3/9/22 Derivation of $\nabla g^*(v)$

$m \, [B]^{n}$ $m \geq n$

(Assuming B has full col. rank)

Let $g(s) = \frac{1}{2}\|Bs - b\|^2$

$$g^*(v) := \underbrace{\sup_s \langle v, s\rangle - \tfrac{1}{2}\|Bs-b\|^2}_{\text{to solve, set gradient } = 0}$$

s^* is sol'n

$$0 = v - B^T(Bs^* - b) \quad \text{i.e.} \quad v + B^Tb = B^TBs^*$$

thus

$$g^*(v) = \langle v, s^*\rangle - \tfrac{1}{2}\|Bs^* - b\|^2$$

$$= \langle v, s^*\rangle - \tfrac{1}{2} s^{*T}\underbrace{B^T(Bs^*-b)}_{v} + \tfrac{1}{2} b^T(Bs^*-b)$$

$$= \langle v, s^*\rangle - \tfrac{1}{2}\langle s^*, v\rangle + \tfrac{1}{2}\langle B^Tb, s^*\rangle - \tfrac{1}{2}\|b\|^2$$

$$= \tfrac{1}{2}\langle v + B^Tb, \underbrace{s^*}_{s^* = (B^TB)^{-1}(v+B^Tb)}\rangle - \tfrac{1}{2}\|b\|^2$$

$$= \tfrac{1}{2}\langle v + B^Tb, (B^TB)^{-1}(v+B^Tb)\rangle - \tfrac{1}{2}\|b\|^2$$

$$= \tfrac{1}{2}\|(B^TB)^{-1/2}(v+B^Tb)\|^2 - \tfrac{1}{2}\|b\|^2.$$

Hence (since if $f(x) = \frac{1}{2}\|Ax - d\|^2$ then $\nabla f = A^T(Ax-d)$)

we have

$$\nabla g^*(v) = \Big(\big(\underbrace{(B^TB)}_{\text{symmetric}}{}^{-1/2}\big)^T \cdot \big((B^TB)^{-1/2}(v+B^Tb)\big)\Big)$$

$$= \underbrace{(B^TB)^{-1}}_{\substack{\text{inverse exists if } B \text{ has full} \\ \text{col. rank, eg. } B = [\;] \text{ so } B^TB \text{ is}}}(v + B^Tb)$$

$[\;] = [\;][\;]$

B like $\overset{m<n}{[\;]}$ won't work since B^TB not invertible.