

MATH4408 Notes

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Definition 1 (Ring). A set \mathcal{R} is called a **ring** if it is non-empty, and $A \cup B$ and $A \setminus B$ are in \mathcal{R} whenever $A, B \in \mathcal{R}$.

Definition 2 (Algebra). A set \mathcal{R} is an **algebra** if it is a ring and contains a universal set R .

Definition 3 (σ -ring). A ring \mathcal{R} is a σ -ring if

$$\bigcup_{n=1}^{\infty} A_n \in \mathcal{R} \quad \text{whenever} \quad (A_n)_{n=1}^{\infty} \subset \mathcal{R}.$$

Definition 4 (σ -algebra). A σ -algebra is a σ -ring containing the universal set R .

Theorem 1 (Ring generated by a collection). If \mathcal{U} is a non-empty collection of subsets of R , then there exists a unique ring $\mathcal{R}(\mathcal{U})$ containing \mathcal{U} and contained in every other ring that contains \mathcal{U} .

Proof. Let Σ be the class of all rings that contain \mathcal{U} . Since $2^R \in \Sigma$, the class is non-empty. Define

$$\mathcal{R}(\mathcal{U}) = \bigcap_{r \in \Sigma} r.$$

By construction it is contained within all rings that contain \mathcal{U} . It is easy to see that it is a ring. \square

Definition 5 (Ring generated by a collection). If \mathcal{R} is a ring containing \mathcal{U} and is contained by all rings which contain \mathcal{U} , then it is called the **ring generated by \mathcal{U}** , denoted $\mathcal{R}(\mathcal{U})$.

Definition 6 (Measure). Let \mathcal{R} be an algebra of sets. A function $\mu : \mathcal{R} \rightarrow [0, \infty]$ is a **measure** if:

1. $\mu(\emptyset) = 0$
2. **σ -additivity.** If $(A_n)_{n=1}^{\infty} \subset \mathcal{R}$ are such that $A_i \cap A_j = \emptyset$ for $i \neq j$, and $\bigcup_{n=1}^{\infty} A_n \in \mathcal{R}$, then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

Definition 7 (Outer measure). Let μ be a measure on an algebra $\mathcal{R} \subset 2^R$. The **outer measure** associated with μ is the function $\mu^* : 2^R \rightarrow \mathbb{R}$ given by

$$\mu^*(A) = \inf \sum_{i=1}^{n(\infty)} \mu(E_i)$$

where the infimum is taken over all possible coverings of A by sets E_1, \dots, E_n, \dots from \mathcal{R} . The coverings may or may not be finite.

Theorem 2. If $A \in \mathcal{R}$ then $\mu^*(A) = \mu(A)$.

Proof. Since $A \in \mathcal{R}$ and it covers itself, we have $\mu^*(A) \leq \mu(A)$. Choose $\epsilon > 0$. There exists a cover $(E_i)_{i=1}^\infty \subseteq \mathcal{R}$ such that

$$\mu^*(A) > \left(\sum_{n=1}^\infty \mu(E_i) \right) - \epsilon.$$

Now, $A = A \cap (\bigcup_{i=1}^\infty E_i) = \bigcup_{i=1}^\infty (A \cap E_i)$ so

$$\mu(A) \leq \sum_{i=1}^\infty \mu(A \cap E_i) \leq \sum_{i=1}^\infty \mu(E_i) \leq \mu^*(A) + \epsilon.$$

As $\epsilon \rightarrow 0$, we must have $\mu(A) \leq \mu^*(A)$. □

Theorem 3. The outer measure μ^* is σ -semi-additive, i.e. given $(A_n)_{n=1}^\infty \subset 2^R$ we have

$$\mu^* \left(\bigcup_{n=1}^\infty A_n \right) \leq \sum_{n=1}^\infty \mu^*(A_n).$$

Proof. For each n and $\epsilon > 0$, there exists a cover $(E_{in})_{i=1}^\infty \subset \mathcal{R}$ such that $A_n \subset \bigcup_{i=1}^\infty E_{in}$ and

$$\mu^*(A_n) > \sum_{i=1}^\infty \mu(E_{in}) - \frac{\epsilon}{2^n}.$$

Clearly,

$$\bigcup_{n=1}^\infty A_n \subset \bigcup_{n=1}^\infty \bigcup_{i=1}^\infty E_{in}$$

and

$$\begin{aligned} \mu^* \left(\bigcup_{n=1}^\infty A_n \right) &< \sum_{i,n=1}^\infty \mu(E_{in}) \\ &\leq \sum_{n=1}^\infty \mu^*(A_n) + \frac{\epsilon}{2^n} \end{aligned}$$

□

Definition 8 (Measureable). A set $A \subset R$ is *measurable* if

$$\mu^* = \mu^*(A \cap E) + \mu^*(\bar{A} \cap E) \quad (0.1)$$

for all $E \subset R$.

Theorem 4. The class $\tilde{\mathcal{R}}$ is a σ -algebra containing \mathcal{R} . The restriction $\tilde{\mu}$ is a measure on $\tilde{\mathcal{R}}$

Definition 9 (Complete). A measure μ on an algebra \mathcal{R} is called *complete* if every subset of a set $A \in \mathcal{R}$ such that $\mu(A) = 0$ lies in \mathcal{R} .

Theorem 5. If μ is a measure on an algebra \mathcal{R} with outer measure μ^* , and $\mu^*(A) = 0$ for some $A \subset R$ then A is measurable.

Proof. We need

$$\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap \bar{A})$$

for $E \subset R$. We know

$$\mu^*(E \cap A) \leq \mu^*(A) = 0$$

and $\mu^*(E \cap \bar{A}) \leq \mu^*(E)$ by monotonicity. □