In this document we compute the double convolution between a regular coefficient function $C(x)$ and a splitting function $P(x)$ containing a regular, a singular and a local part, i.e.

$$P(x) = P^{R}(x) + P^{S}(x) + P^{L}\delta(x-1) \tag{1}$$

where P^{R} is regular in $x \to 1$, while P^{S} is singular in $x \to 1$.

The convolution between $C(x)$ and $P(x)$ is

$$\begin{aligned}(C \otimes P)(x) &= \int_x^{x_{\rm m}} \frac{dz_2}{z_2} C(z_2) P^{R}\left(\frac{x}{z_2}\right) + \int_{\frac{x}{x_{\rm m}}}^1 dz_2 P^{S}(z_2) \left(\frac{C\left(\frac{x}{z_2}\right)}{z_2} - C(x) \right) - C(x) \int_0^{\frac{x}{x_{\rm m}}} dz_2 P^{S}(z_2) + C(x) P^{L} \\ &= \int_x^{x_{\rm m}} \frac{dz_2}{z_2} C(z_2) P^{R}\left(\frac{x}{z_2}\right) + \int_{\frac{x}{x_{\rm m}}}^1 dz_2 P^{S}(z_2) \left(\frac{C\left(\frac{x}{z_2}\right)}{z_2} - C(x) \right) + C(x) \left(P^{L} - \mathcal{P}^{S}\left(\frac{x}{x_{\rm m}}\right) \right)\end{aligned} \tag{2}$$

where we defined

$$\mathcal{P}^{S}(x) = \int_0^x dz P^{S}(z)$$

and we used that $C(x)$ is defined for $0 < x < x_{\rm m}$ and zero outside.

The double convolution is defined as

$$\begin{aligned}(C \otimes P \otimes P)(x) = &\int_x^{x_{\rm max}} \frac{dz_1}{z_1} (C \otimes P)(z_1) P^{R}\left(\frac{x}{z_1}\right) \\ &+ \int_{\frac{x}{x_{\rm m}}}^1 dz_1 P^{S}(z_1) \left(\frac{(C \otimes P)\left(\frac{x}{z_1}\right)}{z_1} - (C \otimes P)(x) \right) \\ &+ (C \otimes P)(x) \left(P^{L} - \mathcal{P}^{S}\left(\frac{x}{x_{\rm m}}\right) \right)\end{aligned} \tag{3}$$

Let's consider each piece separately.

1 Regular term

$$\begin{aligned}&\int_x^{x_{\rm max}} \frac{dz_1}{z_1} (C \otimes P)(z_1) P^{R}\left(\frac{x}{z_1}\right) = \\ &\int_x^{x_{\rm m}} \frac{dz_1}{z_1} P^{R}\left(\frac{x}{z_1}\right) \left[\int_{z_1}^{x_{\rm m}} \frac{dz_2}{z_2} C(z_2) P^{R}\left(\frac{z_1}{z_2}\right) + \int_{\frac{z_1}{x_{\rm m}}}^1 dz_2 P^{S}(z_2) \left(\frac{C\left(\frac{z_1}{z_2}\right)}{z_2} - C(z_1) \right) + C(z_1) \left(P^{L} - \mathcal{P}^{S}\left(\frac{z_1}{x_{\rm m}}\right) \right) \right] \\ &= \int_x^{x_{\rm m}} dz_1 \int_{z_1}^{x_{\rm m}} dz_2 \frac{1}{z_1 z_2} P^{R}\left(\frac{x}{z_1}\right) C(z_2) P^{R}\left(\frac{z_1}{z_2}\right) + \int_x^{x_{\rm m}} dz_1 \int_{\frac{z_1}{x_{\rm m}}}^1 dz_2 \frac{P^{R}\left(\frac{x}{z_1}\right) P^{S}(z_2)}{z_1} \left(\frac{C\left(\frac{z_1}{z_2}\right)}{z_2} - C(z_1) \right) \\ &- \int_x^{x_{\rm m}} \frac{dz_1}{z_1} P^{R}\left(\frac{x}{z_1}\right) C(z_1) \mathcal{P}^{S}\left(\frac{z_1}{x_{\rm m}}\right) + P^{L} \int_x^{x_{\rm m}} \frac{dz_1}{z_1} P^{R}\left(\frac{x}{z_1}\right) C(z_1)\end{aligned}$$

In the second integral of the last line we can perform the change of variable $z_2 \to \frac{z_1}{z_2}$, so that we get

$$\begin{aligned}&= \int_x^{x_{\rm m}} dz_1 \int_{z_1}^{x_{\rm m}} dz_2 \frac{1}{z_1 z_2} P^{R}\left(\frac{x}{z_1}\right) C(z_2) P^{R}\left(\frac{z_1}{z_2}\right) + \int_x^{x_{\rm m}} dz_1 \int_{z_1}^{x_{\rm m}} \frac{dz_2}{z_2} P^{R}\left(\frac{x}{z_1}\right) P^{S}\left(\frac{z_1}{z_2}\right) \left(\frac{C(z_2)}{z_1} - \frac{C(z_1)}{z_2} \right) \\ &- \int_x^{x_{\rm m}} dz_1 \frac{P^{R}\left(\frac{x}{z_1}\right) C(z_1) \mathcal{P}^{S}\left(\frac{z_1}{x_{\rm m}}\right)}{z_1} + P^{L} \int_x^{x_{\rm m}} \frac{dz_1}{z_1} P^{R}\left(\frac{x}{z_1}\right) C(z_1)\end{aligned}$$

Now the double integrals can be trasformed in integrals over rectangles, using the theta functions:

$$\begin{aligned}&= \int \int_x^{x_{\rm m}} dz_1 dz_2 \theta(z_2 - z_1) \frac{1}{z_1 z_2} P^{R}\left(\frac{x}{z_1}\right) C(z_2) P^{R}\left(\frac{z_1}{z_2}\right) \\ &+ \int \int_x^{x_{\rm m}} dz_1 dz_2 \theta(z_2 - z_1) \frac{1}{z_2} P^{R}\left(\frac{x}{z_1}\right) P^{S}\left(\frac{z_1}{z_2}\right) \left(\frac{C(z_2)}{z_1} - \frac{C(z_1)}{z_2} \right) \\ &- \int_x^{x_{\rm m}} dz_1 \frac{P^{R}\left(\frac{x}{z_1}\right) C(z_1) \mathcal{P}^{S}\left(\frac{z_1}{x_{\rm m}}\right)}{z_1} + P^{L} \int_x^{x_{\rm m}} \frac{dz_1}{z_1} P^{R}\left(\frac{x}{z_1}\right) C(z_1)\end{aligned} \tag{4}$$

The last equation is what has been implemented in the code with Monte Carlo integrals.

2 Singular term

$$\int_{\frac{x}{x_{\rm m}}}^{1} dz_1 P^{S}(z_1)\left\{\frac{(C\otimes P)\left(\frac{x}{z_1}\right)}{z_1}-(C\otimes P)(x)\right\}=$$

$$\int_{\frac{x}{x_{\rm m}}}^{1} dz_1 P^{S}(z_1)\Bigg\{\frac{1}{z_1}\left[\int_{\frac{x}{z_1}}^{x_{\rm m}}\frac{dz_2}{z_2}C(z_2)P^{R}\left(\frac{x}{z_1z_2}\right)+\int_{\frac{x}{z_1x_{\rm m}}}^{1}dz_2P^{S}(z_2)\left(\frac{C\left(\frac{x}{z_1z_2}\right)}{z_2}-C\left(\frac{x}{z_1}\right)\right)+C\left(\frac{x}{z_1}\right)\left(P^{L}-\mathcal{P}^{S}\left(\frac{x}{z_1x_{\rm m}}\right)\right)\right]$$

$$-\left[\int_{x}^{x_{\rm m}}\frac{dz_2}{z_2}C(z_2)P^{R}\left(\frac{x}{z_2}\right)+\int_{\frac{x}{x_{\rm m}}}^{1}dz_2P^{S}(z_2)\left(\frac{C\left(\frac{x}{z_2}\right)}{z_2}-C(x)\right)+C(x)\left(P^{L}-\mathcal{P}^{S}\left(\frac{x}{x_{\rm m}}\right)\right)\right]\Bigg\}$$

$$=\int_{\frac{x}{x_{\rm m}}}^{1}dz_1\int_{\frac{x}{z_1}}^{x_{\rm m}}dz_2\frac{1}{z_1z_2}P^{S}(z_1)C(z_2)P^{R}\left(\frac{x}{z_1z_2}\right)+\int_{\frac{x}{x_{\rm m}}}^{1}dz_1\int_{\frac{x}{z_1x_{\rm m}}}^{1}dz_2\frac{P^{S}(z_1)P^{S}(z_2)}{z_1}\left(\frac{C\left(\frac{x}{z_1z_2}\right)}{z_2}-C\left(\frac{x}{z_1}\right)\right)$$

$$-\int_{\frac{x}{x_{\rm m}}}^{1}dz_1\frac{P^{S}(z_1)\mathcal{P}^{S}\left(\frac{x}{z_1x_{\rm m}}\right)}{z_1}C\left(\frac{x}{z_1}\right)+P^{L}\int_{\frac{x}{x_{\rm m}}}^{1}\frac{dz_1}{z_1}P^{S}(z_1)C\left(\frac{x}{z_1}\right)$$

$$-\int_{\frac{x}{x_{\rm m}}}^{1}dz_1\int_{x}^{x_{\rm m}}\frac{dz_2}{z_2}P^{S}(z_1)C(z_2)P^{R}\left(\frac{x}{z_2}\right)-\int_{\frac{x}{x_{\rm m}}}^{1}dz_1\int_{\frac{x}{x_{\rm m}}}^{1}dz_2P^{S}(z_1)P^{S}(z_2)\left(\frac{C\left(\frac{x}{z_2}\right)}{z_2}-C(x)\right)$$

$$+\int_{\frac{x}{x_{\rm m}}}^{1}dz_1P^{S}(z_1)C(x)\mathcal{P}^{S}\left(\frac{x}{x_{\rm m}}\right)-P^{L}\int_{\frac{x}{x_{\rm m}}}^{1}dz_1P^{S}(z_1)C(x)$$

Collecting the terms with same integration boundaries, and using theta functions, we get

$$=\int_{\frac{x}{x_{\rm m}}}^{1}dz_1\int_{x}^{x_{\rm m}}dz_2P^{S}(z_1)\left(\frac{P^{R}\left(\frac{x}{z_1z_2}\right)\theta\left(z_2-\frac{x}{z_1}\right)}{z_1}-P^{R}\left(\frac{x}{z_2}\right)\right)\frac{C(z_2)}{z_2}$$

$$+\int_{\frac{x}{x_{\rm m}}}^{1}dz_1\int_{\frac{x}{x_{\rm m}}}^{1}dz_2P^{S}(z_1)\left[\frac{P^{S}(z_2)}{z_1}\left(\frac{C\left(\frac{x}{z_1z_2}\right)}{z_2}-C\left(\frac{x}{z_1}\right)\right)\theta\left(z_2-\frac{x}{z_1x_{\rm m}}\right)-P^{S}(z_2)\left(\frac{C\left(\frac{x}{z_2}\right)}{z_2}-C(x)\right)\right]$$

$$-\int_{\frac{x}{x_{\rm m}}}^{1}dz_1P^{S}(z_1)\left(\frac{C\left(\frac{x}{z_1}\right)\mathcal{P}^{S}\left(\frac{x}{z_1x_{\rm m}}\right)}{z_1}-C(x)\mathcal{P}^{S}\left(\frac{x}{x_{\rm m}}\right)\right)$$

$$+P^{L}\int_{\frac{x}{x_{\rm m}}}^{1}dz_1P^{S}(z_1)\left(\frac{C\left(\frac{x}{z_1}\right)}{z_1}-C(x)\right)$$

Observe that all these terms are finite since the singularity of P^{S} for $z\to 1$ is integrable since it always appears together with a term of the form $f(x/z)/z-f(x)$ so that the integral

$$\int_{x/x_{\rm m}}^{1}dzP^{S}(z)\left(\frac{f\left(\frac{x}{z}\right)}{z}-f(x)\right)$$

is finite.