



Technical Guide

Density-functional theory and Fourier neural operators
for three-dimensional scalar fields

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Preface

Poraquê is a research code for learning maps between the three-dimensional scalar fields of density-functional theory. It has one organising idea: for a given material, the local external potential $V_{\text{ext}}(\mathbf{r})$, the valence charge density $\rho(\mathbf{r})$ and the kinetic energy density $\tau(\mathbf{r})$ all live on *one* grid, in *one* file format, and are therefore directly comparable, composable, and usable as aligned inputs and targets for a neural operator.

Two maps are learned:

$$V_{\text{ext}} \mapsto \rho \quad \text{and} \quad \rho \mapsto \tau.$$

They are not two unrelated regressions that happen to share a file format. They are the two legs of one variational statement — the first is the Hohenberg–Kohn map, guaranteed to exist by theorem; the second is the kinetic energy density functional, the missing ingredient of orbital-free DFT. Composed, they constitute a complete orbital-free calculation. That relationship is the thread running through this document.

What this guide covers

Chapters 1–2 The physics. Kohn–Sham DFT, the Hohenberg–Kohn theorems, and the orbital-free formulation, with attention to which quantities are exact, which are approximations, and which are the open problem.

Chapter 3 The mathematics of Fourier neural operators: why an operator between function spaces is the right object, how the spectral convolution is constructed, and why it is the natural architecture for a periodic crystal on a plane-wave grid.

Chapters 4–5 Implementation. The shared-grid data model, the code-agnostic ingestion layer, and a detailed account of the interface to VASP — including the exact reconstruction of the local pseudopotential.

Chapter 6 Measured results on a gold data set, with the analytic baselines a learned functional must beat.

Chapter 7 The physics-informed programme: how the governing equations re-enter as constraints, and which of them are already enforced by construction.

Conventions

Two unit systems appear, deliberately.

- **Å and eV** in everything that touches a file. Every plane-wave code writes geometry in Ångström and energies in electronvolt, so the field layer, the readers and the machine-learning code use those units throughout. A charge density is in $e/\text{Å}^3$, a kinetic energy

density in $\text{eV}/\text{\AA}^3$, a potential in eV.

- **Hartree atomic units** in the analytical expressions of Chapters 1–2 and in the legacy solver, where $\hbar = m_e = e = 1$ removes constants that would otherwise clutter every equation.

Conversions are collected in `poraque/fields/constants.py`. The two constants that recur are

$$\frac{e^2}{4\pi\epsilon_0} = 14.399645 \text{ eV} \cdot \text{\AA}, \quad \frac{\hbar^2}{2m_e} = 3.809982 \text{ eV} \cdot \text{\AA}^2.$$

Sign convention for potentials: $V_{\text{ext}}(\mathbf{r})$ is the potential *energy of an electron*, so it is negative near the ions. This matches VASP's `LOCPOT` and `EXTCAR`.

Note

Passages marked **Implementation** connect an equation to the function that evaluates it. Passages marked **Caveat** record an approximation, an untested path, or a limit on what the numbers presented here support. They are not disclaimers added for politeness: each one marks a place where a reader could otherwise draw a conclusion the evidence does not carry.

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CHAPTER 1

Kohn–Sham density-functional theory

This chapter fixes the physical objects the rest of the guide manipulates. The emphasis is on *which statements are theorems and which are approximations*, because that distinction determines what a neural network may legitimately be asked to learn, and what may be enforced on it as a hard constraint.

1.1 The many-body problem

The non-relativistic electronic Hamiltonian for N electrons in the field of fixed nuclei is, in Hartree atomic units,

$$\hat{H} = -\frac{1}{2} \sum_i^N \nabla_i^2 + \sum_i^N V_{\text{ext}}(\mathbf{r}_i) + \sum_{i < j} \frac{1}{|\mathbf{r}_i - \mathbf{r}_j|}, \quad (1.1)$$

where V_{ext} collects the electron–nucleus attraction. The wavefunction $\Psi(\mathbf{r}_1, \dots, \mathbf{r}_N)$ depends on $3N$ coordinates, so a direct solution is hopeless beyond a handful of electrons.

1.2 The Hohenberg–Kohn theorems

Density-functional theory replaces Ψ by the electron density

$$\rho(\mathbf{r}) = N \int |\Psi(\mathbf{r}, \mathbf{r}_2, \dots, \mathbf{r}_N)|^2 d\mathbf{r}_2 \cdots d\mathbf{r}_N, \quad (1.2)$$

a function of three coordinates. Two theorems justify this.

First theorem. The ground-state density determines the external potential up to an additive constant. Consequently, for a fixed electron number, the ground-state density is a functional of V_{ext} alone:

$$V_{\text{ext}}(\mathbf{r}) \longleftrightarrow \rho_0(\mathbf{r}). \quad (1.3)$$

Second theorem. There exists a universal functional $F[\rho]$ such that the total energy

$$E[\rho] = F[\rho] + \int V_{\text{ext}}(\mathbf{r})\rho(\mathbf{r}) d\mathbf{r} \quad (1.4)$$

is minimised, subject to $\int \rho d\mathbf{r} = N$, by the ground-state density, where it equals the ground-state energy.

Equation (1.3) is the formal justification for the first learned model of this work. The target of a $V_{\text{ext}} \mapsto \rho$ regression is not a fitted correlation: it is a single-valued function whose existence and uniqueness are guaranteed. That is a stronger footing than most machine-learning targets enjoy.

Caveat

The map is conditional on the electron number N . In the present data set $N = \sum_a Z_a^{\text{val}}$ follows from the geometry, so a network can infer it; for charged systems or a chemically heterogeneous data set, N must become an explicit input.

1.3 The Kohn–Sham construction

The difficulty is that $F[\rho]$ is unknown, and its dominant part — the kinetic energy — is badly approximated by explicit density functionals. Kohn and Sham sidestep this by introducing an auxiliary system of non-interacting electrons with the *same* density, for which the kinetic energy can be computed exactly from orbitals. Writing

$$F[\rho] = T_s[\rho] + E_H[\rho] + E_{xc}[\rho], \quad (1.5)$$

with the Hartree energy

$$E_H[\rho] = \frac{1}{2} \iint \frac{\rho(\mathbf{r})\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{r} d\mathbf{r}', \quad (1.6)$$

the variational condition becomes a set of single-particle equations,

$$\left(-\frac{1}{2}\nabla^2 + v_s[\rho](\mathbf{r})\right) \psi_i(\mathbf{r}) = \varepsilon_i \psi_i(\mathbf{r}), \quad v_s = V_{\text{ext}} + v_H[\rho] + v_{xc}[\rho], \quad (1.7)$$

with

$$\rho(\mathbf{r}) = \sum_i^{\text{occ}} f_i |\psi_i(\mathbf{r})|^2, \quad T_s = -\frac{1}{2} \sum_i^{\text{occ}} f_i \langle \psi_i | \nabla^2 | \psi_i \rangle. \quad (1.8)$$

Because v_s depends on ρ , which depends on the orbitals, Eq. (1.7) must be solved self-consistently.

1.3.1 Why the map is hard

Two features of Eq. (1.7) shape the choice of neural architecture in Chapter 3.

1. **It is non-local.** The Hartree term is a $1/|\mathbf{r} - \mathbf{r}'|$ convolution: the density at one point depends on the potential everywhere. A network with a finite receptive field must be made very deep to propagate that information across a unit cell.
2. **It is self-consistent.** The density is a fixed point, not a single forward evaluation. A supervised model learns the fixed point directly, which is what makes it fast — and what makes it fragile off the training distribution.

1.4 The kinetic energy density

The quantity stored in TAU CAR is the *positive-definite* kinetic energy density of the Kohn–Sham system,

$$\tau(\mathbf{r}) = \frac{1}{2} \sum_i^{\text{occ}} f_i |\nabla \psi_i(\mathbf{r})|^2. \quad (1.9)$$

It integrates to T_s and is manifestly non-negative. An alternative definition using the Laplacian, $\tilde{\tau} = -\frac{1}{2} \sum_i f_i \psi_i^* \nabla^2 \psi_i$, integrates to the same T_s but differs pointwise by $\frac{1}{4} \nabla^2 \rho$ and is *not* sign-definite. The distinction matters here: the constraints of Section 1.5 apply to Eq. (1.9), and applying them to the Laplacian form would be simply wrong.

Implementation

Chapter 5 shows, from the VASP source, that **TAUCAR** indeed stores Eq. (1.9) — built spectrally as $i(\mathbf{G} + \mathbf{k})$ acting on the plane-wave coefficients — and not the Laplacian form. This was verified rather than assumed.

1.5 Exact properties of τ

Several relations constrain τ exactly. They are theorems, and Chapter 7 uses them as hard constraints rather than as penalties.

Non-negativity. $\tau(\mathbf{r}) \geq 0$, immediately from Eq. (1.9).

The von Weizsäcker bound. For any system,

$$\tau(\mathbf{r}) \geq \tau_{\text{vW}}(\mathbf{r}) \equiv \frac{|\nabla \rho(\mathbf{r})|^2}{8\rho(\mathbf{r})}, \quad (1.10)$$

with equality for a single nodeless orbital. This is the Hoffmann-Ostenhof inequality, and it is the single most useful constraint available for the $\rho \mapsto \tau$ map: τ_{vW} is a closed-form functional of the input, so the bound can be imposed *by construction*.

Uniform-gas limit. As $\nabla \rho \rightarrow 0$,

$$\tau \longrightarrow \tau_{\text{TF}} = C_{\text{TF}} \rho^{5/3}, \quad C_{\text{TF}} = \frac{3}{10} (3\pi^2)^{2/3} \approx 2.8712. \quad (1.11)$$

Coordinate scaling. For $\rho_\lambda(\mathbf{r}) = \lambda^3 \rho(\lambda \mathbf{r})$,

$$T_s[\rho_\lambda] = \lambda^2 T_s[\rho]. \quad (1.12)$$

This is exact and label-free: any training sample can be rescaled and its target T_s is then known in closed form, which is a data augmentation at zero DFT cost.

The decomposition suggested by Eq. (1.10),

$$\tau = \tau_{\text{vW}}[\rho] + \tau_{\text{P}}, \quad \tau_{\text{P}} \geq 0, \quad (1.13)$$

defines the *Pauli term* τ_{P} . It is the part of the kinetic energy that arises purely from Fermi statistics — what distinguishes a many-fermion system from a single-orbital one — and it is the object every orbital-free kinetic functional is really trying to approximate. Section 7.1.3 shows that learning τ_{P} instead of τ is both more accurate and cheaper.

1.6 Pseudopotentials

Core electrons are chemically inert but carry sharp oscillations that would demand an impractically fine grid. A pseudopotential replaces the nucleus and core by an effective ion of charge Z^{val} acting on smooth pseudo-valence orbitals. Its local part behaves as $-Z^{\text{val}}e^2/r$ outside a core radius R_{core} and is softened inside.

Three consequences run through this document:

1. The fields in CHGCAR and TAUCAR are *pseudo* quantities. They are mutually consistent, which is why the bound (1.10) is observed to hold on this data even though it is strictly guaranteed only for all-electron quantities.
2. The external potential in EXTCAR is the *local* part only. Non-local projectors are not functions of $\mathbf{r}$ at all and cannot appear in a volumetric file even in principle.
3. The short-range shape of the local pseudopotential is tabulated, not analytic. Chapter 5 shows that modelling it with a smooth analytic form factor is the dominant error in a naive reconstruction, and how to avoid it.

Orbital-free density-functional theory

Kohn–Sham theory buys an exact kinetic energy at the price of reintroducing N orbitals, and with them the $\mathcal{O}(N^3)$ cost of orthogonalising and diagonalising. Orbital-free DFT removes the orbitals again. It is the reason the second learned map matters.

2.1 The variational problem

Return to Eq. (1.4) and write the energy purely as a functional of the density,

$$E[\rho] = T_s[\rho] + \int V_{\text{ext}}(\mathbf{r}) \rho(\mathbf{r}) \, d\mathbf{r} + E_H[\rho] + E_{\text{xc}}[\rho]. \quad (2.1)$$

Minimising under the constraint $\int \rho \, d\mathbf{r} = N$ with a Lagrange multiplier μ gives the Euler–Lagrange equation

$$\boxed{\frac{\delta T_s}{\delta \rho}(\mathbf{r}) + V_{\text{ext}}(\mathbf{r}) + v_H[\rho](\mathbf{r}) + v_{\text{xc}}[\rho](\mathbf{r}) = \mu} \quad (2.2)$$

with μ the chemical potential — a *constant* in space. A single scalar equation replaces the N coupled equations (1.7), and the cost becomes essentially linear in system size.

Equation (2.2) is the central equation of this work. Every symbol in it is something Poraquê has:

Term	Source
V_{ext}	the input of Model 1 (EXTCAR)
ρ	the output of Model 1 (CHGCAR)
$v_H[\rho]$	computed <i>exactly</i> by FFT, $4\pi e^2 \rho(\mathbf{G})/G^2$
$\delta T_s/\delta \rho$	autograd through Model 2 ($T_s = \int \tau$)
μ	eliminated by removing the cell average

That is what allows Eq. (2.2) to serve as a *label-free* training signal in Chapter 7.

2.2 The kinetic energy functional problem

Everything in Eq. (2.1) has an accurate explicit density functional except $T_s[\rho]$. The classical approximations are:

Thomas–Fermi. $T_s^{\text{TF}}[\rho] = C_{\text{TF}} \int \rho^{5/3}$, exact for the uniform electron gas. It has no shell structure and does not bind molecules.

von Weizsäcker. $T_s^{\text{vW}}[\rho] = \int |\nabla \rho|^2/8\rho$, exact for one orbital, a rigorous lower bound in general.

Gradient expansions. $T_s^{\text{TF}} + \lambda T_s^{\text{vW}}$ with $\lambda = 1/9$ (second order) or $\lambda = 1$ (strongly inhomogeneous densities). Neither is reliable across chemistry.

Chapter 6 measures these on gold. Thomas–Fermi achieves a *negative* coefficient of determination — it is a worse pointwise predictor of τ than the constant mean — which is the expected failure of a uniform-gas limit applied to a d -band metal. That is the gap a learned functional is meant to close.

2.3 Functional derivatives

The quantity Eq. (2.2) actually requires is not τ but $\delta T_s / \delta \rho$. For the two classical forms,

$$\frac{\delta T_s^{\text{TF}}}{\delta \rho} = \frac{5}{3} C_{\text{TF}} \rho^{2/3}, \quad (2.3)$$

$$\frac{\delta T_s^{\text{vW}}}{\delta \rho} = -\frac{1}{2} \frac{\nabla^2 \sqrt{\rho}}{\sqrt{\rho}}. \quad (2.4)$$

Caveat

A model with small error in τ may still have a poor functional derivative, because differentiation amplifies high-frequency error. Training on τ alone therefore optimises a proxy for the quantity that is used. Chapter 7 describes training *through* the derivative to remove this mismatch.

2.4 Electrostatics on a periodic grid

Both V_{ext} and v_{H} are Coulombic, and both are evaluated in reciprocal space, where the kernel is diagonal. Solving $\nabla^2 v_{\text{H}} = -4\pi e^2 \rho$ by Fourier transform gives

$$v_{\text{H}}(\mathbf{G}) = \frac{4\pi e^2 \rho(\mathbf{G})}{G^2}, \quad v_{\text{H}}(\mathbf{G} = 0) \equiv 0. \quad (2.5)$$

The $\mathbf{G} = 0$ component diverges for any charged distribution. In a periodic crystal this divergence cancels exactly between the electron–electron, electron–ion and ion–ion terms, provided the cell is neutral overall. The standard convention — adopted here and by VASP — is to set the $\mathbf{G} = 0$ component of *each* Coulomb term to zero individually, which amounts to adding a uniform neutralising background. The consequence is that potentials are defined only up to a constant, and their cell average vanishes.

Implementation

This convention is why **EXTCAR** written by VASP has a cell average of -2.5×10^{-12} eV, and why Poraquê’s reconstruction reproduces it to $\sim 10^{-16}$. Agreement of the mean was the first check that the two codes share a convention, before any pointwise comparison was meaningful.

2.5 The external potential of pseudo-ions

For pseudo-ions of charge Z_s^{val} at positions $\boldsymbol{\tau}_a$, the local external potential is assembled in reciprocal space as

$$V_{\text{ext}}(\mathbf{G}) = -\frac{4\pi e^2}{\Omega} \sum_s \frac{Z_s^{\text{val}} f_s(G)}{G^2} S_s(\mathbf{G}), \quad S_s(\mathbf{G}) = \sum_{a \in s} e^{-i\mathbf{G} \cdot \boldsymbol{\tau}_a}, \quad (2.6)$$

with $V_{\text{ext}}(\mathbf{G} = 0) \equiv 0$ and Ω the cell volume. The real-space field follows from one inverse FFT. The construction is $\mathcal{O}(N \log N)$, exactly periodic, and free of any real-space cutoff or Ewald parameter.

The form factor $f_s(G)$ encodes the pseudisation:

$f_s = 1$ bare point ions — the exact long-range limit, but its $1/G^2$ tail is truncated at the Nyquist frequency of any finite grid, producing visible ringing near the nuclei.

$f_s = e^{-G^2 \sigma_s^2/2}$ Gaussian ions, equivalent to $-Z_s^{\text{val}} e^2 \text{erf}(r/\sqrt{2}\sigma_s)/r$ in real space: band-limited, smooth, and controlled by one width per species.

f_s **tabulated** the true pseudopotential form factor, read from the **POTCAR**. Chapter 5 shows this is the only choice that reproduces a reference calculation, because the true $f_s(G)$ *oscillates about zero* at large G and no monotonic analytic form can represent it.

Implementation

Implemented in `poraquê.fields.ExternalPotential`. The structure factors are built from separable one-dimensional phase vectors — since $\mathbf{G} \cdot \boldsymbol{\tau}_a = 2\pi \sum_j m_j s_{aj}$ for FFT frequencies m_j and fractional coordinates s_{aj} — so the cost is $\mathcal{O}(N_{\text{at}}(N_1+N_2+N_3))$ complex exponentials rather than $\mathcal{O}(N_{\text{at}}N_1N_2N_3)$, with no full complex grid allocated per atom.

Verification of Eq. (2.6) is by Poisson's equation: the spectral Laplacian of the constructed field is compared against $4\pi e^2(n_{\text{ion}} - \langle n_{\text{ion}} \rangle)$, with n_{ion} built independently by a real-space Gaussian lattice sum. The residual is 4.5×10^{-12} against a scale of $159 \text{ eV}/\text{\AA}^2$ — machine precision, and an independent code path.

Fourier neural operators

3.1 Learning operators, not functions

A conventional network learns a map between finite-dimensional vectors. Retrain it for a different input size and nothing transfers. That is fatal here: the grid (N_1, N_2, N_3) of a plane-wave calculation is fixed by the cell and the cutoff, so *every material has a different one*.

A *neural operator* instead learns a map between function spaces,

$$\mathcal{G} : \mathcal{A} \longrightarrow \mathcal{U}, \quad a(\mathbf{r}) \longmapsto u(\mathbf{r}), \quad (3.1)$$

and is then discretised only for evaluation. Its parameters describe the operator, not a particular sampling of it, so one set of weights serves every grid.

3.2 Kernel integral operators

The construction generalises the affine layer. Lift the input pointwise to a higher-dimensional channel space, $v_0(\mathbf{r}) = P a(\mathbf{r})$, apply L layers of the form

$$v_{\ell+1}(\mathbf{r}) = \sigma(W v_\ell(\mathbf{r}) + b + (\mathcal{K}(v_\ell))(\mathbf{r})), \quad (3.2)$$

and project back pointwise, $u(\mathbf{r}) = Q v_L(\mathbf{r})$. Here W is a local (channel-mixing) linear map and $\mathcal{K}$ is a *kernel integral operator*

$$(\mathcal{K}v)(\mathbf{r}) = \int_D \kappa(\mathbf{r}, \mathbf{r}') v(\mathbf{r}') d\mathbf{r}'. \quad (3.3)$$

The pointwise terms alone would give a network with no spatial coupling at all; $\mathcal{K}$ is what makes it an operator.

3.3 The Fourier layer

Restricting the kernel to be translation invariant, $\kappa(\mathbf{r}, \mathbf{r}') = \kappa(\mathbf{r} - \mathbf{r}')$, turns the integral into a convolution, which the convolution theorem diagonalises:

$$(\mathcal{K}v)(\mathbf{r}) = \mathcal{F}^{-1}[R(\mathbf{G}) \cdot \mathcal{F}[v](\mathbf{G})](\mathbf{r}), \quad (3.4)$$

where $R(\mathbf{G})$ is a learned complex multiplier. Rather than parameterising κ in real space, the Fourier neural operator parameterises R directly and *truncates* it to the lowest $M_1 \times M_2 \times M_3$ modes.

Truncation does three things at once: it regularises (high frequencies are where noise lives), it bounds the parameter count, and — decisively — it makes the weight tensor independent of the grid. Nothing in R refers to (N_1, N_2, N_3) .

3.3.1 Why this suits a crystal

Three properties align the architecture with the physics of Chapters 1–2.

Periodicity is exact, not learned. The FFT imposes Born–von Kármán boundary conditions, which a crystal unit cell already obeys. A padded convolutional network would have to spend capacity imitating that, and would still produce edge artefacts.

The coupling is global in one layer. Multiplying in reciprocal space is convolving over the whole cell in real space. This directly addresses the non-locality of Section 1.2: the Hartree kernel $4\pi e^2/G^2$ that the operator must partly represent is *diagonal* in exactly the basis the layer works in.

Differential operators are free and exact. On a plane-wave grid $\nabla \rightarrow i\mathbf{G}$ and $\nabla^2 \rightarrow -G^2$ are exact for band-limited fields. Since the layer already computes FFTs, the physics-informed terms of Chapter 7 cost almost nothing — and carry no discretisation error that the loss would otherwise misattribute to the network.

3.4 Discretisation invariance

For a real input the transform is taken with `rfft`, halving the last axis. The retained coefficients form four “corners” in the first two axes — combinations of low positive and low negative frequencies — each with its own weight block. The parameter count is

$$4 \times C_{\text{in}} \times C_{\text{out}} \times M_1 M_2 M_3 \quad \text{complex numbers}, \quad (3.5)$$

independent of the grid.

Two further choices are required for weights to be *meaningful* across resolutions, not merely applicable.

3.4.1 Normalisation

The discrete transform must approximate the continuous Fourier series. With coefficients

$$c_{\mathbf{G}} = \frac{1}{N} \sum_n u_n e^{-i\mathbf{G} \cdot \mathbf{r}_n}, \quad (3.6)$$

i.e. `norm="forward"` in both directions, $c_{\mathbf{G}}$ converges to the continuous series coefficient as $N \rightarrow \infty$ and its magnitude does not drift with N . Under the default convention a 120^3 field would enter a layer with amplitudes $\sim 125\times$ those of a 24^3 field and shared weights would be meaningless.

Implementation

Measured: a band-limited field sampled at 16^3 and 32^3 gives operator outputs agreeing to a relative 1.5×10^{-6} at the shared points.

3.4.2 Mode selection

Truncating at a fixed mode *index* keeps a different physical wavevector band for every cell size, since $|\mathbf{G}_{\max}| = 2\pi M/L$. Truncating instead at a fixed $G_{\max}$ — retaining

$$M_i = \left\lfloor \frac{G_{\max} L_i}{2\pi} \right\rfloor \quad (3.7)$$

modes — makes the operator act on the same band of physics in every material. Poraquê offers both; the physical selection is preferred when the data set spans a wide range of cell sizes.

3.5 Handling grids that differ between materials

Three mechanisms together deliver true grid independence.

1. **Dynamic mode truncation.** Weights are allocated for M_i modes; each forward pass uses $\min(M_i, \text{modes available on this grid})$ and leaves the rest untouched. A 24^3 and a 120^3 sample share the same parameters.
2. **Resolution-invariant normalisation**, as above.
3. **Shape-bucketed batching.** Samples of different spatial shape cannot be stacked. Rather than pad — which wastes compute and injects fictitious vacuum into the FFT — materials are grouped by (N_1, N_2, N_3) and batches drawn within a group.

Normalisation layers must also be shape-agnostic: `GroupNorm` is used throughout, since batch statistics are meaningless when every sample has a different spatial extent.

3.6 Cell conditioning

Two materials can share a grid shape and differ entirely in physical scale, so the lattice must enter explicitly. Poraquê supplies it twice: as three fractional-coordinate channels appended to the input (bounded in $[0, 1)$ for every material), and through FiLM conditioning, in which a small network maps rotation-invariant cell descriptors — the three lattice lengths, the three angle cosines, and $\Omega^{1/3}$ — to per-channel scale and shift parameters. Neither touches the spatial dimensions, so both are oblivious to the grid.

3.7 Architecture summary

Stage	Operation
Input	$(B, 1, N_1, N_2, N_3)$, normalised field
Coordinates	append 3 fractional-coordinate channels
Lifting	$1 \times 1 \times 1$ convolution to width C
Fourier layers ($\times L$)	Eq. (3.4) + pointwise + <code>GroupNorm</code> + <code>FiLM</code> , residual
Projection	two $1 \times 1 \times 1$ convolutions with GELU
Output	$(B, 1, N_1, N_2, N_3)$

The residual connection around each Fourier layer stabilises deeper stacks; the pointwise branch supplies the local, high-frequency content that mode truncation removes from the spectral branch.

Implementation

4.1 The shared-grid data model

The organising invariant of `poraque.fields` is that, for one material, every scalar field lives on *one* `FieldGrid` instance and is serialised in *one* format. Everything else follows.

Class	File	Content
<code>ExternalPotential</code>	EXTCAR	V_{ext} , eV
<code>ChargeDensity</code>	CHGCAR	ρ , $e/\text{\AA}^3$
<code>KineticEnergyDensity</code>	TAUCAR	τ , $\text{eV}/\text{\AA}^3$

All three derive from `ScalarField`, which owns the CHGCAR-format reader and writer. One code path therefore handles every field, and reading a field with a supplied grid *imposes* that grid — a mismatch raises rather than passing silently, since a misaligned pair would train the operator on nonsense.

Implementation

`ChargeDensity` and `KineticEnergyDensity` declare `volume_scaled = True`: VASP stores $\rho\Omega$ and $\tau\Omega$, so the conversion happens transparently on read and write, and `integrate()` returns the electron count directly. Verified against the reference data: 297.0000 electrons for 27 gold atoms at $Z^{\text{val}} = 11$. `ExternalPotential` declares it `False`, matching LOCPOT.

4.2 Grid construction

The grid may be obtained in three ways, in decreasing order of reliability:

1. `FieldGrid.from_file(path)` — adopt the grid of an existing volumetric file. The only way to guarantee point-by-point comparability with a reference calculation, and the recommended default.
2. explicit NGXF/NGYF/NGZF tags from the input file;
3. `FieldGrid.from_encut` — derived from the plane-wave cutoff.

The cutoff rule uses $G_{\text{cut}} = \sqrt{E_{\text{cut}}/(\hbar^2/2m_e)}$ and hence $n_i^{\text{max}} = \lfloor G_{\text{cut}}|\mathbf{a}_i|/2\pi \rfloor$, rounded up to even, FFT-friendly sizes.

Caveat

The cutoff-derived shape is an *estimate*. VASP’s exact rounding is version- and setting-dependent: for the gold data set the rule yields 64^3 where VASP used 128^3 , and on a strained cell VASP chose 120 on one axis where the rule gives 128. Every result in this guide reads the grid from the files, so no conclusion depends on the heuristic — but it should not be relied on when a match is required.

4.3 Spectral resampling

Changing the resolution of a plane-wave field is a *basis-truncation* problem, not an interpolation problem. The field is a finite Fourier series, so restricting it to a coarser grid means keeping the coefficients the coarse grid can represent:

$$f(\mathbf{r}) = \sum_{\mathbf{G}} c_{\mathbf{G}} e^{i\mathbf{G} \cdot \mathbf{r}} \longrightarrow \sum_{|\mathbf{G}| \leq \mathbf{G}_{\max}^{\text{coarse}}} c_{\mathbf{G}} e^{i\mathbf{G} \cdot \mathbf{r}}. \quad (4.1)$$

Nothing is approximated: the result is the exact band-limited projection, it remains exactly periodic, and because the $\mathbf{G} = 0$ coefficient is copied unchanged, the cell average — hence $\int \rho \, d\mathbf{r}$ — is preserved to machine precision.

Trilinear or spline interpolation would do none of these things: it aliases high-frequency content onto low frequencies, breaks periodicity at the cell boundary, and shifts the integral.

Implementation

Measured: a band-limited field downsampled $64^3 \rightarrow 32^3$ agrees with direct sampling to 1.3×10^{-15} ; the mean is preserved to 5.6×10^{-17} ; and the operation is idempotent under down-up-down to 6.7×10^{-16} . Implemented in `poraquê/fields/resample.py`.

Caveat

Band-limiting a strictly positive field with sharp core peaks rings slightly (the Gibbs phenomenon), so a small number of voxels can go marginally negative — one point in 32768 for this data. It is an artefact of the truncation, not of the data, and it is why the data pipeline uses a sign-tolerant asinh normalisation rather than a logarithm.

4.4 Code-agnostic ingestion

Support for a new plane-wave code must not ripple through grids, fields, datasets and models. `poraquê.fields.io` therefore defines a narrow contract, and a reader implements exactly four methods:

Method	Returns
<code>read_structure</code>	<code>Structure</code> (Å, fractional, species-grouped)
<code>read_parameters</code>	<code>CalculationParameters</code> , cutoff in eV
<code>read_pseudopotentials</code>	{element: <code>PseudopotentialInfo</code> }
<code>read_field / write_field</code>	<code>ScalarField</code>

Units are normalised at the boundary: a Quantum ESPRESSO reader converts its Rydberg `ecutwfc` in `read_parameters`, and nothing downstream needs to know. Field kinds are referred

to by neutral names (`external`, `density`, `kinetic`) rather than by filename, since `CHGCAR` has no meaning outside VASP.

`VaspReader` is the reference implementation. `EspressoReader` and `GpawReader` are scaffolded, each documenting the specific traps: for Quantum ESPRESSO the Rydberg cutoff, the `ibrav` cell construction, the `alat/crystal` coordinate units, and the absence of a volume pre-factor; for GPAW the coarse-versus-fine grid distinction and the pseudo-versus-all-electron density ambiguity, which would be a silent physics error if mixed across a data set.

4.5 Numerical kernels

The classical electrostatic lattice sums — the pairwise Coulomb energy and the real-space Ewald term — are the scalar hot spots of the legacy solver. They live in `poraque/potentials/kernels.py` as vectorised NumPy, with no JIT and no optional dependency. Each is a pure function (arrays in, float out) carrying no Poraquê types, so a compiled backend can replace the bodies without touching any caller.

Implementation

Correctness is pinned by physics rather than by an implementation reference: the rock-salt Madelung constant is reproduced as 1.74756459 against the exact 1.7475645946, and is invariant under the Ewald splitting parameter α — the signature of a correctly balanced real/reciprocal decomposition.

4.6 Hardware acceleration

`resolve_device` prefers CUDA, then Apple Metal (MPS), then CPU, and degrades gracefully: an explicit request for an absent backend warns and falls back rather than raising, so a configuration written on a workstation still runs on a laptop.

Apple Metal required three genuine workarounds, all verified by regression tests:

No float64, no linalg.det. Both are used for the cell metric. Since a 3×3 inverse and determinant are $\mathcal{O}(1)$ work beside an FFT, they are evaluated *on the host in double precision* and only the result is moved to the device — which also keeps the extra accuracy on every backend. Note the ordering: a tensor must be moved to the host *before* widening, since casting an MPS tensor to float64 in place raises.

No complex einsum. The spectral contraction lowers to an `mps.gather` over complex values, which Metal rejects by aborting the process. Splitting the product into real arithmetic, $(a + ib)(c + id) = (ac - bd) + i(ad + bc)$, uses only real contractions and is numerically identical.

Strided complex views are silently wrong. Taking `.real` of a *non-contiguous* complex tensor yields a strided real view over which MPS miscomputes `einsum` — with no error and results 40–90 % off. The operands here are precisely such views (high-frequency corners such as `x_ft[..., nx-m1:, ny-m2:, :m3]`). Materialising them first reduces the error to $\sim 10^{-7}$.

Gradient clipping also required replacement: `clip_grad_norm_` calls `linalg.vector_norm`, unsupported for complex dtypes on MPS, and the spectral weights are complex. Viewing a complex tensor as its stacked (Re, Im) representation is exact — $\sum_k |z_k|^2 = \sum_k (a_k^2 + b_k^2)$ — so

the clip is identical on every backend.

Grid	CPU	MPS	speed-up
32^3	42.0 ms	19.2 ms	2.19×
48^3	124.3 ms	38.3 ms	3.24×
64^3	262.2 ms	45.9 ms	5.71×

Forward pass, five-run average, synchronised. CPU and MPS agree to $\sim 10^{-6}$ relative on a single forward pass with identical weights.

4.7 Configuration

A training run is defined by a YAML file with four sections — **data**, **model**, **training**, **output** — mirroring the four things a run needs. Command-line flags override individual entries, so one committed config can be swept from the shell without being edited:

```
python scripts/train_fno.py --config configs/train_config.yaml --epochs 500
```

Unknown keys are rejected rather than ignored: a typo such as **learn_rate** would otherwise be silently dropped, and the run would quietly use the default while appearing to be described by the file. The *resolved* configuration is written beside the results, recording what actually ran, overrides included.

The VASP interface

Reference data for this work comes from a modified VASP 6.2.0 that writes two additional volumetric files. Reproducing them exactly required reading the Fortran source; this chapter records what it says, because the details decide whether a Python reconstruction agrees with the reference or misses it by 13%.

5.1 The modified tags

Two INCAR flags were added, declared in `base.F` and parsed in `main.F`:

Tag	Variable	Writes
EXTCAR	LEXTCAR	local external (ionic) potential
TAUCAR	LTAUCAR	kinetic energy density

5.2 EXTCAR: the local pseudopotential

5.2.1 What is written

`main.F` calls `POTION` to build the potential in reciprocal space, transforms it to real space, and writes it with `OUTPOS + OUTPOT`. The choice of `OUTPOT` rather than `OUTCHG` is what makes `EXTCAR` a *potential* file in the `LOCPOT` sense: **no multiplication by Ω** . The source comment states the conventions directly: values in eV, no volume scaling, $\mathbf{G} = 0$ set to zero, a single spin-independent block, no PAW one-centre terms and no non-local projector contribution.

5.2.2 The formula

`POTION` (`pot.F`) contains, per species and per $\mathbf{G}$ vector:

```

ARGSC  = NPSPTS/P(NT)%PSGMAX
PSGMA2 = P(NT)%PSGMAX - P(NT)%PSGMAX/NPSPTS
ZZ      = -4*PI*P(NT)%ZVALF*FELECT

G = SQRT(GX**2+GY**2+GZ**2)*2*PI
IF ( (G /= 0) .AND. (G < PSGMA2) ) THEN
  I    = INT(G*ARGSC)+1
  REM  = G - P(NT)%PSP(I,1)
  VPST = PSP(I,2)+REM*(PSP(I,3)+REM*(PSP(I,4)+REM*PSP(I,5)))
  CVPS(N) = CVPS(N) + ( VPST + ZZ/G**2 ) / LATT_CUR%OMEGA * CSTRF(N,NT)
ELSE
  CVPS(N) = 0._q
ENDIF

```

which is

$$V_{\text{ext}}(\mathbf{G}) = \frac{1}{\Omega} \sum_s S_s(\mathbf{G}) \left[v_{\text{short}}^s(G) - \frac{4\pi Z_s^{\text{val}} e^2}{G^2} \right], \quad V_{\text{ext}}(\mathbf{0}) = 0. \quad (5.1)$$

Comparing with Eq. (2.6): the second term is *exactly* the bare-Coulomb form factor $f_s = 1$. The long-range physics was never the problem. The first term, v_{short} , is a cubic spline through a *tabulated* function read from the POTCAR — and that is the entire discrepancy.

Two further details are easy to miss and both are observable:

- **Hard truncation.** For $G \geq \text{PSGMA2}$ the *whole* contribution is zeroed, Coulomb tail included; VASP does not extrapolate the table. For the gold potential $\text{PSGMAX} = 75.6 \text{ \AA}^{-1}$ while a 128^3 grid reaches $|\mathbf{G}|_{\text{max}} = 106 \text{ \AA}^{-1}$, so the truncation is genuinely active and must be reproduced.
- **G carries the 2π .** `LATT_CUR%B` is the reciprocal lattice *without* 2π , so $\mathbf{G}$ is the ordinary angular wavevector in $\text{\AA}^{-1}$ — the same convention as `FieldGrid.get_g_vectors`.

5.2.3 The POTCAR table layout

The layout of the local part block is given by the reader in `pseudo.F`:

```

READ(10, '(1X,A1)') CSEL          ! the "local part" marker line
READ(10,*) P(NTYP)%PSGMAX         ! <-- FIRST number is PSGMAX, not ZVAL
ALLOCATE(P(NTYP)%PSP(NPSPTS,5))
READ(10,*) (P(NTYP)%PSP(I,2), I=1, NPSPTS)
DO I=1, NPSPTS
    P(NTYP)%PSP(I,1) = (P(NTYP)%PSGMAX/NPSPTS) * (I-1)
ENDDO
P(NTYP)%PSCORE = P(NTYP)%PSP(1,2)
CALL SPLCOF(P(NTYP)%PSP(1,1), NPSPTS, NPSPTS, 0._q)
```

with `NPSPTS = 1000`. The mesh is therefore uniform,

$$q_i = \frac{\text{PSGMAX}}{1000} (i - 1), \quad i = 1, \dots, 1000, \quad (5.2)$$

and the values are in $\text{eV} \cdot \text{\AA}^3$.

Note

The first number after the marker is `PSGMAX`, the table's maximum wavevector — *not* the valence charge. The two can look alike in some files, and the misreading was the secondary cause of the original discrepancy. Confirmed against the gold POTCAR: the value is 75.5890395431569, the block holds 1001 numbers (= `PSGMAX` + exactly 1000 samples), and `ZVAL` is 11.

`PSCORE` = $v_{\text{short}}(q \rightarrow 0) = 105.89 \text{ eV} \cdot \text{\AA}^3$ for gold feeds the `PSCENC` energy correction and does not enter the potential, since $V_{\text{ext}}(\mathbf{G} = 0)$ is zeroed.

5.2.4 Result

Implementing Eq. (5.1) verbatim — natural cubic spline on the `PSGMAX` mesh, analytic Coulomb tail, $\mathbf{G} = 0$ zeroed, truncation at `PSGMA2`, structure factors per species — gives:

Structure	grid	Gaussian model	tabulated
struct_000	128 ³	0.1338	2.13 × 10 ⁻⁵
struct_001	120×128×128	0.1303	5.95 × 10 ⁻⁵
struct_002	128×120×128	0.1285	5.75 × 10 ⁻⁵

(relative L^2 against the reference EXTCAR; Pearson $r = 1.000000$ in all three, MAE 0.34 meV.) The improvement is a factor of roughly 6000.

Caveat

The residual $\sim 6 \times 10^{-5}$ is the spline boundary condition: VASP calls SPLCOF(PSP(1,1),NPSPTS,NPSPTS,0._q) whereas the Python implementation uses a natural cubic spline. The difference is confined to the ends of the table and is worth about 0.6 meV. It is documented rather than chased.

5.2.5 Why an analytic form factor cannot work

For a single-species cell Eq. (5.1) can be inverted to recover the empirical form factor directly from a reference file,

$$f(\mathbf{G}) = -\frac{\Omega G^2 V_{\text{ext}}^{\text{ref}}(\mathbf{G})}{4\pi e^2 Z^{\text{val}} S(\mathbf{G})}. \quad (5.3)$$

Doing so for gold shows $f(G)$ decaying far more slowly than a Gaussian at low G and then *oscillating about zero* at large G — the signature of a repulsive pseudopotential core. No monotonic $e^{-G^2\sigma^2/2}$ can reproduce that, which is why fitting σ saturates near a relative L^2 of 0.13 regardless of the value chosen. The best-fit width came out at $\sigma^* = 0.374 \text{ \AA}$ on all three structures independently, a consistency that identified the limitation as structural rather than accidental.

5.3 TAUCAR: the kinetic energy density

5.3.1 Definition

The source comment gives $\text{tau_s}(\mathbf{r}) = \hbar^2/2m \sum_{\mathbf{k}} w_{\mathbf{k}} f_{\mathbf{k}} |\text{grad } \psi_{\mathbf{k},s}(\mathbf{r})|^2$, i.e. the positive-definite gradient form of Eq. (1.9), and *not* the Laplacian form. TAU_PW in metagga.F evaluates it per Cartesian direction:

```
DO I=1,NPL
  G1=WDES%IGX(I,NK)+WDES%VKPT(1,NK)
  GC=(G1*LATT_CUR%B(IDIR,1)+G2*LATT_CUR%B(IDIR,2)+G3*LATT_CUR%B(IDIR,3))
  CFA(I)=GC*W%CPTWFP(...)*CITPI          ! i*2pi*(G+k)_IDIR * c_G
ENDDO
CALL FFTWAV(NPL,WDES%NINDPW(1,NK),CFAR(1),CFA(1),GRID)
DO I=1,GRID%RL%NP
  CKIN(I)=CKIN(I)+HSQDTM*REAL(CONJG(CFBR(I))*CFAR(I))*WEIGHT
ENDDO
```

The gradient is applied *spectrally* as $i(\mathbf{G} + \mathbf{k})$ on the plane-wave coefficients, inverse-transformed to real space, and accumulated as $|\partial_{\alpha}\psi|^2$ weighted by $w_{\mathbf{k}}f_{\mathbf{k}}$, with $\text{HSQDTM} = \hbar^2/2m_e$.

5.3.2 Storage

CTAUC = CTAUC*LATT_CUR%OMEGA followed by OUTCHG means TAUCAR stores $\tau\Omega$, exactly as CHGCAR stores $\rho\Omega$, so that $\sum_{\mathbf{r}} \text{TAUCAR}(\mathbf{r})/(N_1N_2N_3)$ is the plane-wave kinetic energy of the cell in eV.

For ISPIN=2 the two blocks are (total, magnetisation), matching the CHGCAR layout.

5.3.3 Verdict

Poraquê's handling of TAUCAR required no change: the positive-definite definition, the Ω factor, the units, the shared grid and the plane-wave-only content all matched already. That τ is the pseudo quantity also explains an earlier observation — the bound (1.10) was found to hold at essentially every grid point (one violation in 32768, at the single voxel where spectral downsampling had rung τ slightly negative), because τ and ρ are pseudised *consistently*. The bound is strictly guaranteed only for all-electron quantities, so this was worth measuring rather than assuming.

5.4 What is absent from EXTCAR

Worth stating explicitly, because it bounds what the first learned map can be asked to do:

- no PAW one-centre terms — EXTCAR is a pseudo quantity;
- no non-local projectors — not local functions of $\mathbf{r}$, so they cannot appear in a volumetric file even in principle;
- no Hartree and no exchange–correlation, unlike LOCPOT which is $v_H + V_{\text{ext}}$. EXTCAR is the bare ionic term, which is precisely the right input for the $V_{\text{ext}} \mapsto \rho$ map;
- **no Ewald or ion–ion contribution.** That energy is a scalar (PSCENC, TEWEN), not a field; none is missing, because none belongs in this file.

Caveat

Two paths remain untested. The multi-species sum in Eq. (5.1) uses a per-species PSGMAX and is implemented as such, but the present data set is pure gold. And ISPIN=2 writes a second block in both CHGCAR and TAUCAR which the reader currently ignores — correct for these ISPIN=1 runs, a silent truncation otherwise.

CHAPTER 6

Results

6.1 The data set

Three gold structures, each a $3 \times 3 \times 3$ supercell of the face-centred cubic primitive cell (27 atoms, $Z^{\text{val}} = 11$, so 297 valence electrons), computed with the modified VASP 6.2.0 of Chapter 5 at $E_{\text{cut}} = 450 \text{ eV}$, `PREC=Accurate`, and a $5 \times 5 \times 5$ Γ -centred k -mesh.

Structure	Geometry	$\Omega [\text{\AA}^3]$	Native grid
struct_000	pristine fcc	484.53	$128 \times 128 \times 128$
struct_001	rattled + strained	477.65	$120 \times 128 \times 128$
struct_002	rattled + strained	476.92	$128 \times 120 \times 128$

The three native grids *differ*, which is not an inconvenience but the central test: it is exactly the situation Section 3.4 was designed for, and it arises automatically because the cell shape enters VASP’s grid selection.

Implementation

Fields are reduced to a longest axis of 32 points by the spectral truncation of Section 4.3, giving 32^3 , $30 \times 32 \times 32$ and $32 \times 30 \times 32$. The shape difference *survives* the reduction — had a fixed target shape been used instead, the ragged-grid capability would have gone untested.

6.2 Verification before learning

Nothing was trained until the field machinery was checked against quantities known independently.

Check	Expected	Measured
Poisson residual, $\nabla^2 V_{\text{ext}}$ vs $4\pi e^2 n_{\text{ion}}$	0	4.5×10^{-12}
Translation covariance	exact	2.1×10^{-14}
Linearity in Z^{val}	exact	0.0
$\int \rho \text{ d}\mathbf{r}$ from CHGCAR	297	297.0000
Spectral resampling, band-limited	exact	1.3×10^{-15}
Resampling, mean preserved	exact	5.6×10^{-17}
Madelung constant (rock salt)	1.7475645946	1.74756459
EXTCAR vs VASP, tabulated	—	2.1×10^{-5} rel. L^2

The electron count is the sharpest of these: it exercises the CHGCAR parser, the Ω convention and the grid metric simultaneously, and returns the exact integer.

6.3 Model 1: $V_{\text{ext}} \mapsto \rho$

Leave-one-out over the three materials; metrics on the held-out material in physical units ($e/\text{\AA}^3$).

Held out	MAE	RMSE	rel. L^2	R^2
struct_000	0.01667	0.03097	0.0318	0.9983
struct_001	0.02433	0.03984	0.0405	0.9973
struct_002	0.02423	0.04112	0.0418	0.9971
mean	0.02174	0.03731	0.0380	0.9976
baseline: mean density	0.5199	0.7589	0.7779	0.0000

The learned map is a factor of 20 better than predicting a constant density. The electron count, which nothing in the objective constrains, is recovered to 1.1–2.2% — precisely the single global degree of freedom that the charge-normalisation head of Section 7.1.3 would fix exactly and for free.

6.4 Model 2: $\rho \mapsto \tau$

Same protocol, in $\text{eV}/\text{\AA}^3$, with the Pauli head of Eq. (7.2) enabled. The analytic orbital-free functionals of Chapter 2 are evaluated on the same input as baselines.

Predictor	MAE	RMSE	rel. L^2	R^2
FNO (learned)	0.6600	1.2002	0.0620	0.9924
von Weizsäcker	8.99	14.22	0.7367	0.0150
Thomas–Fermi	9.38	25.99	1.3454	−2.2908
TF + vW/9	9.35	26.18	1.3554	−2.3395

The learned functional is better than the best classical approximation by a factor of about 12 in relative L^2 .

Note

Thomas–Fermi attains a *negative* coefficient of determination: as a pointwise predictor of τ it is worse than the constant mean. This is not a defect of the implementation but the expected failure of a uniform-electron-gas limit applied to a d -band metal, whose core and d -shell regions are about as far from uniform as matter gets. It is the gap that motivates a learned functional.

6.4.1 Integrated kinetic energy

The quantity that enters a total energy is $T_s = \int \tau \, \text{d}\mathbf{r}$, not τ pointwise:

Held out	predicted [eV]	reference [eV]	error
struct_000	6277.16	6207.07	1.13 %
struct_001	6198.07	6214.80	0.27 %
struct_002	6180.76	6217.06	0.58 %

6.4.2 Constraint satisfaction

With the head enabled, $\tau \geq \tau_{\text{vW}}[\rho]$ held at every one of the $\sim 3 \times 10^4$ grid points on all three folds, with minimum margins of 1.045, 0.950 and 1.024 eV/Å³. The margins are the informative number: the model is not sitting on the bound but predicting a Pauli term comfortably above it.

The fitted scale s came out at 5.22, 5.02 and 5.01 eV/Å³ across folds — consistent with the median τ_{P} measured directly from the data (4.81–5.21), which is a check that `fit_pauli_scale` estimates what it claims to.

Section 7.2 gives the matched comparison against an unconstrained backbone: the head improves every fold and every metric, by 18.3% in mean relative L^2 , while training 17.8% *faster*.

6.5 The inference pipeline

Chaining both models turns a bare crystal geometry into predicted ρ and τ with no wavefunctions and no self-consistency cycle:

$$\{\text{POSCAR}, \text{INCAR}, \text{POTCAR}\} \xrightarrow{\text{analytic}} V_{\text{ext}} \xrightarrow{\text{FNO 1}} \rho \xrightarrow{\text{FNO 2}} \tau. \quad (6.1)$$

Only the two field-to-field maps are learned; the first step is closed-form.

The importance of Chapter 5’s correction shows up here most sharply. Evaluating on `struct_002` with models that never saw it:

Field	Gaussian V_{ext}	tabulated V_{ext}
EXTCAR	0.6645	0.0014
CHGCAR	0.4482	0.0647
TAUCAR	0.2456	0.1190

(relative L^2 against the reference files.) The models were always *trained* on VASP’s own `EXTCAR`; the error was entirely in the inference path, which had been feeding them a potential from a different distribution. Correcting the analytic step alone improved the predicted density sevenfold, with no retraining.

Implementation

All predicted fields are written in `CHGCAR` format and open directly in VESTA. `-compare` additionally brings reference files onto the prediction grid by spectral truncation — the exact restriction — rather than failing on a shape mismatch.

6.6 Hardware

The full run — two tasks, three folds each, 200 epochs, spectral downsampling, figures and checkpoints — completes in roughly 20 minutes on Apple Silicon via MPS. Forward-pass throughput is given in Section 4.6; the advantage grows with grid size, from $2.2\times$ at 32^3 to $5.7\times$ at 64^3 , because kernel-launch overhead dominates at small sizes while the FFTs dominate at large ones.

6.7 What these numbers do and do not support

Caveat

With three closely related structures of a single element, leave-one-out measures **interpolation between nearby geometries of gold**. It demonstrates that the pipeline is correct, that the architecture has sufficient capacity, and that the learned kinetic functional beats the classical approximations on this system. It says **nothing** about transfer to other chemistry, other coordination environments, or other cutoffs. The training log carries this caveat in its own footer so that it travels with the numbers.

What *is* established independently of data-set size:

- the field machinery is verified against closed-form physics (Poisson, Madelung, electron count) to machine precision;
- the EXTCAR reconstruction matches a reference implementation to 2×10^{-5} ;
- one set of weights serves three different grid shapes, with CPU/accelerator agreement at 10^{-6} ;
- the von Weizsäcker bound is structural, verified at random initialisation and under adversarial optimisation;
- the classical functionals' failure on this system is quantified rather than asserted.

The obvious next step is more materials. The ingestion layer, the shape-bucketed sampler and the configuration system were all built for a heterogeneous data set; the binding constraint is now the data, not the code.

Physics-informed operators

7.1 Constrain by construction, not by penalty

A soft penalty $\lambda \|\text{violation}\|^2$ trades accuracy against constraint satisfaction and fully achieves neither: the weight must be tuned, the constraint holds only approximately, and a badly scaled term degrades the model while looking principled. Where a constraint can be built into the *parameterisation* it should be. It then holds for every weight configuration, at initialisation and after any optimiser step, with nothing to tune and no inference cost.

Three constraints move into the architecture.

7.1.1 Positivity

Predict $\log \rho$ and exponentiate. The inverse transform returns $e^y - \epsilon$, so $\rho > -\epsilon$ *identically*.

7.1.2 Electron count

$$\rho_{\text{out}}(\mathbf{r}) = N_{\text{val}} \frac{\tilde{\rho}(\mathbf{r})}{\int \tilde{\rho} \, d\mathbf{r}} \quad (7.1)$$

makes $\int \rho = N_{\text{val}}$ hold to machine precision, differentially, at the cost of one reduction. Strictly better than a penalty on the same quantity.

7.1.3 The von Weizsäcker bound

Using the exact decomposition (1.13), write the network output as

$$\tau_{\theta} = \tau_{\text{vW}}[\rho] + s \cdot \text{softplus}(f_{\theta}(\rho)) \quad (7.2)$$

so the network learns only the Pauli term. This buys two things at once:

1. **The bound becomes structural.** `softplus` is non-negative, so $\tau_{\theta} \geq \tau_{\text{vW}}$ identically. The inequality is enforced *non-strictly*, exactly as Hoffmann-Ostenhof states it: for a strongly negative output `softplus` underflows and the head returns $\tau = \tau_{\text{vW}}$ exactly — the correct one-orbital limit, which is *reachable* rather than merely approachable. A soft penalty can only approach it from above.
2. **The network stops re-deriving known physics.** On the gold data τ_{vW} accounts for $\sim 31\%$ of τ ; that fraction is now supplied analytically and exactly by a spectral gradient, leaving only the genuinely unknown remainder to learn.

Implementation

The scale s is fitted on the *training split only* — fitting it on the held-out material would

leak the answer — and is then optimised as $\log s$, which cannot change sign. $\tau_{\text{vW}}[\rho]$ is a fixed function of the *input*, so it contributes no gradient to θ : it is a per-sample offset, not a second trainable branch.

Caveat

The bound is a theorem for all-electron densities; CHGCAR and TAUCAR are pseudo quantities, so it must be *measured* before the head is enabled on a new data set. `pauli_bound_violation` does this. On the present data it holds at every point of two structures and at all but one point of the third.

7.2 Measured effect of the head

Identical data, seed, architecture and epoch count; only the head differs.

Held out	baseline	Pauli head	change
struct_000	0.0534	0.0363	−32.0 %
struct_001	0.0840	0.0762	−9.3 %
struct_002	0.0902	0.0734	−18.6 %
mean	0.0759	0.0620	−18.3 %

(relative L^2 on the held-out material.) The head wins on every fold and every metric; MSE falls 30.6 %.

Two results deserve more attention than the headline number.

T_s improves more than τ does. The integrated kinetic energy — the quantity that actually enters the total energy — improves from 2.55 $\rightarrow$ 1.13 %, 0.53 $\rightarrow$ 0.27 % and 0.90 $\rightarrow$ 0.58 %: roughly halved on every fold, a larger relative gain than the pointwise error. That is the expected direction, since τ_{vW} is supplied exactly and contributes no model error to the integral.

It trains faster. 148.6 s per fold against 180.8 s, and to a lower final training loss. The extra FFT for τ_{vW} is more than repaid by conditioning: the network fits only τ_{P} , whose dynamic range is far narrower than τ ’s. This is not an accuracy-versus-cost trade.

Caveat

The constraint did *not* bind. The unconstrained baseline violated the bound at zero points in all three folds, with margins of 0.83–1.05 eV/Å³. On converged reference data the head’s gain therefore comes from handing the model 31 % of the answer analytically, not from rescuing violations. The hard guarantee is insurance that costs nothing here and becomes load-bearing inside an SCF loop, where the functional sees non-converged densities far from the training distribution. These three structures cannot distinguish “the head enforces the bound” from “the bound is easy here”; the enforcement claim rests on the tests, which check it at random initialisation, at 50× weights and after training at a destructive learning rate.

7.3 Soft constraints

Where a constraint cannot be built in, it enters the objective:

$$\mathcal{L} = \mathcal{L}_{\text{data}} + \lambda_{\text{EL}} \mathcal{L}_{\text{EL}} + \lambda_{\text{TF}} \mathcal{L}_{\text{TF}} + \lambda_T \mathcal{L}_T + \lambda_{\text{scale}} \mathcal{L}_{\text{scale}}. \quad (7.3)$$

Every physics weight defaults to zero, so the objective is exactly the supervised baseline until a term is enabled deliberately. Three design rules:

1. **Physics terms act on the prediction decoded to physical units**, never on the normalised representation — a constraint is a statement about physics, not about preprocessing.
2. **Every term is dimensionless**. A raw $(\text{eV}/\text{\AA}^3)^2$ penalty varies by orders of magnitude between a light semiconductor and a transition-metal oxide, and no single weight could serve both.
3. **The data term is H^1 , not L^2** , when gradients matter: τ_{vW} depends on $\nabla \rho$, and gradient noise is amplified by $|\nabla \rho|^2/8\rho$ exactly where ρ is small.

7.3.1 The Euler–Lagrange residual

The most valuable soft term is Eq. (2.2) itself. Since μ is a constant, subtracting the cell average eliminates it and leaves a residual that must vanish for the exact density:

$$r(\mathbf{r}) = \frac{\delta T_s}{\delta \rho} + V_{\text{ext}} + v_{\text{H}}[\rho] + v_{\text{xc}}[\rho] - \overline{(\cdots)}. \quad (7.4)$$

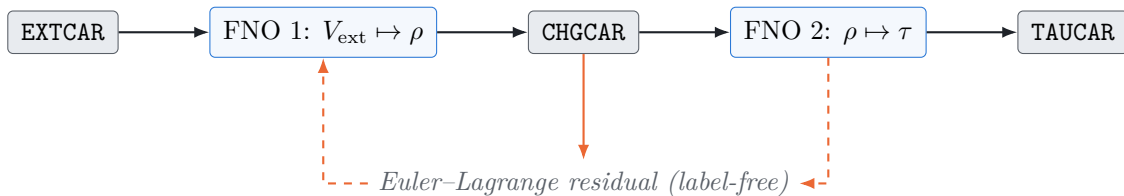
It contains only V_{ext} (the network input) and ρ (its output), so it requires **no additional labels**.

Caveat

The residual is only as good as the kinetic functional used to build it. With the analytic $\text{TF} + \lambda \text{vW}$ surrogate it supplies a physically correct *inductive bias*, not ground truth, and must carry a modest weight.

7.4 The closed loop

The end state is not two models but one differentiable orbital-free cycle. The link is Eq. (2.2) with $\delta T_s/\delta \rho$ obtained by autograd through Model 2, since $T_s = \int \tau_{\theta_B}[\rho] \text{d}\mathbf{r}$:



Three consequences, in order of practical value:

1. **Model 2 is regularised on Model 1's output distribution**. A functional trained only on converged reference densities fails when placed inside an SCF loop, because it then sees non-converged densities it was never shown. Joint training on the residual exposes it to exactly that distribution.

2. **The optimised quantity becomes the used quantity.** Orbital-free DFT needs $\delta T_s/\delta\rho$, not τ ; training through the derivative removes the objective mismatch of Chapter 2.
3. **Inference becomes physically consistent.** A predicted (ρ, τ) pair satisfies the variational condition it is supposed to satisfy, rather than merely resembling the training data.

Beyond that, replacing Model 1’s forward pass with a fixed-point solve of Eq. (2.2) using the learned functional — a deep-equilibrium model, differentiated by the implicit function theorem — would make the network an orbital-free *solver* rather than a regressor, with transferability bounded by the functional alone.

7.5 Why this is cheap here

Physics-informed training is normally expensive: derivatives must come from autograd through the network or from finite differences, and the latter injects a truncation error that the loss misattributes to the model. On a plane-wave grid neither problem arises.

Operator	On this grid	Cost
∇	$i\mathbf{G}$	one FFT, exact
∇^2	$-G^2$	one FFT, exact
$v_H[\rho]$	$4\pi e^2 \rho(\mathbf{G})/G^2$	one FFT, exact
τ_{vW}	$ \nabla\rho ^2/8\rho$	one FFT, exact

These are exact for band-limited fields — precisely what a plane-wave grid carries — and the operator is already computing FFTs in every layer. The physics and the architecture share one mathematical substrate. That is the strongest technical argument for a Fourier neural operator here, independent of accuracy.

7.6 Roadmap

Phase	Work	Status
0	supervised baseline, honest split by material	done
1	hard constraints: log head, charge normalisation, Pauli head	Pauli head done
2	H^1 data loss; scaling-relation augmentation	available
3	Euler–Lagrange residual with the analytic surrogate	available
4	$\delta T_s/\delta\rho$ by autograd; joint training	planned
5	cycle consistency via density-to-potential inversion	planned
6	deep-equilibrium orbital-free solver	stretch

Phases 1–3 are low risk and independently valuable. Phase 4 is the research contribution.

The role of Model 2: a learned kinetic energy functional

8.1 The question, stated without charity

A skeptic’s version of the objection is worth writing down, because the weak answers to it are the ones most often given:

*You already have Model 1, which predicts ρ from the geometry. If you have a converged DFT calculation you have τ for free; and if you do not have one, you have no ρ to feed Model 2 either. What is **chg2tau** for?*

Three answers are available. Two are weak.

“ **τ is a useful observable**” — **weak**. It enters meta-GGA functionals, the electron localisation function and bonding analysis. True, but anyone holding a converged **CHGCAR** also holds a **TAUCAR**. Predicting a quantity one already has is a benchmark, not an application.

“**It completes the pipeline**” — **weak**. Chaining $V_{\text{ext}} \rightarrow \rho \rightarrow \tau$ yields τ for a structure never computed, which is useful for screening. But it makes Model 2 a convenience: nothing in it is more than interpolation, and its errors feed back into nothing.

“**It is the orbital-free kinetic energy functional**” — **the actual reason**. See below.

$T_s[\rho]$ is the *only* term of the total energy with no accurate explicit density functional. Kohn–Sham theory evades the problem by reintroducing orbitals and pays $\mathcal{O}(N^3)$ for it. Orbital-free DFT removes them and scales near-linearly, but is only as good as its functional. A model of $\rho \mapsto \tau$ is such a functional, because

$$T_s[\rho] = \int \tau[\rho](\mathbf{r}) \, d\mathbf{r}. \quad (8.1)$$

Model 2 is therefore not a predictor of a convenient observable; it is a candidate solution to the central open problem of orbital-free DFT, in the one form a network can represent and autograd can differentiate. **Model 2 is the scientific contribution; Model 1 is the infrastructure around it.**

8.2 Is τ a functional of ρ ?

Worth settling, because the answer is subtler than “Hohenberg–Kohn says so”.

For the integrated quantity the argument is clean: HK gives $\rho \rightarrow v_s$ uniquely for non-interacting v -representable densities, v_s determines the orbitals, and the orbitals determine T_s . The same

chain $\rho \rightarrow v_s \rightarrow \{\psi_i\} \rightarrow \tau$ establishes $\tau[\rho]$ as well. But note what that chain contains — solving the Kohn–Sham equations. The functional is thus

- **exact but non-local:** τ at one point depends on ρ everywhere, because the orbitals are delocalised Bloch states;
- **not semi-local:** no finite gradient expansion converges to it, which is precisely why Thomas–Fermi and its gradient corrections fail;
- **expensive to evaluate exactly:** the definition requires the very diagonalisation orbital-free DFT exists to avoid.

Two architectural consequences follow. The architecture *must* be non-local, which rules out a finite-receptive-field convolutional network and motivates the globally coupled Fourier layer of Chapter 3. And the target is well posed: unlike most machine-learning targets, $\tau[\rho]$ is a genuine single-valued functional, so failure to learn it is a capacity or data problem, not an ill-posedness problem.

Caveat

τ is a functional of ρ for ground-state, non-interacting v -representable densities. Inside a self-consistency loop the intermediate densities are not ground states of any v_s , so strictly $\tau[\rho]$ is undefined there. Every orbital-free calculation uses the functional anyway — but it means a model trained only on converged densities is *extrapolated* every time it is used variationally. This is the single largest risk in the programme.

8.3 What the model must get right — and it is not τ

Orbital-free DFT does not consume τ . It consumes the functional derivative

$$v_s^{\text{kin}}(\mathbf{r}) \equiv \frac{\delta T_s}{\delta \rho(\mathbf{r})}, \quad (8.2)$$

which is the term appearing in Eq. (2.2) and therefore the term that determines the density.

A model can have small $\|\tau_\theta - \tau\|$ and a *useless* derivative, because differentiation amplifies high-frequency error: a ripple of amplitude ϵ and wavevector G contributes ϵ to the value but ϵG to the gradient. Three consequences:

1. Reporting only pointwise τ error reports a **proxy**. The relative L^2 figures of Chapter 6 are necessary, not sufficient.
2. The H^1 data loss exists for this reason: it penalises the gradient error that plain L^2 ignores.
3. The definitive test is **downstream** — insert the functional into an orbital-free minimiser and measure the converged density and energy.

8.4 The mechanism: $\delta T_s / \delta \rho$ by autograd

This is what makes a *neural* functional more useful than a tabulated one, and it is mechanically simple: T_s is a scalar computed from ρ by a differentiable program, so one backward pass yields $\partial T_s / \partial \rho_i$ at every grid point — no finite differences and no derivative to derive by hand.

```
rho = rho_physical.detach().requires_grad_(True)      # (B,1,Nx,Ny,Nz)

tau = target_transform.inverse(model2(input_transform(rho), cell))
T_s = integrate(tau, cell)                            # (B,), eV

grad, = torch.autograd.grad(T_s.sum(), rho, create_graph=True)
dT_s_drho = grad / volume_element                   # see below
```

8.4.1 The discretisation factor

With $T_s = \sum_i \tau_i \Delta v$, and the functional derivative defined by $\delta T_s = \int (\delta T_s / \delta \rho) \delta \rho \, d\mathbf{r} \approx \sum_i (\delta T_s / \delta \rho)_i \delta \rho_i \Delta v$, while autograd returns $\delta T_s = \sum_i (\partial T_s / \partial \rho_i) \delta \rho_i$,

$$\boxed{\frac{\delta T_s}{\delta \rho}(\mathbf{r}_i) = \frac{1}{\Delta v} \frac{\partial T_s}{\partial \rho_i}} \quad \Delta v = \frac{\Omega}{N_1 N_2 N_3}. \quad (8.3)$$

Caveat

Omitting the $1/\Delta v$ rescales the entire kinetic potential by the number of grid points — a factor of 3×10^4 on these grids — which would silently destroy any Euler–Lagrange residual built from it, without raising anything. `create_graph=True` is likewise required if the derivative is to appear in a loss that is itself backpropagated.

8.4.2 Validation of the derivative

The derivative is meaningful only if verified independently:

Test	Expectation
Substitute analytic Thomas–Fermi for Model 2	recovers $\frac{5}{3} C_{\text{TF}} \rho^{2/3}$
Substitute von Weizsäcker	recovers $-\frac{1}{2} \nabla^2 \sqrt{\rho} / \sqrt{\rho}$
Finite-difference a few random voxels	agrees to FD accuracy
Uniform density	$\delta T_s / \delta \rho$ constant in space

These are cheap and they catch the Δv error, a broken transform chain, and any accidental `detach()`.

8.5 Coupling: how Model 2 trains Model 1

8.5.1 The equation

The Euler–Lagrange condition (2.2) holds pointwise at the ground state with μ constant in space. Every term is available:

Term	Source	Exact?
V_{ext}	Model 1’s <i>input</i>	exact (tabulated pseudopotential)
ρ	Model 1’s <i>output</i>	learned
$v_{\text{H}}[\rho]$	$4\pi e^2 \rho(\mathbf{G}) / G^2$, one FFT	exact
$\delta T_s / \delta \rho$	autograd through Model 2	learned
$v_{\text{xc}}[\rho]$	LDA/GGA, or omitted	approximate
μ	eliminated by the cell average	—

Subtracting the cell average removes the unknown μ exactly, leaving the residual of Eq. (7.4), which must vanish for the true ground-state density.

8.5.2 Why this is worth the trouble

The residual is computable from **Model 1’s own input and output**, plus Model 2. It requires **no additional labels** — no extra DFT calculations, no targets — so it can be evaluated on unlabelled structures, on perturbed or interpolated geometries generated on the fly, and on the model’s own predictions during training.

That converts Model 2 from a passive predictor into an **active physical constraint on Model 1**. This coupling is what the architecture is built around.

8.5.3 The loss

$$\mathcal{L}_1 = \underbrace{\|\rho_\theta - \rho^{\text{DFT}}\|_{\text{rel}}}_{\text{data}} + \lambda_{\text{EL}} \underbrace{\left\langle (r(\mathbf{r})/s)^2 \right\rangle}_{\text{Euler-Lagrange}} + \lambda_N \underbrace{\left(\frac{\int \rho_\theta - N}{N} \right)^2}_{\text{particle number}} \quad (8.4)$$

with $s = \text{std}(V_{\text{ext}})$ rendering the residual dimensionless and comparable across materials whose potentials differ by an order of magnitude. Three rules apply, each established elsewhere in this project: physics terms act on the prediction decoded to *physical* units; every term is dimensionless; and λ_{EL} is ramped only after the data term has converged, since a residual evaluated at random initialisation is noise.

8.5.4 Freeze Model 2, or train jointly?

Frozen. Model 2 is a fixed physics oracle. Simple, stable, and the constraint is exactly as good as the functional. Recommended first.

Joint. Mutual regularisation, and Model 2 is exposed to the densities Model 1 actually produces — which directly attacks the distribution-shift problem. But there is a real hazard.

Caveat

The Euler–Lagrange residual alone has trivial solutions. With both models free, the pair can drive $r \rightarrow 0$ by co-adapting: Model 2 can learn a functional whose derivative cancels $V_{\text{ext}} + v_{\text{H}} + v_{\text{xc}}$ for whatever ρ Model 1 emits, without either being correct. The residual is a *consistency* condition, not a *correctness* condition. Joint training is safe only with both data terms retained and dominant; the diagnostic of collapse is a residual that falls while the held-out data error rises.

8.6 Critical assessment

8.6.1 Genuine strengths

- The target is a well-posed functional, not a fitted correlation.
- It attacks the actual bottleneck of orbital-free DFT.
- The derivative is free and exact via autograd — removing the traditional obstacle to proposing new kinetic functionals, which is deriving $\delta T_{\text{s}}/\delta \rho$ by hand.
- Exact constraints exist and can be imposed structurally: $\tau \geq \tau_{\text{vW}}$ already is, supplying

$\sim 31\%$ of τ analytically.

- On this data it beats every classical functional by roughly an order of magnitude, with Thomas–Fermi at negative R^2 .

8.6.2 Real weaknesses

- **Distribution shift** is the central risk, and it is a domain question rather than a practical inconvenience: the functional is strictly undefined off the v -representable manifold.
- **We optimise τ and use $\delta T_s/\delta\rho$** (Section 8.3). Until training runs through the derivative, the objective is a proxy.
- **The derivative is unvalidated.** Nothing in the results of Chapter 6 bears on $\delta T_s/\delta\rho$; the Section 8.4 checks are not yet implemented.
- **Data scale.** Three structures of one element.
- **The Pauli potential constraint is unused.** Levy–Ou–Yang gives $v_P = \delta T_P/\delta\rho \geq 0$ — a constraint on the *derivative*, which is exactly the object that should be constrained.

8.7 Verification plan

Falsifiable, in dependency order:

1. **Derivative correctness** — the substitution tests above. Blocking: nothing downstream is meaningful until they pass.
2. **Residual floor** — evaluate $r(\mathbf{r})$ with the *reference* ρ and a classical functional. It will not vanish, since the functional is approximate; record the floor, because a learned functional must beat it to be adding anything.
3. **Ablation** — $\lambda_{\text{EL}} = 0$ versus > 0 on held-out material, matched otherwise. Same protocol as the Pauli-head comparison of Section 7.2.
4. **Label efficiency** — the real promise. Train Model 1 on k labelled structures plus m unlabelled ones under the residual, and show the curve beats k labelled alone. If the residual cannot buy label efficiency, its main practical argument fails.
5. **Downstream** — insert the functional into an orbital-free minimiser and measure converged density and energy error. The only test that matters.

8.8 Status

Component	State
$\rho \mapsto \tau$ supervised	done (rel. L^2 0.0620, R^2 0.9924)
$\tau \geq \tau_{\text{vW}}$ structural	done (PauliResidualOperator)
Exact spectral ∇ , ∇^2 , v_{H}	done
Euler–Lagrange residual, analytic functional	done
$\delta T_{\text{s}}/\delta \rho$ via autograd	not implemented
Joint training	not implemented
Pauli potential $v_{\text{p}} \geq 0$	not implemented
Orbital-free minimiser integration	not implemented

The gap between the fourth and fifth rows is the whole programme: everything up to the analytic residual is infrastructure, and the autograd derivative is what turns two regressions into a differentiable orbital-free solver.