

Splitting methods for Intermediate Long Wave and perturbed Benjamin–Ono models

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August 7, 2026

Abstract

Recently, novel integrability techniques, and most notably Birkhoff coordinates followed by an explicit formula, have been introduced to solve the Benjamin–Ono (BO) equation, leading to major theoretical and computational advances. We propose to perturb this explicit BO formula via a novel splitting method for numerically solving a class of PDEs, which includes the Intermediate Long Wave (ILW) equation, as well as other quasilinear equations that are not necessarily integrable. Using the Birkhoff coordinates of BO we establish first-order convergence in the H^s -norm ($s \geq 0$) while requiring only one additional derivative on the initial data $u_0 \in H_0^{s+1}$, significantly improving upon the regularity requirements of typical splitting methods for nonlinear dispersive equations. Furthermore, we prove that in the deep-water limit, the scheme for the ILW equation converges to the BO solution. Computational advantages of these schemes are shown in simulations: unlike classical splitting methods, our approach does not require a restrictive quadratic time step condition to nearly preserve energy over long time scales, thus rendering efficient and accurate long-time simulations feasible. As an application, we numerically explore the soliton resolution conjecture for both the ILW and a KdV–BO equation.

Keywords: Nonlinear Fourier transform, quasilinear equations, splitting methods, structure-preserving schemes, low-regularity, soliton resolution

Mathematics Subject Classification: Primary – 37K10, 65M70; Secondary – 65M15, 35Q55, 35Q35

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1 Introduction

In this paper, we consider the approximation to the following class of perturbed Benjamin–Ono equation

$$\partial_t u = \partial_x |D|u - 2u\partial_x u + Au, \quad u|_{t=0}(x) = u_0(x), \quad (t, x) \in \mathbb{R} \times \mathbb{T}, \quad (1)$$

where $u(t, x)$ is real-valued, $D = \frac{1}{i} \frac{d}{dx}$, $\widehat{|D|f}(k) = |k|\widehat{f}(k)$ and A is a Fourier multiplier of order zero satisfying the following conditions

$$\widehat{Af}(k) = a(k)\widehat{f}(k), \quad a(-k) = \overline{a(k)}, \quad k \in \mathbb{Z} \quad \text{and} \quad \|a\|_{\ell^\infty} = \sup_{k \in \mathbb{Z}} |a(k)| < \infty. \quad (2)$$

We emphasize that no smallness assumptions or smoothing properties are required for the perturbative term A ; for relevant PDE results in the former setting we refer to Remark 1.4.

We will be considering initial data of nonnegative Sobolev regularity, for which global well-posedness for $u_0 \in H_0^s := \{u_0 \in H^s(\mathbb{T}) : \int_{\mathbb{T}} u_0 = 0\}$, $s \geq 0$, has recently been obtained in Gassot–Laurens [20]. For a discussion on the zero-mean condition, see Remark 1.5.

We start by giving three examples of equations belonging to the class (1).

The intermediate long wave equation. The (ILW) equation in the moving frame reads as

$$\partial_t u = -i \coth(\delta D) \partial_x^2 u - 2u\partial_x u, \quad (\text{ILW})$$

with $\delta > 0$. We can write (ILW) in the form (1) with

$$a(k) = i(\coth(\delta k) - \text{sign}(k))k^2. \quad (3)$$

Equation (ILW) is an integrable model for long internal waves in fluids of finite total depth; we refer to Klein–Saut [30, Chapter 3] for a comprehensive overview of existing results. Regarding well-posedness, the recent work of Chapouto–Li–Oh–Pilod [9] establishes global well-posedness in $H^s(\mathbb{T})$ for $s \geq 0$ by exploiting the regularizing effect of the perturbation A . Subsequently, Gassot–Laurens [20] extended this global well-posedness theory to $s > -1/2$, which was identified as the scaling-critical regularity in Chapouto–Forlano–Li–Oh–Pilod [10], and for which uniform-in-time a priori bounds were recently established by Harrop–Griffiths–Killip–Viřan [26].

The Smith equation. The equation reads

$$\partial_t u = \partial_x (1 + |D|^2)^{\frac{1}{2}} u - 2u\partial_x u. \quad (\text{Smith})$$

We can write (Smith) in the form (1) with

$$a(k) = ik\sqrt{1 + k^2} - i \text{sign}(k)k^2.$$

This equation is a model for continental-shelf waves [37], which is not known to be integrable, see also [30] and references therein.

The last example we present is a recent model which can be seen as a Korteweg de Vries (KdV) perturbation of the (BO) equation and can be derived following a similar approach to that in [32]. It is also not known to be integrable.

A KdV–BO equation. The equation is given by

$$\partial_t u = -\frac{\partial_x + \partial_x^3}{\sqrt{1 + |D|^2}} u - 2u\partial_x u, \quad (\text{KdVBO})$$

which can be written in the form (1) with

$$a(k) = -i\frac{k - k^3}{\sqrt{1 + k^2}} - i\operatorname{sign}(k)k^2.$$

We notice that all the above examples of perturbations are in the case where A is skew-symmetric.

In the case where $a(k) = 0$ for all $k \in \mathbb{Z}$, we recover the Benjamin–Ono equation. Both the design and the convergence analysis for our scheme are based on new tools for solving this equation.

The Benjamin–Ono equation. The (BO) equation reads

$$\partial_t v = \partial_x |D|v - 2v\partial_x v. \quad (\text{BO})$$

This equation models the propagation of unidirectional long internal waves at the interface of a two-layer fluid where the lower layer is of infinite depth, as rigorously justified by Paulsen [34].

Using techniques from the theory of integrable systems, numerous advances have been obtained for (BO). First, through the discovery of Birkhoff coordinates [23], a nonlinear Fourier transform, Gérard–Kappeler–Topalov [25] demonstrated the global well-posedness of (BO) in $H^s(\mathbb{T})$ for $s > -\frac{1}{2}$ and ill-posedness otherwise. This nonlinear transformation will be essential for the convergence analysis of our scheme, see Section 2. Furthermore, circumventing the need for any nonlinear transform, Gérard [22] established the existence of an explicit formula for solving (BO). This remarkable formula allows one to express the solution at time t in terms of the initial data v_0 , through the composition of linear operators. Here, we present it as a characterization of the Fourier coefficients of the solution (see also Remark 1.6):

$$\widehat{v}(t, k) = \langle (e^{it} e^{2itL_{v_0}} S^*)^k \Pi v_0, 1 \rangle, \quad k \geq 0, \quad (4)$$

with the Lax and shift operators

$$L_{v_0} f = Df - \Pi(v_0 f), \quad S^* f = \Pi(e^{-ix} f), \quad f \in L_+^2,$$

where $\widehat{\Pi f}(k) = \mathbb{1}_{k \geq 0} \widehat{f}(k)$ and $L_+^2 = \Pi L^2 = \{f \in L^2 : \widehat{f}(k) = 0, k < 0\}$. Moreover, since v is real-valued, we simply have $\widehat{v}(t, k) = \overline{\widehat{v}(t, -k)}$ for $k < 0$.

Recently, the works [3] and [4] demonstrated the significant advantage of approximating this formula (4) rather than applying existing numerical methods to (BO). Namely, these works show that approximating (4) yields schemes that are exact in time (i.e. they do not require a time discretization) as the formula provides an explicit representation of the solution at any time $t \in [0, T]$. Thus, not only do these schemes solely require a spatial discretization to be computable, but their computational cost is independent of the final time T (see equation (21) and Remark 4.2 for more detail). This observation is the starting point for the *construction* of our scheme: as (BO) can be computed exactly in time, yielding stable and spectrally accurate schemes, we propose to iteratively solve the (BO) flow and compose it with the flow generated by the perturbation over small time intervals, leading to a highly efficient and stable computational approach for solving (1). This is further detailed in the next section.

1.1 Splitting schemes

We consider approximating (1) using a novel first-order splitting scheme, which partitions the terms in a manner distinct from traditional Lie splitting methods. Specifically, we decompose the right-hand side of (1) into the Benjamin-Ono part $B(u) = \partial_x |D|u - 2u\partial_x u$ and the linear perturbation Au :

$$\partial_t u = B(u) + Au, \quad u(0) = u_0; \quad \partial_t v = B(v), \quad v(0) = v_0; \quad \partial_t z = Az, \quad z(0) = z_0,$$

where

$$u(t) = \Psi_{\text{full}}^t(u_0), \quad v(t) = \varphi_{\text{BO}}^t(v_0) \quad \text{and} \quad z(t) = e^{tA}z_0$$

denote the solutions at time t to the respective initial value problems defined in H_0^s , $s \geq 0$.

A crucial advantage of this decomposition (as opposed to classical Lie splittings) is that both sub-flows φ_{BO}^t and e^{tA} can be computed *exactly in time* and are well defined in H_0^s for all $t \geq 0$. Indeed, Gérard's explicit formula (4) provides a closed-form expression for $\varphi_{\text{BO}}^t w_0$, and [3, 4] allow for its efficient implementation in Fourier space. For the linear part, since A is a diagonal operator in Fourier space, the evolution $e^{tA}z_0$ is evaluated exactly in this basis. Consequently, the only time-discretization error in our method stems from the splitting procedure itself. Our scheme is presented next.

The perturbative splitting scheme. Let $\tau \in (0, 1]$ be the time step. Our splitting scheme approximates the solution at time $t_{n+1} = t_n + \tau$ by composing these exact flows over each time step:

$$u^{n+1} = \varphi_{\text{BO}}^\tau e^{\tau A} u^n, \quad u^0 = u_0. \quad (5)$$

For a discussion on the implementation of the fully-discrete scheme we refer to Section 4 (see also Remark 3.9).

Remark 1.1. We emphasize that for initial data $u_0 \in H_0^s$ satisfying the zero-mean condition, i.e., $\int_{\mathbb{T}} u_0 = 0$, the splitting schemes (as well as its fully discrete counterpart) exactly preserve this property, ensuring that $\int_{\mathbb{T}} u^n = 0$ for all $n \geq 0$. This is necessary to obtain convergence,

which we establish in H_0^s , by exploiting the analyticity properties of the Birkhoff map in these spaces [20, 24], see Section 2 for more detail.

Next, we motivate our choice of splitting by comparing it with classical Lie splitting schemes.

Comparison with the classical Lie splitting method. Two effects are present in (1): the Burgers nonlinearity $\partial_x(u^2)$, which set alone forms shocks in finite time, and the linear terms $\mathcal{L} := \partial_x|D| + A$, whose evolution is bounded in all Sobolev norms on any compact time interval, and even preserves the L^2 norm when A is skew-adjoint, as is the case in all the above examples. Consequently, in the error analysis of time-discretization schemes for (1), the central challenge lies in establishing stability bounds to control the derivative loss in the nonlinearity.

The Lie splitting scheme relies on decomposing the governing equation $\partial_t u - \mathcal{L}u = -\partial_x(u^2)$ into the linear equation $\partial_t u = \mathcal{L}u$ and the inviscid Burgers equation $\partial_t u = -\partial_x(u^2)$. The scheme advances the solution over a time step τ by composing their respective flows, yielding

$$u^{n+1} = (\Phi_{\mathcal{N}}^\tau \circ \Phi_{\mathcal{L}}^\tau)(u^n), \quad (6)$$

where $\Phi_{\mathcal{L}}^\tau = e^{\tau\mathcal{L}}$ is the (exact) linear propagator and $\Phi_{\mathcal{N}}^\tau$ is the nonlinear Burgers flow.

The error analysis of such splitting methods is typically conducted assuming that the Burgers flow $\Phi_{\mathcal{N}}^\tau$ is computed exactly in time, and the key difficulty in the analysis lies in the fact that Burger's flow eventually produces discontinuities independently of the smoothness of the initial data, as the underlying equation is not well posed in any Sobolev space with positive regularity. For smooth solutions we further refer to [27, 28] in the context of the KdV equation and to [13] in the context of the (BO) equation, where energy estimates are employed. For the approximation to low-regularity solutions to KdV we further refer to [36], which introduces discrete Bourgain spaces and a quadratic time step condition to guarantee stability of the method.

We emphasize that, a priori, our analysis is even more challenging than that for the KdV equation. While the KdV equation features the third-order Airy dispersive operator ∂_{xxx} , our linear dispersive operator $\partial_x|D| + A$ is only of second order. Consequently, the high-frequency dispersion present in the equation is significantly reduced, rendering the control of the derivative in the nonlinearity a much harder problem. Unlike the KdV equation, where the dispersion is strong enough to permit semilinear arguments, when working with solutions that lack high smoothness (for which energy estimates cannot be applied), this weaker dispersion cannot overcome the derivative loss induced by the nonlinearity using standard Duhamel or Picard arguments. As a result, the equations studied here are of quasilinear nature, rendering the analysis more challenging and requiring different techniques.

To enable the analysis of our splitting scheme (5), we introduce a novel approach that uplifts the Birkhoff coordinates, see Sections 2 and 3. Remarkably, this, together with our choice of splitting, allows us to obtain a significantly better convergence result than that of classical splitting methods, even when compared to the KdV, BO, and NLS equations.