied to policy by Bullard and Mitra (2002) and connected to the Lucas critique by Evans and Ramey (2006): that literature has the stability theory but no causal semantics, the cyclic-SCM literature the reverse; T1 is the bridge.

Learning in games and econometrics of learning agents. No-regret convergence to CCE is classical (Foster and Vohra, 1997; Hart and Mas-Colell, 2000; Cesa-Bianchi and Lugosi, 2006; Fudenberg and Levine, 1998); Nekipelov et al. (2015) use it as an identifying assumption; Bergemann et al. (2022), Magnolfi and Roncoroni (2023), and Syrgkanis et al.

(2017) compute LP identified sets over equilibrium polytopes; Kline and Tamer (2024) set-identify counterfactual outcomes in empirical games with the point-vs-set boundary driven by equilibrium selection, not dynamics; and measured regret appears as a regulatory audit statistic in Hartline et al. (2024). The nearest neighbors are Lomys and Magnolfi (2025): ε -CCE set estimators for learning populations, with support-function LPs, duality, and counterfactual bounds. Four choices separate our certificate from their estimator: ε is the *measured realized regret* of the actual trajectory, with no behavioral or algorithm-class assumption (they inflate by assumed worst-case algorithm-class rates, themselves noting that realized regrets concentrate better); containment is carried directly to the *intervened* game’s polytope rather than re-solved from identified primitives; the certificate *abstains* when vacuous; and the duals are reported as regret prices. In causal games, Hammond et al. (2023) define pre/post-policy interventions with set-valued answers quantified over the equilibrium set (including a credal-bounds reading) and present mechanised games as relational cyclic causal models, a semantic bridge we build on; Mishra et al. (2024) characterise a sound-and-complete family of primitive interventions. Absent from that line is the identification layer supplied here: conditions for point/interval/vacuous answers, a finite-time route from *measured play* rather than a fully specified model, and bounds actually computed. Non-convergent learning is classical (Sato et al., 2002; Mertikopoulos et al., 2018; Pangallo et al., 2019); we use it as the BOUNDED rung’s test bed. Performative prediction (Perdomo et al., 2020) is the single-learner cousin: performative stability is case 1’s analogue, and case 3 has no analogue there because a single learner suppresses equilibrium selection.

8. Limitations and open problems

Exactness class. T1 is exact for the linear class; T1’ is local and requires hyperbolicity; the global nonlinear statement leans on Assumption 12, parts of which are guarded at runtime rather than decided a priori — typed degradation to EMPIRICAL is the honest response, but a sharper decidable characterization for structured nonlinear classes is open, as is a bifurcation-aware treatment of the refused boundary cases. **Game scale.** The CCE LP is exponential in the number of players; sampled and relaxed bounds with certified gaps are the scale path. **Stage-game scope.** The demarcation of which learning families escape the stage-game polytope (Remark 11) is companion work. **Economics-facing model.** E4 is a bespoke static-timing toy; a standard-timing calibrated New-Keynesian exercise is required before policy conclusions. **Diagnostics not yet shipped.** E7’s basin masses are computed by the experiment, not a shipped instrument, and E2’s “non-convergent” label lacks a Lyapunov estimate of the learning flow; both are cheap follow-ups.

9. Conclusion

“When can you trust an equilibrium counterfactual?” has, for well-delimited classes, a three-part answer a machine can check: *trust it* exactly when the mean dynamics are E-stable and the stable solution is globally unique; *bound it* by the measured-regret CCE interval when the system is a finite game, with no assumptions beyond the trajectory log; *and otherwise do not* — but demand the diagnostics, for certified failure is usable knowledge: the Lucas critique not as a slogan but as a computable property of an intervened system, firing exactly where it should.

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Appendix A. Proof of Theorem 2 (stochastic parts)

Throughout, $e_k = x_k - x^*$ and $\nu_1, \dots, \nu_n$ are the eigenvalues of $A := B_{\text{do}} - I$ (so margin > 0 iff A is Hurwitz).

(2), noise bound. With constant gain γ and $u_{\text{do},k} = \mathbb{E}[u_{\text{do}}] + \xi_k$, where $\{\xi_k\}$ is a martingale-difference sequence with $\sup_k \mathbb{E}[\|\xi_k\|^2 \mid \mathcal{F}_{k-1}] \leq \sigma^2$, the error recursion is

$$e_{k+1} = M_\gamma e_k + \gamma \xi_k, \quad M_\gamma = I + \gamma A.$$

Let $\Sigma_k = \mathbb{E}[e_k e_k^\top]$. Then $\Sigma_{k+1} = M_\gamma \Sigma_k M_\gamma^\top + \gamma^2 \mathbb{E}[\xi_k \xi_k^\top]$, and for $\gamma < \gamma^*$ we have $\rho(M_\gamma) < 1$, so Σ_k converges to the unique solution $\Sigma_\infty(\gamma)$ of the discrete Lyapunov equation $\Sigma = M_\gamma \Sigma M_\gamma^\top + \gamma^2 \Sigma_\xi$ with $\|\Sigma_\xi\| \leq \sigma^2$, namely $\Sigma_\infty(\gamma) = \sum_{j \geq 0} \gamma^2 M_\gamma^j \Sigma_\xi (M_\gamma^\top)^j$. Choose a norm in which $\|M_\gamma\| \leq 1 - c\gamma$ for some $c > 0$ depending only on the spectrum of A (possible for all $\gamma \leq \bar{\gamma} < \gamma^*$ since $\rho(M_\gamma)^2 \leq 1 - 2\gamma c_0 + \gamma^2 c_1$ with $c_0 = \min_i (-\text{Re } \nu_i) > 0$). Then $\|\Sigma_\infty(\gamma)\| \leq \gamma^2 \sigma^2 \sum_j (1 - c\gamma)^{2j} \leq \gamma^2 \sigma^2 / (2c\gamma - c^2 \gamma^2) = O(\gamma)$, which is the claim $\limsup_k \mathbb{E}\|e_k\|^2 = O(\gamma)$.

(1) $\Rightarrow$ (3). For decreasing gains satisfying $\sum \gamma_k = \infty$, $\sum \gamma_k^2 < \infty$, the recursion (3) is a Robbins–Monro scheme whose mean ODE is $\dot{e} = Ae$. For linear A Hurwitz, x^* is the globally asymptotically stable equilibrium of the mean ODE; the iterates are bounded almost surely (the quadratic Lyapunov function $V(e) = e^\top P e$ with $A^\top P + PA = -I$ yields a supermartingale argument standard for linear SA; see [Kushner and Yin, 2003](#), Ch. 5 or [Borkar, 2008](#), Ch. 3), and the ODE-method limit-set theorem ([Ljung, 1977](#); [Benaïm, 1999](#); [Kushner and Yin, 2003](#)) gives $x_k \rightarrow x^*$ a.s.

(3) $\Rightarrow$ (1) and the converse claim. Suppose margin < 0 , i.e. some $\text{Re } \nu_i > 0$; by hyperbolicity we may assume no eigenvalue has zero real part (the boundary case is refused by the instruments, Table 1). Then x^* is a linearly unstable zero of the mean field. Under the excitation condition (A4) of Appendix B (nondegenerate noise in the unstable directions) and gains $\gamma_k = \Theta(k^{-\alpha})$, $\alpha \in (1/2, 1]$, the nonconvergence theorem of [Pemantle \(1990\)](#) (repeller case) and [Brandière and Dufo \(1996\)](#) (saddle/“trap” case) yields $\mathbb{P}(x_k \rightarrow x^*) = 0$. Hence (3) fails, and contrapositively (3) $\Rightarrow$ (1). For the constant-gain converse, Lemma 1 gives $|1 + \gamma \nu_i| > 1$ for every $\gamma > 0$, so the mean iteration diverges for every initial condition with a nonzero component in the corresponding invariant subspace — all but a proper (measure-zero) subspace. $\blacksquare$

Appendix B. Assumptions and proof of Theorem 4

Assumptions.

- (A1) $f \in C^1$ on an open set containing x^* ; $F = f - \text{id}$; x^* is a hyperbolic zero of F : $J = DF(x^*)$ has no eigenvalue on the imaginary axis. DF is locally Lipschitz at x^* .
- (A2) $x_{k+1} = x_k + \gamma_{k+1}(F(x_k) + M_{k+1})$ with $\{M_k\}$ a martingale-difference sequence with respect to a filtration $\{\mathcal{F}_k\}$: $\mathbb{E}[M_{k+1} \mid \mathcal{F}_k] = 0$ and $\mathbb{E}[\|M_{k+1}\|^2 \mid \mathcal{F}_k] \leq \sigma^2(1 + \|x_k\|^2)$ a.s.
- (A3) Gains: either constant $\gamma_k \equiv \gamma$, or decreasing with $\sum_k \gamma_k = \infty$ and $\sum_k \gamma_k^2 < \infty$.

(A4) (For part (ii) only.) There exist $b, \delta > 0$ such that on $\{\|x_k - x^*\| \leq \delta\}$, $\mathbb{E}[\langle M_{k+1}, v \rangle^+ | \mathcal{F}_k] \geq b$ for every unit vector v , the noise is a.s. bounded on that event, and $\gamma_k = \Theta(k^{-\alpha})$ with $\alpha \in (1/2, 1]$.

Part (i), constant gain (deterministic core). The damped map is $T_\gamma(x) = x + \gamma F(x)$ with $DT_\gamma(x^*) = I + \gamma J$. By Lemma 1 applied to the eigenvalues of J , $\rho(I + \gamma J) < 1$ for every $\gamma \in (0, \gamma^*(J))$. Pick a norm with $\|I + \gamma J\| \leq 1 - c\gamma < 1$; by (A1) there is a ball $\mathcal{B}$ around x^* on which $\|T_\gamma(x) - x^*\| \leq (1 - c\gamma/2)\|x - x^*\|$, giving local exponential stability with basin at least $\mathcal{B}$. With noise, the argument of Appendix A applied to the linearization, plus a standard perturbation bound for the locally Lipschitz remainder (Benveniste et al., 1990, Ch. 9), yields $\limsup_k \mathbb{E}[\|x_k - x^*\|^2 \mathbf{1}_{\{x_j \in \mathcal{B} \forall j \leq k\}}] = O(\gamma)$: iterates that remain in the basin concentrate at the mean-square $O(\gamma)$ scale.

Part (i), decreasing gain. J Hurwitz makes x^* a (locally) asymptotically stable equilibrium of $\dot{x} = F(x)$ with domain of attraction $\mathcal{D}$. Let $K \subset \mathcal{D}$ be compact. On the event $\Omega_K = \{x_k \in K \text{ for all large } k\}$, the SA limit-set theorem (Kushner and Yin, 2003, Thm. 5.2.1; Benaïm, 1999; Borkar, 2008, Ch. 2) applies: under (A1)–(A3) the interpolated iterate trajectory is almost surely an asymptotic pseudotrajectory of the mean flow, whose limit set on Ω_K is a compact, invariant, internally chain-transitive subset of K ; since the only such subset of a compact set contained in the basin of a hyperbolic attractor is $\{x^*\}$ itself, $x_k \rightarrow x^*$ a.s. on Ω_K . This is the precise sense of “locally, on the basin of x^* ”: the theorem conditions on remaining in a compact subset of the basin, and Section 6.4 is the measured demonstration of why the conditioning cannot be dropped when multiple attractors exist.

Part (ii). x^* hyperbolic with an unstable eigenvalue is a repeller or a saddle of the mean flow. Under (A2)–(A4), Pemantle (1990) (repellers) and Brandière and Dufo (1996) (saddles; see also Benaïm, 1999) give $\mathbb{P}(x_k \rightarrow x^*) = 0$. The excitation condition (A4) is what rules out the degenerate case of noise supported on the stable manifold; it is satisfied, e.g., by any noise with a density bounded below on a neighborhood of the origin, and in particular by the Gaussian noise used in Section 6.4. ■

Appendix C. Proofs for Section 4

Corollary 7. The feasible set $\text{CCE}_\varepsilon(G_{\text{do}})$ is a polytope $\{\mu \geq 0, \mathbf{1}^\top \mu = 1, G\mu \leq \varepsilon\}$ whose constraint matrix is fixed. It is nonempty for every $\varepsilon \geq 0$: any Nash equilibrium of the finite game G_{do} — which exists — lies in CCE_0 , and the set only grows with ε . The optimal value of a bounded feasible LP is a piecewise-linear, hence locally Lipschitz, function of the constraint right-hand side, with one-sided derivatives given by the optimal dual multipliers (standard LP sensitivity). If $\varepsilon_T \rightarrow 0$, any weak limit μ_∞ of $\{\mu_T\}$ satisfies $G\mu_\infty \leq 0$ by continuity of $\mu \mapsto G\mu$ on the simplex, so $\mu_\infty \in \text{CCE}_0(G_{\text{do}})$, and the LP values at ε_T converge to the values at 0 by the Lipschitz property. ■

Corollary 8. If h is constant over CCE_0 , the interval is a point and contains the limit of $\mathbb{E}_{\mu_T}[h]$ for any no-regret population by Corollary 7. Conversely, if h is non-constant over CCE_0 , pick $\mu \in \text{CCE}_0$ with $\mathbb{E}_\mu[h]$ different from the equilibrium point prediction; a population playing i.i.d. draws from μ is no-regret (its realized regrets converge to the nonpositive expected deviation gains) and its time-average converges to $\mathbb{E}_\mu[h]$. Zero-sum

example: for any $\mu \in \text{CCE}_0$ of a two-player zero-sum game, the no-deviation constraints give $\mathbb{E}_\mu[u_1] \geq \max_{a'_1} \mathbb{E}_\mu[u_1(a'_1, s_2)] \geq v$ and symmetrically $\mathbb{E}_\mu[u_1] \leq v$, so every CCE achieves the value v . ■

Corollary 9. Parametric LP: the max-side value $V(\varepsilon) = \max\{c^\top \mu : \mu \in \text{CCE}_\varepsilon\}$ is concave and piecewise-linear in the constraint right-hand side (standard RHS-perturbation concavity of a maximization LP), and nondecreasing because relaxing constraints can only increase the value. Along the uniform inflation direction $\varepsilon = \varepsilon_0 + t\mathbf{1}$, at any t where the optimal dual is unique the derivative is $\sum_j y_j^*$, the sum of the optimal dual multipliers on the inflated constraints; at kinks the one-sided derivatives are the extreme values of that sum over the optimal dual face. The min side is the max side of $-h$, hence convex with the symmetric statement. ■

Remark 14 *Our LP solver returns the dual vector alongside the primal optimum, and the shipped certificates carry $\{\partial_{\text{lower}}, \partial_{\text{upper}}, \partial_{\text{width}}\}$ at the measured ε_T . All experimental LP values in this paper are additionally verified by independent weak-duality certificates: a dual-feasible $y \geq 0$ with $(G^\top y)_j \geq v - c_j$ for all j certifies that no feasible μ exceeds v , with slack $\sim 10^{-14}$ in all reported cases.*

Appendix D. Proof details for Theorem 13

Exhaustiveness. Fix the class. (a)/(b): by (R1) the intervened mean field has finitely many zeros, each hyperbolic, so each is either stable (margin > 0) or unstable (margin < 0); the number of stable zeros is 0, 1, or ≥ 2 . If (R2)/(R3) fail at runtime (divergence, cycles, positive Lyapunov estimate), case 3 fires by its second clause. (c): the LP width is a number; it is either $\leq \text{tol}$ (case 1(ii)) or $> \text{tol}$, and in the latter case the measured- ε_T interval is either informative (case 2) or vacuous (case 3). Every system in the stated classes therefore lands in some case.

Exclusivity. Case 1(i) requires a globally unique stable σ -solution, which the multiplicity clause of case 3 negates; case 1(ii) and case 2 split on width $\leq \text{tol}$ versus $> \text{tol}$; the remaining clauses of case 3 (all-unstable, (R)-failure, vacuity) each contradict the preconditions of cases 1 and 2 directly.

Warrant. Case 1(i): in class (a), Theorem 2 gives global convergence for every $\gamma \in (0, \gamma^*)$ and a.s. convergence under decreasing gain — globality is free because the mean dynamics are linear. In class (b), let x^* be the unique stable zero. By (R2) the iterates a.s. remain in a compact set K ; by the limit-set theorem the a.s. limit set is an internally chain-transitive subset of the flow restricted to K , which by (R3) is a subset of the zero set. By Theorem 4(ii) (whose excitation condition we adopt here) the unstable zeros attract with probability zero, so the limit set is a.s. $\{x^*\}$. Case 1(ii) is Corollary 8; case 2 is Theorem 6 plus Corollary 9; case 3’s clauses carry, respectively, the divergence certificate of Theorem 2’s converse, the runtime diagnostics, the abstention rule of Remark 10, and the basin-local certificates plus the representational impossibility of Blom et al. (2019) for the multiplicity clause.

Decidability scope. In class (a) every condition is a spectral or LP computation. In class (b) the theorem is decidable *given* the enumeration of zeros (available by bisection for the

scalar systems used here, and by homotopy or interval methods more generally); (R2)–(R3) are guarded, not decided — the honest scope stated in Assumption 12. In class (c) both conditions are single LP solves. ■

Appendix E. Experimental details and reproducibility

All experiments are single-file scripts against the released instruments; seeds are fixed in-script; every number in Section 6 is pasted from an actual run. LP values carry independent weak-duality verification (Appendix C). The instruments — one per row of Table 1: the equilibrium-vs-unrolling comparator, the spectral diagnostics (stability margin and maximal stable learning rate), and the game instruments (CCE-polytope constructor, LP bounds, measured realized regret, certifier) — and the experiment scripts are provided as an anonymized code artifact accompanying the submission; their public API identifiers are redacted from this draft for double-blind review.

E1 (cobweb). Demand $D(p) = 10 - p$, supply intercept $c = 1$, tax $\text{do}(\tau=1)$ entering as the supply-intercept shift; SCM $p = -(s/d)p^e + (a - c + s\tau)/d$, $p^e = p$. Regimes: A $s/d = 0.5$; B $s/d = 1.5$ (naive, then damped $\gamma = 0.4 = \gamma^*/2$); C trend-extrapolating demand $b = 1.2$; D arctan supply (scale 2.2, steepness 12) under adaptive expectations $p^{e\ell} = (1 - g)p^e + g f(p^e)$, with the chaotic gain $g = 0.14$ found by a Lyapunov scan over $g \in [0.08, 0.52]$ and the largest Lyapunov exponent $+0.222$ computed over 8,000 steps. The linearized certificate at p^* hedges at $\gamma^* = 0.073 < g$. Honesty note (regime D): the *time-averaged* price (5.8747) happens to sit near p^* (5.9562) because the chaotic attractor is roughly symmetric around the fixed point; the pointwise prediction fails completely (the trajectory never settles), and the certificate hedges exactly there.

E2 (perturbed RPS). Payoffs as in Sato et al. (2002) with $\epsilon_X = 0.5$, $\epsilon_Y = -0.1$; intervention: payoff tax 0.3 on X 's action 0 (the intervened game built explicitly). Hedge (multiplicative-weights) population, $T = 200,000$ rounds, seed 7. Exact CCE interval for X 's payoff $[0.0667, 0.3091]$; measured- ε_T interval $[0.0646, 0.3120]$ at $\varepsilon_T = 0.0021$.

E4 (macro loop). $\beta = 0.99$, $\sigma = 1$, $\kappa = 0.3$, demand shock $\text{do}(u=+1)$; $T(\varphi) = (\beta + \kappa\sigma)/(1 + \kappa\sigma\varphi)$; cyclic SCM $B = [[0, T], [1, 0]]$ closed by naive expectations; comparator horizon 2,000, tolerance 10^{-6} ; learning paths are the naive-expectations recursion (thirty steps shown). $\varphi = 1.5$: margin = $+0.057$, gap 7×10^{-15} , $\pi^* = +9.06$, $\pi_{30} = +8.79$. $\varphi = 0.5$: margin = -0.059 , $\gamma^* = 0$, $\pi^* = -8.21$, $\pi_{30} = +249.6$.

E7 (bistable selection). $F(x) = \tanh(3x) - x + u$; roots by bisection on a grid of 6×10^5 points over $[-3, 3]$; margins = $-F'(\text{root})$. SA ensemble: $N = 10^5$ learners, constant gain 0.05, additive Gaussian noise (s.d. 0.1), 4,000 steps, $x_0 \sim \mathcal{N}(0, s_0^2)$, seed 0; dispersion curve over $s_0 \in \{0.25, 0.5, 1.0, 1.5, 2.0, 3.0\}$. Figures regenerated deterministically by the released figure script from the same code paths and seeds.