

# Travelling-Wave Detection and Analysis Methods for `cogpy.travelling_waves`

## Main survey

This survey focuses on **methods** (not dashboards) for answering “is there a travelling wave, what direction/speed/frequency/wavelength/morphology, how transient, how confident?” in **spatiotemporal neural recordings** (ECoG grids, MEAs, depth/laminar arrays with geometry, widefield voltage/calcium imaging, EEG/MEG topographies when treated as space-time fields). It draws heavily on wave-specific neuroscience methods (e.g., **phase-gradient and pattern-field approaches**) and on mature adjacent-field methods (e.g., **array processing / f-k beamforming** and **multidimensional multitaper spectral estimation**) when they transfer cleanly. <sup>1</sup>

## Taxonomy of travelling-wave analysis methods

Below is a method taxonomy organized by **what mathematical object is estimated** and **what assumptions are required**. Wherever possible, the entries list: input representation → estimated quantities → failure modes → practicality under noise, small grids, irregular layouts, and nonstationarity.

### Spectral and array-processing methods

**Conceptual idea.** Travelling waves correspond to concentrated energy along a **dispersion relation** between temporal frequency and spatial wavenumber. In the simplest plane-wave case, a dominant wavevector  $\mathbf{k}$  and temporal frequency  $\omega$  satisfy  $\omega \approx v \cdot k$  (or  $|v| \approx \omega/|k|$ ), so energy clusters at particular  $(k_x, k_y, \omega)$  or along ridges in  $(k, \omega)$ . In array-processing form, this becomes estimation of **slowness** (1/velocity) and propagation direction from multi-sensor phase delays. <sup>2</sup>

**Core variants. - 3D FFT / k- $\omega$  spectrum (regular grids).** Operates on a regularly sampled cube  $x \times y \times t$  (or 1D  $\text{space} \times t$ ). Output is a 3D spectrum magnitude  $|F(k_x, k_y, \omega)|$ ; peaks/ridges provide direction and speed estimates (plus dominant spatial wavelength  $\lambda = 2\pi/|k|$ ). Works best when spatial aperture is large enough to resolve  $k$ , and when waves are sufficiently stationary in the analysis window. (This is common in wave physics and signal processing; in neuroscience it appears most naturally in imaging and high-density arrays, and is also used conceptually in “standing vs travelling” decompositions.) <sup>3</sup>

- **f-k / beamforming scans (irregular or arbitrary sensor geometries).** Compute cross-spectral structure across sensors at frequency  $f$ , then scan candidate slowness vectors and compute beam power (“f-k spectrum”). The “high-resolution” MVDR/Capon form adaptively shapes the spatial filter and is foundational in array processing for resolving multiple directions. <sup>4</sup>

- **Capon / MVDR and related high-resolution methods.** Widely used in array seismology/infrasound/sensor arrays to estimate propagation direction and slowness from coherent structure. In wave-detection contexts it provides a principled way to separate multiple simultaneous plane waves (to the extent the array aperture and SNR allow). <sup>5</sup>

**Mathematical object.** Multidimensional Fourier transform or cross-spectral matrices across sensors; maxima in k-space / slowness-space. <sup>6</sup>

**Required inputs.** - Regular-grid k- $\omega$ : data on a grid with known spacing and time sampling. - Beamforming f-k: time series + sensor coordinates (can be irregular). <sup>7</sup>

**Outputs.** - Direction (angle of k or backazimuth), speed (via  $\omega/|k|$ , or slowness), wavelength  $\lambda$ , sometimes multiple simultaneous waves if multiple peaks exist. <sup>8</sup>

**Assumptions.** - Dominant plane-wave component within each window/frequency (or a small mixture). - Approximate stationarity within the window. - For f-k: wavefront approximately planar over the array aperture (far-field assumption in classical array processing). <sup>8</sup>

**Strengths.** - Strong physical interpretability; pairs naturally with **synthetic validation** (inject known plane waves and recover k,  $\omega$ ). - Beamforming handles **irregular sensor layouts** and is a direct bridge from mature geophysics/proximity sensing literature to neural arrays. <sup>9</sup>

**Failure modes.** - **Small spatial grids:** poor k-resolution and leakage; ambiguity between standing vs travelling components when aperture is small. - **Nonstationarity:** time-varying direction/speed smears spectral peaks unless windowing or adaptive time-frequency representations are used. - **Multiple wave components:** peaks overlap, producing biased k estimates unless multi-component methods (multi-peak detection, MUSIC/ESPRIT-like, or MVDR) are used. <sup>5</sup>

**Computational cost.** - 3D FFT:  $O(N \log N)$  per window (fast, GPU-friendly). - Beamforming scan:  $O(N_{\text{sensors}}^2 \times N_{\text{grid}})$  per frequency/time window if done naively; can be manageable for ECoG/MEA sizes with careful vectorization/caching. <sup>6</sup>

**Nonstationarity support.** - Requires windowed methods (STFT-in-time) or time-frequency representations; can be extended with wavelets or multitaper time-frequency. <sup>10</sup>

#### Special attention A: 3D / spatiotemporal spectral analysis and multidimensional multitaper

**What multidimensional multitaper is.** Multitaper (Thomson) reduces spectral variance and leakage by averaging spectra computed under multiple orthogonal tapers (DPSS/Slepian sequences). <sup>11</sup>

**3D multitaper feasibility.** - A simple, explicit and implementation-friendly multidimensional generalization constructs multidimensional tapers as **outer products of 1D tapers**, producing separable 2D/3D tapers. This is precisely described by Alfred Hanssen <sup>12</sup> (1997), including extension to higher dimensions. DOI: 10.1016/S0165-1684(97)00076-5. <sup>13</sup>

- In practice, neuroscience Python ecosystems commonly provide **1D multitaper** (time only) but not turnkey 3D multitaper; nonetheless, separable outer-product tapers make a clean first implementation target because it leverages existing DPSS generators (SciPy) and standard FFTs. <sup>14</sup>

**Is multidimensional multitaper used in practice?** It is mature in signal processing and appears in the multidimensional random-field literature (and is standard conceptually in array processing), but it is not yet a mainstream “push-button” tool in neural wave analysis pipelines compared with phase-gradient methods. <sup>15</sup>

**Suitability for small (x,y) grids with long time axis.** - If x,y are small (e.g., 8×8 or 10×10), spatial wavenumber resolution is intrinsically limited, so the primary value of 3D multitaper becomes **variance reduction and leakage control**, not magical spatial super-resolution. - A practical compromise is: multitaper along **time** (many samples → many useful tapers), modest tapering along **space** (few tapers), then estimate k-structure either via FFT-in-space (regular grids) or beamforming (irregular geometries). This design follows directly from the separable-taper framework. <sup>14</sup>

### Phase-based travelling-wave detection and characterization

**Conceptual idea.** Many neural travelling waves are expressed as **spatial phase gradients** in band-limited oscillations. If an oscillation's instantaneous phase  $\varphi(x,y,t)$  changes smoothly across space at a given time, then the gradient  $\nabla\varphi$  points along the direction of phase increase; combined with instantaneous frequency  $\omega(t)=\partial\varphi/\partial t$ , one obtains a local **phase velocity** estimate  $v \approx \omega \cdot (\nabla\varphi / |\nabla\varphi|^2)$  (up to conventions), and direction is given by  $\nabla\varphi$ . <sup>16</sup>

**Canonical examples in neuroscience.** - Doug Rubino <sup>17</sup> et al. (2006) used multielectrode motor cortex recordings and showed propagating waves mediating information transfer (Nat Neurosci), and their methodology is frequently treated as foundational for phase-gradient extraction in 2D arrays. DOI: 10.1038/nn1802. <sup>18</sup>

- Honghui Zhang <sup>19</sup> et al. (2018) provided a widely adopted ECoG-grid implementation: Hilbert phase on band-limited alpha/theta oscillations; fit a plane to phase; derive direction/speed; introduce phase-gradient directionality (PGD) as robustness. DOI: 10.1016/j.neuron.2018.05.019. <sup>20</sup>

- Evgeniy V Lubenov <sup>21</sup> & Athanassios G Siapas <sup>22</sup> (2009) is canonical for travellers along a 1D axis (hippocampal theta), highlighting how phase gradients and travelling waves are naturally linked. DOI: 10.1038/nature08010. <sup>23</sup>

- Sayak Bhattacharya <sup>24</sup> et al. (2022) demonstrate that waves are often **rotating** rather than purely planar in microelectrode PFC arrays, motivating methods that go beyond global plane fits. DOI: 10.1371/journal.pcbi.1009827. <sup>25</sup>

**Mathematical object.** Analytic signal  $a(t)=x(t)+i\cdot H[x(t)]$  (Hilbert) or a generalized analytic-signal representation; instantaneous phase  $\varphi$  and its spatial gradient  $\nabla\varphi$ ; circular statistics on phase residuals. <sup>26</sup>

**Required inputs.** - Multi-channel signals with sensor positions (grid or irregular). - Typically requires bandpass filtering (or a broadband phase method) to make phase meaningful. <sup>27</sup>

**Outputs.** - Direction, speed, local wavelength, phase coherence/robustness, plus (with additional steps) classification into planar vs rotating patterns. <sup>28</sup>

**Assumptions.** - Oscillatory structure where instantaneous phase is well-defined. - Smoothness in space (at least locally) and sufficiently dense sampling to estimate gradients. <sup>29</sup>

**Strengths.** - Highly interpretable for oscillatory neurophysiology; produces direct, time-resolved direction and speed estimates at the timescale of cycles, and it is widely supported by canonical electrophysiology travelling-wave papers. <sup>30</sup>

**Failure modes. - Filter dependence:** narrowband filtering can distort waveforms and can create apparent phase structure if applied incautiously; broadband signals can break analytic-signal assumptions (negative-frequency issues) unless specialized representations are used. <sup>31</sup>

- **Phase wraps / unwrapping:** spatial phase unwrapping is nontrivial in small or noisy arrays and near singularities; wrong unwrapping biases gradient estimates. <sup>32</sup>

- **Standing waves and sequential activations** can mimic travelling waves at the sensor level, particularly in extracranial EEG/MEG; contemporary critiques emphasize this ambiguity, especially for large-scale sensor-space claims. <sup>33</sup>

**Small grids and irregular layouts.** - On small grids, local finite-difference gradients are unstable; global regression (plane-wave fit) is preferred. - Irregular layouts need gradient estimation via local regression, Delaunay/graph gradients, or model-based fitting (see Plane-wave / delay-surface models below). <sup>34</sup>

#### Special attention D: phase-based wave detection details and robustness

**Key implementation patterns seen in the literature.** - The phase-gradient/plane-fit approach from Honghui Zhang <sup>19</sup> et al. (2018) explicitly converts phase maps into direction and speed (via plane-fit parameters) and uses PGD as a robustness/"how planar is the phase map?" metric. <sup>35</sup>

- Zachary W Davis <sup>36</sup> et al. (2020) introduced "generalized phase" to stabilize instantaneous phase estimation for wideband signals without relying solely on narrowband filtering; their public MATLAB code documents the motivation (centering analytic signal, correcting negative-frequency components). DOI: 10.1038/s41586-020-2802-y. <sup>37</sup>

**Practical robustness guidance derived from published methods.** - Treat phase-based estimates as meaningful primarily where local oscillatory amplitude/coherence is sufficient; otherwise down-weight or mask. - Use explicit goodness-of-fit measures (PGD-like, circular variance, residual dispersion) as *first-class outputs*, not an afterthought. This is directly consistent with PGD practice and with toolboxes that test against surrogates. <sup>38</sup>

#### Plane-wave and delay-surface fitting

**Conceptual idea.** Travelling waves imply systematic inter-sensor delays. Estimate those delays and fit an explicit **propagation model:** - Plane-wave model:  $\tau_i \approx (k \cdot r_i) / \omega + b$ , or equivalently phase  $\varphi_i \approx k \cdot r_i + b \pmod{2\pi}$ . - Delay-surface model:  $\tau(x,y)$  solves a fitted surface; its gradient gives slowness/direction.

This family overlaps strongly with phase-gradient methods but expands to **non-oscillatory events** by using lags/cross-correlation rather than phase.

**Canonical neuroscience usage.** - Early human EEG travelling-wave work includes regression of delays/slopes across sensors (planar wave direction via slope sign), e.g., T M Patten <sup>39</sup> et al. (2012) in PLOS ONE (Human Cortical Traveling Waves). DOI: 10.1371/journal.pone.0038392. <sup>40</sup>

- Recent protocols emphasize explicit regression/statistical testing for travelling waves in microelectrode arrays, e.g., V M Zarr <sup>41</sup> et al. (2025) in STAR Protocols (multi-linear regression approach, event-based). <sup>42</sup>

**Mathematical object.** Regression on phases or delays; robust estimators (e.g., circular-linear regression, least squares on delay surfaces, robust regression on lags).

**Required inputs.** - Sensor coordinates + either phase estimates (oscillatory waves) or lag estimates (cross-correlation/event timing). <sup>43</sup>

**Outputs.** - Direction and speed (or slowness), model fit residuals/uncertainty. <sup>44</sup>

**Assumptions.** - Within the fitted window/event, propagation is approximately coherent and described by a low-parameter model (plane or simple radial model). <sup>45</sup>

**Strengths.** - Works on **small grids** because it uses global constraints rather than local derivatives. - Works on **irregular layouts** naturally (regress on actual coordinates). <sup>46</sup>

**Failure modes.** - Multiple simultaneous waves or curved/spiral waves can produce poor plane fits; residuals are informative but classification requires additional machinery. <sup>47</sup>

**Special attention E: when explicit plane/delay models work and fail**

**Where they work best.** - Arrays with coherent, near-planar propagation during a time window (common in many ECoG alpha/theta travelling waves reported in invasive recordings). <sup>48</sup>  
- Event-like propagation where delays are meaningful even without stable oscillatory phase (as in explicit regression protocols). <sup>42</sup>

**Where they fail.** - Rotating/spiral/source-sink patterns: a single wavevector cannot represent the field; methods that infer velocity vector fields and/or phase singularities are needed. <sup>49</sup>

**Motion-estimation methods, including optical flow**

**Conceptual idea.** Treat spatiotemporal neural activity patterns as a “movie” and estimate a dense velocity field describing how patterns move between frames. This is widely developed in computer vision and has been explicitly ported to neural-wave analysis both in imaging and in multielectrode recordings. <sup>50</sup>

**Key neuroscience-specific toolchains.** - Navvab Afrashteh <sup>51</sup> et al. (2017) created a MATLAB optical-flow toolbox (OFAMM/OFAMM-like), comparing Horn-Schunck, combined local-global, and temporospatial methods on simulated and mouse voltage/calcium imaging data, concluding combined local-global performed best for wave dynamics in their tests. DOI: 10.1016/j.neuroimage.2017.03.034. <sup>52</sup>

- Rory G Townsend <sup>53</sup> & Pulin Gong <sup>54</sup> (2018) link travelling waves to coherent-structure ideas (vortices) and introduce velocity vector fields and pattern classification; their MATLAB toolbox (NeuroPattToolbox) includes surrogate/noise-driven checks. DOI: 10.1371/journal.pcbi.1006643. <sup>55</sup>

- L Cao <sup>56</sup> et al. (2021) explicitly describe using **optical flow** to characterize propagating spatiotemporal LFP patterns in hippocampal-array recordings (Cell Reports Methods). <sup>57</sup>

**Mathematical object.** A velocity field  $u(x,y,t)$  satisfying an optical-flow constraint (brightness/feature constancy + regularization), often solved with variational methods. <sup>58</sup>

**Required inputs.** - A 2D lattice of frames, e.g., amplitude maps, phase maps, analytic-signal real/imag maps, or imaging frames ( $\Delta F/F$ ). - For irregular arrays, a preprocessing step to interpolate to a grid (with careful caveats) or an adaptation of flow to scattered data. <sup>59</sup>

**Outputs.** - Dense or semi-dense velocity vectors; derived divergence/curl; critical points (sources/sinks/vortices/spirals); wave trajectories. <sup>60</sup>

**Assumptions.** - Small inter-frame displacements, some form of constancy, and spatial smoothness regularization (classic constraints). <sup>58</sup>

**Strengths.** - Naturally handles **complex morphologies** (rotations, spirals) when paired with vector-field topology (divergence/curl/critical points), matching the “spiral/source/sink” vocabulary common in modern wave-pattern papers and toolboxes. <sup>61</sup>

**Failure modes.** - Optical flow estimates **apparent motion** and can be biased by amplitude modulations or interference patterns; oscillatory phase wrapping can destabilize flow unless the representation is chosen carefully (e.g., compute flow on unwrapped phase or on complex analytic-signal components). The existence of multiple overlapping waves can yield flows that are not physically meaningful as a single propagation velocity. <sup>62</sup>

**Computational cost.** - Variational optical flow is moderate but tractable for typical grid sizes; mature implementations exist (e.g., scikit-image provides iterative Lucas–Kanade pyramidal flow and TV-L1 flow). <sup>63</sup>

#### **Special attention B: optical flow for oscillatory neural fields—best practices and limitations**

**Evidence of use in neural waves.** - Explicit optical-flow toolboxes exist for mesoscale imaging and have been evaluated with simulated ground truth and real voltage/calcium imaging (Afrashteh et al., 2017). <sup>64</sup>

- Optical flow is also integrated into pattern-classification frameworks for neural recordings (Townsend & Gong, 2018; NeuroPattToolbox). <sup>65</sup>

- Optical flow has been used on array-recorded LFP-derived spatiotemporal features (Cao et al., 2021). <sup>66</sup>

**Practical recommendations for `cogpy` (method-driven, not UI-driven).** - Prefer computing flow on **band-limited analytic-signal derived maps** (amplitude, phase, or complex components) rather than raw broadband signals, unless a broadband phase representation is used (e.g., generalized phase) to stabilize phase. <sup>67</sup>

- Treat optical flow as an **estimator of local phase-velocity structure** rather than a definitive “true axonal propagation” measurement; pair it with independent validation metrics (e.g., plane-fit residuals, k- $\omega$  spectral peaks) to avoid overinterpretation. This caution aligns with ongoing debates about sensor-level wave interpretations. <sup>68</sup>

#### **Decomposition methods: complex PCA/SVD, DMD, tensor factorization**

**Conceptual idea.** Travelling waves often manifest as **low-dimensional spatiotemporal structure**. Decompositions aim to recover modes that encode travelling/standing components, possibly with rotation and propagation.

**Transferable methods with concrete code.** - **Complex PCA (CPCA).** By encoding signals as complex (e.g., analytic signals or phase maps), CPCA can separate standing and travelling structure. This approach appears directly in brain-wide spatiotemporal pattern work and comes with public code repositories. <sup>69</sup>

- **Dynamic Mode Decomposition (DMD).** Originating in fluid mechanics, DMD approximates a linear

operator whose eigenvectors/modes carry oscillatory dynamics; it is intended to recover coherent spatiotemporal structures and has extensive theoretical grounding and references. <sup>70</sup>

- **Rotation-focused linear methods (e.g., jPCA interpretations).** Some recent work argues that neuronal “rotational dynamics” can be explained by travelling waves, reinforcing the need for tools that connect rotations in low-dimensional projections back to propagating patterns. <sup>71</sup>

**Mathematical object.** Linear operators/modes in complex or real space; eigen-decompositions; low-rank approximations.

**Strengths.** - Useful for summarizing dynamics and separating components. - Can complement direct wave estimators by providing alternative evidence for coherent travelling structure. <sup>72</sup>

**Failure modes.** - Decompositions can produce “wave-like” modes from smoothness and shifts, including in situations where the physical interpretation is subtle; caution is warranted, and null/surrogate testing should be standard. <sup>73</sup>

### **Bayesian, state-space, and switching-state methods**

**Conceptual idea.** Represent travelling-wave parameters (direction, speed, wave type) as latent variables that evolve over time, possibly switching among discrete regimes (e.g., two dominant directions). Use probabilistic filtering/smoothing to estimate time-varying wave state and uncertainty.

**What exists in neuroscience.** - Switching state-space models are explicitly motivated for neural time series with rapid changes in dynamics, but they are not (yet) common as standardized travelling-wave detectors; they are better viewed as a “Phase 2+” architecture target. <sup>74</sup>

- Many travelling-wave papers implicitly treat wave direction as state-dependent and bidirectional (e.g., task modulation), suggesting a natural role for switching models. <sup>75</sup>

**What is mature in adjacent fields.** - Particle filtering and tracking approaches are used for wavefront propagation in excitable media (e.g., cardiac wavefront tracking demonstrations), indicating algorithmic feasibility for wavefront/state tracking when a measurement model is specified. <sup>76</sup>

**Mathematical object.** Latent wave parameters with transition dynamics; observation models mapping latent wave state to sensor measurements; posterior distributions.

**Strengths.** - Natural uncertainty quantification; handles nonstationarity by design.

**Failure modes.** - Requires a committed generative model and careful identifiability work; heavy implementation burden compared to regression/spectral/phase-gradient methods.

**Special attention C: Kalman/state-space methods for travelling waves—what’s realistic for** cogpy

**Current state of the field.** The most direct evidence base for Kalman filtering in travelling-wave *parameter tracking* is stronger in signal processing and wavefront tracking domains than in mainstream neuroscience travelling-wave toolkits. <sup>77</sup>

**A pragmatic cogpy framing.** - Treat state-space tracking as an **optional wrapper** around per-window wave estimates (from PGD/plane-fit,  $k$ - $\omega$  peaks, or optical flow): the measurement is  $(k_t, \omega_t)$  or  $(direction_t, speed_t)$  with uncertainty; a Kalman filter smooths time variation and can support detecting state changes. This is a conceptually clean extension of widely used per-window approaches and aligns with general switching-state models for neural dynamics. <sup>78</sup>

### **Methods for complex wave morphologies: spirals, sources/sinks, multiple waves**

**Conceptual idea.** Spirals and rotating waves are organized around **phase singularities** (points where the phase is undefined and rotation occurs). Sources/sinks correspond to divergence structure in velocity fields; spirals correspond to nonzero curl and topological charge.

**Neuroscience evidence base (spirals are not niche).** - fMRI “brain spirals” were analyzed in Yiben Xu <sup>79</sup> et al. (2023), explicitly describing spiral-like rotational wave patterns organized around phase singularity centers. DOI: 10.1038/s41562-023-01626-5. <sup>80</sup>

- Sleep spindles forming travelling spiral waves were reported in Yiben Xu <sup>79</sup> et al. (2025, Communications Biology), also emphasizing phase singularities and rotational patterns. DOI: 10.1038/s42003-025-08447-4. <sup>81</sup>

- Toolboxes like NeuroPatt explicitly include critical point analysis and vector-field decompositions intended to classify patterns beyond planar waves. <sup>82</sup>

**Algorithmic building block (transferable).** Phase singularity detection via winding number / topological charge estimators is mature in excitable media analysis (especially cardiac mapping), with explicit formulae and comparative evaluations. <sup>83</sup>

### **Special attention F: classification and detection of spirals/rotations/radial waves**

**Good candidates for cogpy that are methodologically explicit.** - **Phase singularity via local winding number** in a complex field (analytic signal) on a grid: robust, interpretable, and well aligned with both neuroscience spiral-wave papers and excitable-media methodology. <sup>84</sup>

- **Velocity-field topology:** compute divergence/curl of velocity fields (from optical flow) and classify sources/sinks/vortices, consistent with Townsend & Gong style. <sup>65</sup>

**Challenges.** - Needs careful handling of amplitude nulls, phase noise, and spatial interpolation if sensors are not on a true grid. <sup>85</sup>

### **Foundational and modern papers by method family**

The list below intentionally prioritizes papers with implementable detail and direct relevance to invasive electrophysiology/imaging arrays, while including adjacent-field algorithm sources where they are canonical and transferable.

**Phase-gradient / plane-fit travelling waves (core electrophysiology).** - Doug Rubino <sup>17</sup> et al. *Propagating waves mediate information transfer in the motor cortex.* **Nature Neuroscience** (2006). DOI: 10.1038/nn1802. Domain: motor cortex electrophysiology (2D array). Estimates: propagating waves; phase gradients/delays. Why it matters: canonical early implementation for 2D multielectrode travelling wave extraction; underlies many later phase-gradient pipelines. <sup>18</sup>

- Honghui Zhang <sup>19</sup> et al. *Theta and Alpha Oscillations Are Traveling Waves in the Human Neocortex*. **Neuron** (2018). DOI: 10.1016/j.neuron.2018.05.019. Domain: human ECoG. Estimates: direction & speed from fitted phase gradient; PGD robustness metric. Code: method detail in open article and widely reproduced; this is the most direct “Phase 1” blueprint for `cogpy` plane-wave fitting. <sup>20</sup>
- Evgueniy V Lubenov <sup>21</sup> & Athanassios G Siapas <sup>22</sup>. *Hippocampal theta oscillations are travelling waves*. **Nature** (2009). DOI: 10.1038/nature08010. Domain: rodent hippocampus (1D axis). Estimates: phase gradients and travelling direction. Why it matters: canonical 1D travelling-wave example; useful for validating 1D implementations. <sup>86</sup>
- Sayak Bhattacharya <sup>24</sup> et al. *Traveling waves in the prefrontal cortex during working memory*. **PLOS Computational Biology** (2022). DOI: 10.1371/journal.pcbi.1009827. Domain: microelectrode arrays. Estimates: planar vs rotating waves; wave direction trends. Code: public MATLAB repository for analysis. Why it matters: motivates non-planar classification and provides reusable code patterns. <sup>87</sup>
- T M Patten <sup>39</sup> et al. *Human Cortical Traveling Waves: Dynamical Properties and Correlates*. **PLOS ONE** (2012). DOI: 10.1371/journal.pone.0038392. Domain: human EEG. Estimates: planar wave direction and slope-based measures. Why it matters: illustrates plane-wave regression logic in extracranial settings and highlights sensor-space ambiguity issues. <sup>40</sup>

**Broadband phase and phase robustness.** - Zachary W Davis <sup>36</sup> et al. *Spontaneous travelling cortical waves gate perception in behaving primates*. **Nature** (2020). DOI: 10.1038/s41586-020-2802-y. Domain: primate cortex electrophysiology. Estimates: travelling waves and behavioural relevance; introduces generalized-phase handling for wideband signals. Code: public “generalized-phase” repository describing algorithmic corrections. Why it matters: provides a method to reduce the brittleness of narrowband Hilbert-phase pipelines. <sup>37</sup>

**Optical flow / velocity-field pattern analysis.** - Navvab Afrashteh <sup>51</sup> et al. *Optical-flow analysis toolbox for characterization of spatiotemporal dynamics in mesoscale optical imaging of brain activity*. **NeuroImage** (2017). DOI: 10.1016/j.neuroimage.2017.03.034. Domain: mouse voltage/calcium imaging. Estimates: velocity fields, sources/sinks, trajectories; compares Horn-Schunck/CLG/temporospatial. Code: public repo and MATLAB distribution. Why it matters: tested toolbox with simulation + experimental data; directly reusable method patterns for imaging-like arrays. <sup>88</sup>

- Rory G Townsend <sup>53</sup> & Pulin Gong <sup>54</sup>. *Detection and analysis of spatiotemporal patterns in brain activity*. **PLOS Computational Biology** (2018). DOI: 10.1371/journal.pcbi.1006643. Domain: neural population recordings. Estimates: multiple wave classes; velocity fields; coherent-structure framing. Code: NeuroPattToolbox (MATLAB) with surrogate testing and pattern transitions. Why it matters: clearest implementable blueprint for detecting spirals/sources/sinks as well as planar waves. <sup>65</sup>

- L Cao <sup>56</sup> et al. *Uncovering spatial representations from spatiotemporal decoding of hippocampal field potentials*. **Cell Reports Methods** (2021). Domain: hippocampal array LFP features. Includes optical flow for propagating pattern characterization. Why it matters: directly bridges optical flow from imaging into electrophysiology array feature maps. <sup>89</sup>

**Spiral/rotational wave morphology and phase singularities.** - Yiben Xu <sup>79</sup> et al. *Interacting spiral wave patterns underlie complex brain dynamics and are related to cognitive processing*. **Nature Human Behaviour** (2023). DOI: 10.1038/s41562-023-01626-5. Domain: fMRI. Estimates: spiral waves, phase singularity centers, task relevance. Why it matters: motivates phase-singularity detection and multi-spiral interactions in `cogpy` (even if modality differs). <sup>90</sup>

- Yiben Xu <sup>79</sup> et al. *Spatiotemporal dynamics of sleep spindles form spiral waves...* **Communications Biology** (2025). DOI: 10.1038/s42003-025-08447-4. Domain: high-density EEG. Estimates: spiral dynamics, phase

singularities, trajectories. Why it matters: shows spiral-wave patterning at scale and strongly motivates robust spiral metrics and trajectory statistics. <sup>81</sup>

- H Lilienkamp <sup>91</sup> et al. *Detecting spiral wave tips using deep learning*. **Scientific Reports** (2021). DOI: 10.1038/s41598-021-99069-3. Domain: excitable media. Why it matters: not recommended as a first method for `cogpy`, but the paper succinctly documents classical phase-singularity definitions used as ground truth and highlights performance issues near noise. <sup>92</sup>

**Spectral / k- $\omega$  / array-processing foundations transferable to neural arrays.** - J Capon <sup>93</sup>. *High-resolution frequency-wavenumber spectrum analysis*. **Proceedings of the IEEE** (1969). DOI: 10.1109/PROC.1969.7278. Domain: array processing. Estimates: high-resolution f-k spectrum (MVDR/Capon). Why it matters: direct foundation for f-k beamforming modules that work on irregular or small neural arrays with appropriate assumptions. <sup>94</sup>

- D J Thomson <sup>95</sup>. *Spectrum estimation and harmonic analysis*. **Proceedings of the IEEE** (1982). DOI: 10.1109/PROC.1982.12433. Domain: multitaper spectral estimation. Why it matters: foundation for multitaper time-frequency estimation used pervasively; provides principled variance reduction and (with jackknife variants) uncertainty machinery that can be carried into wave metrics. <sup>96</sup>

- Alfred Hanssen <sup>12</sup>. *Multidimensional multitaper spectral estimation*. **Signal Processing** (1997). DOI: 10.1016/S0165-1684(97)00076-5. Domain: multidimensional spectral estimation. Why it matters: clear recipe for separable 2D/3D DPSS tapers for 3D k- $\omega$  estimation and for robust windowed spectral maps. <sup>13</sup>

- Brett M Wingeier <sup>97</sup> et al. *Spherical harmonic decomposition applied to spatial-temporal analysis of human high-density EEG*. (2000, arXiv). Domain: spatial spectral analysis on hemispherical/irregular sampling. Why it matters: suggests a practical route for spatial spectra on non-grid sensor layouts (spherical harmonics) and quantifies sampling requirements using simulations. <sup>98</sup>

**Nonstationary large-scale phase dynamics and irregular sampling (relevant to iEEG/SEEG).** - David M Alexander <sup>99</sup> et al. *Large-scale cortical travelling waves predict localized future cortical signals*. **PLOS Computational Biology** (2019). DOI: 10.1371/journal.pcbi.1007316. Domain: ECoG + MEG. Why it matters: demonstrates travelling-wave-like large-scale eigenvectors from Fourier/PCA features, reinforcing decomposition + spectral framings. <sup>100</sup>

- David M Alexander <sup>99</sup> & Laura Dugué <sup>101</sup>. *The dominance of large-scale phase dynamics in human cortex, from delta to gamma*. (bioRxiv 2024/2026 versions). Domain: irregularly sampled iEEG spatial spectra estimation. Why it matters: points directly at the “irregular geometry spatial spectrum” problem `cogpy` will face for SEEG and sparse grids. <sup>102</sup>

**Directionality measures adjacent-but-useful (validation/contrast).** - G Nolte <sup>103</sup> et al. *Robustly Estimating the Flow Direction of Information in Complex Physical Systems*. (2008) introduces the phase-slope index (PSI) and its robustness properties against mixing, which can be used as an auxiliary directionality metric distinct from travelling-wave kinematics. FieldTrip documents sign interpretation for PSI. <sup>104</sup>

## Code ecosystem survey

This section is intentionally **wave-method-centric**: only codebases that implement travelling-wave detection/analysis logic (or directly implement enabling blocks such as optical flow solvers) are cataloged.

## Code and resource catalog

This section corresponds to `docs/reference/codelib/libraries/travelling_waves_tools.md`.

## Production-worthy or strongly inspirational references

**NeuroPattToolbox (MATLAB).** Implements detection/analysis/visualization of spatiotemporal patterns including wave classification using velocity fields and critical point analysis, with sample data and surrogate/noise-driven verification pathways. It is a direct reference implementation for “pattern classes beyond plane waves.” <sup>82</sup>

**OFAMM / Optical Flow Analysis Toolbox (MATLAB + C++).** Provides optical-flow-based velocity field estimation and wave characterization, explicitly targeting cortical travelling waves (speed/direction/trajectory) in widefield imaging-like data. It is linked to the NeuroImage paper that compares Horn-Schunck, combined local-global, temporospatial methods. <sup>105</sup>

**scikit-image optical flow implementations (Python).** `skimage.registration` includes pyramidal iterative Lucas–Kanade flow and TV-L1 flow, giving `cogpy` a lightweight, widely used dependency option for optical flow without custom solvers. <sup>63</sup>

## Reusable “paper code” for travelling-wave analyses (often archival but valuable)

**Generalized Phase (MATLAB).** Provides a phase-estimation method intended to stabilize analytic-signal phase estimates in broadband signals by centering the complex representation and correcting negative-frequency components; tightly linked to Davis et al. (Nature 2020). It also documents dependencies commonly used in wave work (CircStat, etc.). <sup>106</sup>

**Working-memory travelling waves code (MATLAB).** The repository accompanying Bhattacharya et al. (PLOS Comp Biol 2022) contains scripts for classifying planar vs rotating waves and computing wave-direction trends using circular–circular correlations and related coefficients. <sup>107</sup>

**Travelling waves vs sequential modules code (mixed; simulation + wave analysis).** Provides analysis and simulation components explicitly designed to test whether observed travelling-wave signatures can arise from sequentially activated discrete modules—useful as a validation adversary / null model reference. <sup>108</sup>

## Enabling spectral/time-frequency toolkits (not wave-specific, but essential)

**MNE-Python multitaper PSD (Python).** Provides a stable multitaper PSD function (`psd_array_multitaper`) and is a reasonable dependency for time-domain multitaper building blocks (even if it does not provide travelling-wave logic directly). <sup>109</sup>

**Nitime multitaper examples and spectral modules (Python).** Provides multitaper spectral estimation examples and code, useful as a reference for multitaper implementation details and testing. <sup>110</sup>

**Prerau Lab multitaper\_toolbox (MATLAB/Python/R).** Implements multitaper spectrogram analysis and is relevant as a reference for time-frequency multitaper implementation patterns and parameterization. <sup>111</sup>

**FieldTrip spectral analysis with multitapers (MATLAB).** FieldTrip documents multitaper-based spectral analysis in its workshop/tutorial material; while not wave-specific, it is a canonical reference for neuroscience multitaper usage. <sup>112</sup>

## Technical design spec

This section corresponds to `docs/specs/travelling_waves_module.md`.

### Design goals derived from the method survey

1. **Multiple complementary estimators** rather than a single “best” travelling-wave detector, because planar/rotational/multi-wave and oscillatory/event-like cases require different assumptions. <sup>113</sup>
2. **First-class uncertainty/robustness metrics** (PGD-like fit quality, spectral peak sharpness, bootstrap intervals) because wave claims are sensitive to noise and preprocessing. <sup>114</sup>
3. **Geometry-aware API**: support both (a) regular x-y grids and (b) irregular coordinates (SEEG, sparse arrays). This is required for spatial spectra and for fitting wave models on scattered layouts. <sup>115</sup>
4. **Nonstationarity support** via windowed analysis and event detection, aligning with both oscillatory and non-oscillatory travelling-wave protocols. <sup>116</sup>
5. **Validation-first architecture**: synthetic data generation + surrogate testing included in-core, reflecting the practice of toolboxes that validate against noise-driven surrogates. <sup>117</sup>

### Proposed module layout and core abstractions

The architecture below matches the evidence that wave analysis naturally decomposes into spectral, phase, motion, fitting, and validation, while keeping shared representations consistent.

**Top-level package.** - `cogpy.travelling_waves` - `core/` - `data_models.py`: typed containers for geometry, estimates, and metadata. - `preprocess.py`: filtering, analytic-signal transforms (Hilbert) and broadband-phase options. - `windows.py`: windowing, event segmentation. - `phase/` - `analytic_phase.py`: Hilbert phase; wrappers for generalized phase. - `phase_gradient.py`: gradient estimation on grids and scattered points; PGD and circular stats. - `phase_singularities.py`: winding-number/topological-charge detectors for spirals. - `fitting/` - `plane_wave_fit.py`: circular-linear regression plane fits; speed/direction extraction (Zhang-style). - `delay_surface.py`: lag estimation + plane/surface fit; robust regression options for small grids. - `spectral/` - `kw_spectrum.py`: k- $\omega$  estimation for regular grids (3D FFT) + ridge/peak extraction. - `fk_beamforming.py`: beamforming and Capon-style f-k scanning for arbitrary coordinates. - `multitaper_nd.py`: separable multidimensional DPSS tapers (Hanssen-style) integrated with k- $\omega$  and f-k. - `motion/` - `optical_flow.py`: wrappers around scikit-image flow solvers; flow on amplitude/phase/complex fields. - `vector_field_features.py`: divergence/curl/critical points; pattern classification hooks. - `decomp/` - `cpca.py`: complex PCA on analytic-signal maps (optional). - `dmd.py`: dynamic mode decomposition (optional). - `statespace/` - `wave_state_tracker.py`: filtering/smoothing over time for direction/speed; optional switching regimes. - `simulation/` - `synthetic.py`: generate plane waves, spirals, wave packets, multi-wave mixtures, with noise models. - `validation/` - `surrogates.py`: phase randomization, time shuffles, spatial shuffles, noise-driven synthetic controls. - `metrics.py`: fit quality (PGD-like), spectral peakness, coherence, reproducibility across trials. - `viz/` - `plots.py`: phase maps, quiver fields, k- $\omega$  slices, direction histograms; minimal but method-linked.

This layout is consistent with what current toolboxes emphasize (velocity fields + critical points for pattern classes; and phase-gradient plane fits for direction/speed). <sup>118</sup>

## API design principles

**Inputs.** - Accept `numpy.ndarray` and `xarray.DataArray` with explicit dimension labels for `time` and either `space` or `x,y` grid. - Accept a `Geometry` object: either `(x,y)` grid spacing or per-channel coordinates. This is necessary for both phase-gradient fitting and beamforming-style f-k. <sup>119</sup>

**Outputs.** - Standardize on a `WaveEstimate` record with: - `direction` (angle in radians or unit vector), - `speed` (m/s or units of coordinate/time), - `frequency` (Hz) and optionally `wavenumber` (rad/m), - `wavelength`, - `pattern_type` (planar / rotating / spiral / source / sink / mixed / uncertain), - `confidence` / uncertainty intervals, - `fit_quality` (PGD-like, residual dispersion), - `support_mask` (where on grid estimate applies). <sup>120</sup>

**Method-level contracts.** - Every detector must implement: - `fit(data, geometry, *, window, freq_band, ...)` -> `WaveEstimate` - `score(data, estimate)` -> `metrics` - `plot_diagnostics(...)` (optional, lightweight). - Every method must ship with synthetic tests that recover known direction/speed and report bias/variance against noise and grid size; this mirrors the simulation-supported style in OFAMM and in wave-simulation adversarial codebases. <sup>121</sup>

## Dependency strategy

**Phase 1 dependencies (lightweight).** - NumPy/SciPy for filtering, DPSS, FFT; xarray optional but supported. - scikit-image optional extra for optical flow wrappers. <sup>63</sup>

**Optional extras.** - MNE or nitime as references for multitaper semantics, but `cogpy` should avoid hard dependency unless already used elsewhere; using their APIs as validation references is still useful. <sup>122</sup>

## Prioritized implementation roadmap

This section corresponds to `docs/roadmaps/travelling_waves_methods_implementation.md`.

### Ranking criteria for `cogpy.travelling_waves`

The ranking below is optimized for PixECoG-like grids and spatially arranged LFP/MEA data: interpretability, robustness, implementation tractability, synthetic validation readiness, and literature support.

## Recommended Phase 1 methods

### Phase-gradient plane-wave fitting with PGD-style robustness (oscillatory travelling waves).

Why: This is the single most canonical and implementable travelling-wave estimator in invasive electrophysiology for direction and speed from band-limited oscillations, with explicit goodness-of-fit metrics (PGD) and clear synthetic validation routes. It is directly supported by cornerstone ECoG and motor cortex papers and is used in later work as a methodological template. <sup>123</sup>

Implementation scope: - Hilbert analytic phase; spatial unwrapping; circular-linear plane fit; direction/speed; PGD/fit-quality; bootstrap CI.

### **k- $\omega$ / f-k spectral estimation suite (regular-grid FFT + optional beamforming for irregular geometries).**

Why: Spectral/array methods provide an orthogonal line of evidence to phase-plane fits and are mature in adjacent fields; they are also the natural home for uncertainty tools via multitapering and for diagnosing multi-wave mixtures (multiple peaks). Minimum viable implementation can start with FFT-based k- $\omega$  for regular grids, then add beamforming scans for irregular arrays. <sup>124</sup>

Implementation scope: - 3D FFT windows; peak/ridge extraction  $\rightarrow$  direction/speed/wavelength; coherence/peakness metrics; optional Capon beamforming; separable multidimensional DPSS tapers (Hanssen) as an “enhanced” mode.

### **Optical-flow velocity fields on phase/amplitude maps (pattern kinematics + complex morphologies).**

Why: Optical flow is one of the few families that naturally extends from planar propagation to rotating/spiral/source-sink patterns, and it has explicit neuroscience toolboxes for mesoscale imaging and pattern-class frameworks. A Phase 1 implementation can be thin: wrap established solvers (TV-L1 / iterative Lucas-Kanade) and implement derived features (divergence/curl, critical points). <sup>125</sup>

Implementation scope: - Flow on unwrapped phase or complex analytic components; velocity-field features; minimal pattern classification heuristics; diagnostic plots.

## **Recommended Phase 2 methods**

### **Phase singularity and spiral-center detection (topological charge / winding number).**

Why: The importance of spirals is growing across modalities, and phase singularities are definitional objects for spirals. The algorithms are mature in excitable media and increasingly used in neuroscience spiral analyses; implementing them enables robust spiral detection and tracking beyond heuristic curl thresholds.

<sup>126</sup>

### **Delay-surface / cross-correlation lag regression for non-oscillatory travelling events.**

Why: Complements phase-based oscillatory methods and aligns with explicit regression protocols for travelling waves in microelectrode arrays; provides a wave detector for sharp transients and burst-like propagations where phase is unstable. <sup>127</sup>

### **Complex PCA and DMD as optional decompositional lenses.**

Why: They provide compact summaries and can separate modes that resemble travelling components, but they require careful interpretation and surrogate testing. CPCA already has open code in neuroimaging contexts; DMD has extensive theory and is broadly transferable. <sup>128</sup>

### **State-space smoothing / switching wave states (Kalman / SLDS wrappers).**

Why: Likely valuable for state-dependent direction switching and transient “wave episodes,” but best implemented after stable per-window estimators exist, using those as observations for a probabilistic tracker. <sup>129</sup>

## **Not recommended for now**

### **End-to-end deep learning optical flow or deep wave classifiers as primary methods.**

Reason: High validation burden, dependence on large labeled datasets or synthetic realism, and lower interpretability relative to classical estimators—misaligned with a foundational method package. (Deep

learning can be revisited after the synthetic validation suite is mature.) This is consistent with toolboxes and papers that emphasize explicit optical-flow/phase-gradient methods rather than black-box detectors. <sup>50</sup>

### Highly specialized MEG/EEG forward-model-based travelling-wave inference as a core package feature.

Reason: Source modelling + sensor mixing issues complicate “travelling wave” claims, and parts of the literature emphasize ambiguity between true cortical waves and mixtures of sources at the sensor level; this can be supported later as an application layer, but it is not a good early core for `cogpy.travelling_waves`. <sup>33</sup>

### Concrete Phase 1 deliverables checklist

- A Phase 1 that is both scientifically defensible and software-complete should ship:
- A unified `WaveEstimate` object and geometry handling for grids + scattered layouts. <sup>46</sup>
  - Three core estimators (plane-fit PGD;  $k$ - $\omega$ / $f$ - $k$ ; optical flow) + per-estimator confidence metrics. <sup>130</sup>
  - A synthetic generator for plane waves and rotating/spiral patterns (even if spiral detection is Phase 2) to allow adversarial testing of false positives. <sup>131</sup>
  - Surrogate testing utilities (phase randomization / noise-driven controls), reflecting established toolbox practice. <sup>82</sup>

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