Regression

3. Locally Weighted Regression Methods

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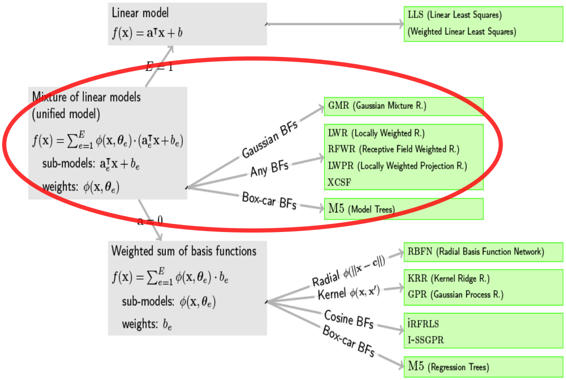
Reminder: Outline of methods



- Multiple local and weighted least square regressions (shown with LWR)

Stulp, F. and Sigaud, O. (2015) Many regression algorithms, one unified model: A review. *Neural Networks*, 69:60–79

Locally Weighted Regression

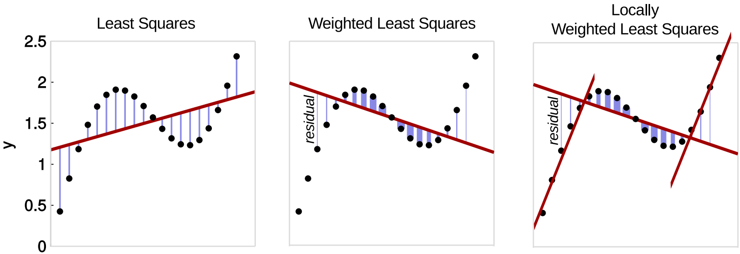
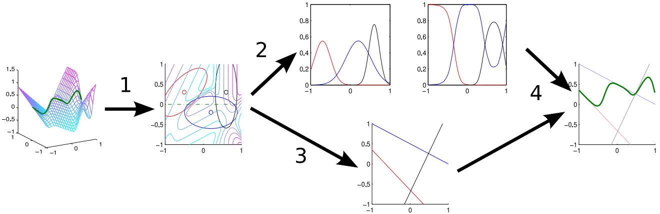


Figure: The thickness of the lines indicates the weights.

- General idea:
  - Split the function domain into linear parts
  - Give more weight to the centers
- Local linear models are tuned with Least Squares
- The importance of datapoints is represented by a Gaussian function

William S Cleveland and Susan J Devlin (1988) Locally weighted regression: an approach to regression analysis by local fitting. *Journal of the American statistical association*, 83(403):596–610.

Locally Weighted Regression: Processes



1. Define the split into regions (receptive fields): by hand, data-driven, evolutionary
2. Determine relative importance of domains
3. Find linear models in the regions
4. Combine all linear models

Atkeson, C. (1991) Using locally weighted regression for robot learning. In *Proceedings of the IEEE International Conference on Robotics and Automation (ICRA)*, vol. 2, pp. 958–963

Combining linear models

- There are E features, or receptive fields (RF)
- Each RF is defined as a Gaussian $\phi(\mathbf{x}, \boldsymbol{\theta}_i) = e^{-\frac{1}{2}(\mathbf{x}-\mu_i)^T \boldsymbol{\Sigma}_i^{-1}(\mathbf{x}-\mu_i)}$ with $\boldsymbol{\theta}_i = (\mu_i, \boldsymbol{\Sigma}_i)$
- Each RF tunes a local linear model

$$\Psi_i(\mathbf{x}) = \mathbf{a}_i^\intercal \mathbf{x} + b_i$$

- Gaussians tell you how much each RF contributes to the output

$$y = \frac{\sum_{i=1}^{E} \phi(\mathbf{x}, \boldsymbol{\theta}_i) \Psi_i(\mathbf{x})}{\sum_{i=1}^{E} \phi(\mathbf{x}, \boldsymbol{\theta}_i)}$$

Batch learning (1)

- Consider a batch of N $\{(\mathbf{x}^{(i)}, y^{(i)})\}_{1 \leq i \leq N}$ data.
-
$$y = f(\mathbf{x}) = \frac{\sum_{i=1}^{E} \phi(\mathbf{x}, \boldsymbol{\theta}_i)\Psi_i(\mathbf{x})}{\sum_{i=1}^{E} \phi(\mathbf{x}, \boldsymbol{\theta}_i)}$$
with $\Psi_i(\mathbf{x}) = w(\mathbf{x})^\mathsf{T}\boldsymbol{\theta}_i$ and $w(\mathbf{x}) = (\mathbf{x}_1 \ \mathbf{x}_2 \cdots \mathbf{x}_d \ 1)^\mathsf{T}$.
- Each local model is computed using the following locally weighted error:

$$\begin{aligned}\epsilon_i(\boldsymbol{\theta}_i) &= \frac{1}{2N}\sum_{j=1}^{N} \phi(\mathbf{x}^{(j)}, \boldsymbol{\theta}_i)\left(y^{(j)} - \Psi_i(\mathbf{x}^{(j)})\right)^2 \\ &= \frac{1}{2N}\sum_{j=1}^{N} \phi(\mathbf{x}^{(j)}, \boldsymbol{\theta}_i)\left(y^{(j)} - w(\mathbf{x}^{(j)})^\mathsf{T}\boldsymbol{\theta}_i\right)^2.\end{aligned}$$

Batch learning (2)

- As with the least squares method, we try to cancel out the gradient:

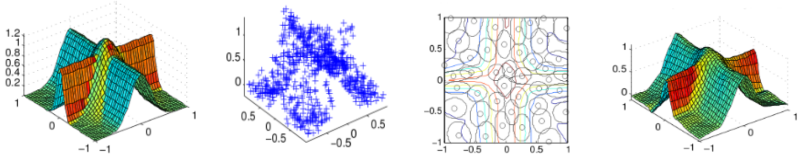
$$-\frac{1}{N}\sum_{j=1}^{N}\phi(\mathbf{x}^{(j)},\boldsymbol{\theta}_i)w(\mathbf{x}^{(j)})\left(y^{(j)}-w(\mathbf{x}^{(j)})^{\mathsf{T}}\boldsymbol{\theta}_i\right)=0.$$

- Therefore, we pose $\boldsymbol{\theta}_i = A_i^{\sharp} b_i$, with:

$$A_i = \sum_{j=1}^{N}\phi(\mathbf{x}^{(j)},\boldsymbol{\theta}_i)w(\mathbf{x}^{(j)})w(\mathbf{x}^{(j)})^{\mathsf{T}}$$

$$b_i = \sum_{j=1}^{N}\phi(\mathbf{x}^{(j)},\boldsymbol{\theta}_i)w(\mathbf{x}^{(j)})y^{(j)}.$$

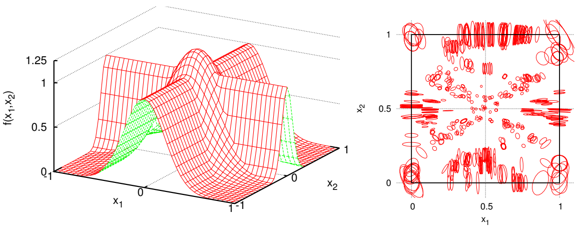
LWPR: general goal



- Non-linear function approximation in very large spaces
- Using PLS to project linear models in a smaller space
- Adds receptive fields around non covered datapoints, moves the shape, cannot remove them
- Good along local trajectories

Schaal, S., Atkeson, C. G., and Vijayakumar, S. (2002). Scalable techniques from nonparametric statistics for real time robot learning. *Applied Intelligence*, 17(1):49–60.

XCSF: overview

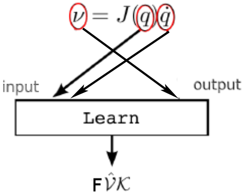


- XCSF is a Learning Classifier System [Holland(1975)]
- Linear models weighted by Gaussian functions (similar to LWPR)
- Linear models are updated using RLS
- Gaussian functions adaptation: Σ_i^{-1} and c_i are updated using a GA
- Condensation: reduce population to generalize better

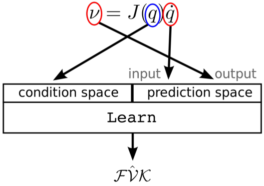
Wilson, S. W. (2001) Function approximation with a classifier system. In *Proceedings of the Genetic and Evolutionary Computation Conference (GECCO-2001)*, pages 974–981, San Francisco, California, USA. Morgan Kaufmann

XCSF: key feature

- Distinguish the space of linear models (prediction space) and the space of weights (condition space)
- Forward kinematics: $\dot{\boldsymbol{\xi}} = F_\theta(\mathbf{q}, \dot{\mathbf{q}})$ $\dot{\boldsymbol{\xi}} = J(\mathbf{q})\,\dot{\mathbf{q}}$
- Forward dynamics: $\ddot{\mathbf{q}} = G_\theta(\mathbf{q}, \dot{\mathbf{q}}, \boldsymbol{\Gamma})$ $\ddot{\mathbf{q}} = A(\mathbf{q})^{-1}(\boldsymbol{\Gamma} - \boldsymbol{n}(\mathbf{q}, \dot{\mathbf{q}}))$



Forward kinematics with LWPR

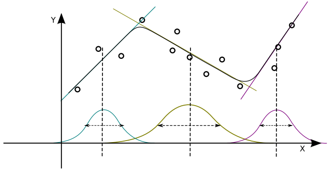


Forward kinematics with XCSF

- Example: learning forward kinematics $(\mathbf{x} = < \mathbf{q}, \dot{\mathbf{q}} >)$
- LWPR: $\hat{f}(\mathbf{x}) = \sum_{i=1}^{E} \phi((\mathbf{q}, \dot{\mathbf{q}}), \boldsymbol{\theta}_i) \cdot (b_i + \mathbf{a}_i^\intercal(\mathbf{q}, \dot{\mathbf{q}}))$
- XCSF: $f(\mathbf{x}) = \sum_{i=1}^{E} \phi(\mathbf{q}, \boldsymbol{\theta}_i) \cdot (b_i + \mathbf{a}_i^\intercal \dot{\mathbf{q}})$

Butz, M., Pedersen, G., and Stalph, P. (2009) Learning sensorimotor control structures with XCSF: redundancy exploitation and dynamic control. In *Proceedings of the 11th Annual conference on Genetic and evolutionary computation*, pages 1171–1178. ACM

GMR



$$y = \sum_{k=1}^{K} h_k(\mathbf{x})(\mu_{k,Y} + \Sigma_{k,YX}\Sigma_{k,Y}^{-1}(\mathbf{x} - \mu_{k,X}))$$

With

$$\mu_k = [\mu_{k,X}^T, \mu_{k,Y}^T]^T \text{ and } \mathbf{\Sigma}_k = \begin{pmatrix} \Sigma_{k,X} & \Sigma_{k,XY} \\ \Sigma_{k,YX} & \Sigma_{k,YX} \end{pmatrix}$$

- From input-output manifold to input-output function
- Same representation as the others, using $\boldsymbol{\theta}^\intercal = \Sigma_{i,YX}\Sigma_{i,Y}^{-1}$ and $b_i = \mu_{i,Y} - \Sigma_{i,YX}\Sigma_{i,X}^{-1}\mu_{i,X}$
- We get

$$\tilde{y} = \sum_{i=1}^{E} \frac{\pi_i \phi(\mathbf{x}, \boldsymbol{\theta}_i)}{\sum_{l=1}^{E} \pi_l \phi(\mathbf{x}, \boldsymbol{\theta}_l)} (\boldsymbol{\theta}^\intercal \mathbf{x} + b_i),$$

- Same as usual + scaling with the priors $\pi_i \rightarrow \pi_i = 1$ in standard model.
- Incorporates Bayesian variance estimation $\rightarrow$ The richest representation

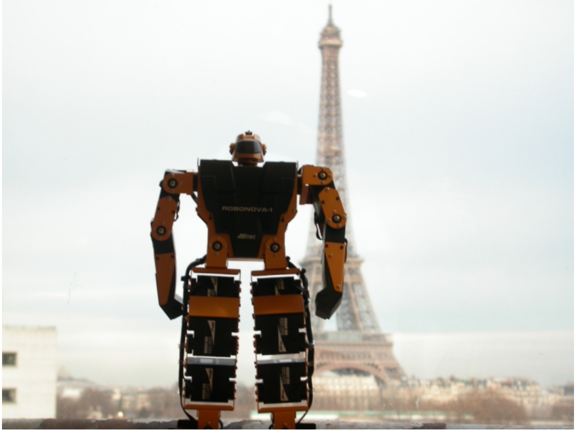
Hersch, M., Guenter, F., Calinon, S., & Billard, A. (2008) "Dynamical system modulation for robot learning via kinesthetic demonstrations." *IEEE Transactions on Robotics*, 24(6), 1463–1467

LWR methods: main features

Algo	LWR	LWPR	GMR	XCSF
Number of RFs	fixed	growing	fixed	adaptive
Position of RFs	fixed	fixed	adaptive	adaptive
Size of RFs	fixed	adaptive	adaptive	adaptive

- The main differences are in meta-parameter tuning
- Fewer hyperparameters is better, but less flexibility

Any question?



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